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# $\mathcal{H}_2$ state-feedback control for LPV systems with input saturation and matched disturbance

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**Abstract** This paper proposes a controller design for linear parameter-varying (LPV) systems with input saturation and a matched disturbance. On the basis of the feedback gain matrix $K(\theta(t))$ and the Lyapunov function $V(x(t))$, three types of controllers are suggested under $\mathcal{H}_2$ performance conditions. To this end, the conditions used for designing the $\mathcal{H}_2$ state-feedback controller are first formulated in terms of parameterized linear matrix inequalities (PLMIs). They are then converted into linear matrix inequalities (LMIs) using a parameter relaxation technique. The simulation results illustrate the effectiveness of the proposed controllers.

**Keywords** Gain scheduling · Input saturation · Linear parameter-varying (LPV) system · Parameter-dependent Lyapunov function (PDLF) · Disturbance rejection

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## 1 Introduction

Gain-scheduling approaches for a linear parameter-varying (LPV) system have been widely studied over the past several years [1–14]. For instance, gain-scheduling approaches have been reviewed in [1]. The authors of [2, 3] proposed a stabilization problem for LPV systems. In [4], a gain-scheduled output feedback controller was designed to stabilize an LPV system. Gain-scheduled $\mathcal{H}_2$ and $\mathcal{H}_\infty$ controls were proposed in [5–9]. In [12], the use of a gain-scheduling method for handling actuator saturation was proposed. The authors of [13] presented stability conditions and guaranteed cost controllers via a parameter relaxation technique. The authors of [14] introduced the gain-scheduled $\mathcal{H}_\infty$ control for LPV systems with input saturation and disturbances.

However, in these former studies, $\mathcal{H}_2$ controllers have not been designed for LPV systems with input saturation and disturbances. The authors of [2, 3, 13] derived an $\mathcal{H}_2$ controller for LPV systems; however, they did not consider input saturation and disturbances. Further, although the authors of [4–11] handled disturbances in their LPV systems, they did not deal with input saturation. In [12], gain-scheduling $\mathcal{H}_2$

controllers for an LPV system with input saturation were designed using only vertices of the polytope to solve parameterized linear matrix inequality (PLMI) conditions, which resulted in a deterioration of performance.

Therefore, in this paper, the design details for an $\mathcal{H}_2$ state-feedback controller for LPV systems with input saturation and a matched disturbance is provided. The proposed controllers consist of the main control part and the secondary control part. The main part is used for achieving proper $\mathcal{H}_2$ performance, and the secondary part is used for rejecting a matched disturbance. Further, the proposed controllers are divided into three types according to the different forms of feedback gain matrix $K(\theta(t))$ and the Lyapunov function $V(x(t))$. The first type is a quadratic constant state-feedback controller that is developed using a conservative and classical method, the second type is a parameter-dependent state-feedback controller based on a common quadratic Lyapunov function (CQLF), and the third type is a parameter-dependent state-feedback controller based on a parameter dependent Lyapunov function (PDLF). The conditions used for designing the $\mathcal{H}_2$ state-feedback controller are first formulated in terms of the PLMI conditions. They are then converted into linear matrix inequality (LMI) conditions using a parameter relaxation technique [13].

The paper is organized as follows. Section 2 addresses the problem statement and presents some preliminary results. Section 3 provides the PLMI and LMI conditions used in designing an $\mathcal{H}_2$ state-feedback controller. Section 4 presents the simulation results obtained for a two-mass-spring example [15, 16]. Finally, Sect. 5 presents the conclusions of this study.

The notations used in this paper are fairly standard. For example, $x \in \mathcal{R}^n$, $x^{\mathrm{T}}$ indicates the transpose of $x$, and $[x]_k$ denotes the $k$th element of $x$. Furthermore, the notations $X \geq Y$ and $X > Y$, where $X$ and $Y$ are symmetric matrices, denote that $X - Y$ is positive semi-definite and positive definite, respectively. The notation $e_k$ indicates a unit vector with a single nonzero entry at the $k$th position, i.e., $e_k \triangleq [0 \cdots \underbrace{1}_{k\text{th}} \cdots 0]^{\mathrm{T}}$.

## 2 Problem statement

### 2.1 System description

Consider the following LPV system with input saturation and a matched disturbance:

$$\dot{x}(t) = A\big(\theta(t)\big)x(t) + B\big(\theta(t)\big)\big\{sat\big(u(t)\big) + d(t)\big\}, \quad (1)$$

$$y(t) = Cx(t), \quad (2)$$

where

$$A\big(\theta(t)\big) = A_0 + \sum_{i=1}^{r} \theta_i(t) A_i,$$

$$B\big(\theta(t)\big) = B_0 + \sum_{i=1}^{r} \theta_i(t) B_i,$$

$A_0, A_i, B_0, B_i$, and $C$ are known real constant matrices with appropriate dimensions, $x(t) \in \mathcal{R}^n$ is the state, $\theta(t)$ is the time-varying parameter, $u(t) \in \mathcal{R}^m$ is the control input, $d(t) \in \mathcal{R}^m$ is the matched disturbance, and $y(t) \in \mathcal{R}^q$ is the output. Assumed that the time-varying parameter $\theta(t)$ satisfies the following conditions:

$$\bar{\theta}_{\min} \leq \sum_{i=1}^{r} \theta_i(t) = \bar{\theta}(t) \leq \bar{\theta}_{\max}, \quad (3)$$

$$0 \leq \alpha_i \leq \theta_i(t) \leq \beta_i \quad \forall i \in [1, r], \quad (4)$$

$$\eta_i \leq \dot{\theta}_i(t) \leq \nu_i \quad \forall i \in [1, r], \quad (5)$$

and each component of $d(t)$ is bounded by $\varepsilon$, i.e.,

$$\big|e_k^{\mathrm{T}} d(t)\big| \leq \varepsilon. \quad (6)$$

Further, $sat(\cdot)$ denotes a saturation operator, which is defined as

$$\big[sat(\sigma)\big]_i \triangleq \begin{cases} [\sigma]_i & \text{if } |[\sigma]_i| < \mu, \\ \mu & \text{if } [\sigma]_i \geq \mu, \\ -\mu & \text{if } [\sigma]_i \leq -\mu, \end{cases} \quad (7)$$

where $\mu (> \varepsilon)$ is the saturation level. To handle the saturation nonlinearity, the following representation method is employed.

**Lemma 1** (Cao et al. [12] and Hu and Lin [17]) *Let* $u, v \in \mathcal{R}^m$,

$$u = [u_1\ u_2 \cdots u_m]^{\mathrm{T}}, \qquad v = [v_1\ v_2 \cdots v_m]^{\mathrm{T}}.$$

*If* $|e_k^{\mathrm{T}} v| \le \mu$ *for all* $k \in [1, m]$ *and* $\mu$ *is a positive real number, then*

$$sat(u) \in \mathbf{co}\{E_i u + E_i^- v | i \in [1, 2^m]\}, \tag{8}$$

*where* **co** *denotes the convex hull,* $E_i$ *is a diagonal matrix whose diagonal elements have all possible combinations of* 1 *and* 0, *and* $E_i^- \triangleq I - E_i$. *We can rewrite* $sat(u)$ *as*

$$sat(u) \equiv \sum_{j=1}^{2^m} \xi_j \{E_j u + E_j^- v\}, \tag{9}$$

*where* $\sum_{j=1}^{2^m} \xi_j = 1, \xi_j \ge 0$.

### 2.2 Three types of controllers

Let us form a controller for system (1) as

$$u(t) = f(x(t), \theta(t), t) + \bar{u}(x(t), \theta(t), t), \tag{10}$$

where $f(x(t), \theta(t), t)$ is the main control part for achieving $\mathcal{H}_2$ performance, and $\bar{u}(x(t), \theta(t), t)$ is the secondary control part for rejecting a matched disturbance. In this paper, on the basis of the different forms of $f(x(t), \theta(t), t)$ and the Lyapunov function $V(x(t))$, the control methods can be divided into the following three types [13]:

- Type 1: Quadratic constant controller design method

$$\begin{aligned} &u(t) = Kx(t) + \bar{u}(x(t), \theta(t), t), \\ &V(x(t)) = x^{\mathrm{T}}(t) P^{-1} x(t), \end{aligned} \tag{11}$$

  where $K$ is a real constant matrix, and $P$ is a constant positive definite matrix.
- Type 2: Parameter-dependent controller design method with a CQLF

$$\begin{aligned} &u(t) = K(\theta(t))x(t) + \bar{u}(x(t), \theta(t), t), \\ &V(x(t)) = x^{\mathrm{T}}(t) P^{-1} x(t), \end{aligned} \tag{12}$$

  where $K(\theta(t))$ is a parameter-dependent matrix, and $P$ is a constant positive definite matrix.
- Type 3: Parameter-dependent controller design method with a PDLF

$$\begin{aligned} &u(t) = K(\theta(t))x(t) + \bar{u}(x(t), \theta(t), t), \\ &V(x(t)) = x^{\mathrm{T}}(t) P^{-1}(\theta(t)) x(t), \end{aligned} \tag{13}$$

  where $K(\theta(t))$ is a parameter-dependent matrix, and $P(\theta(t))$ is a parameter-dependent positive definite matrix.

The Types 2 and 3 controllers correspond to gain-scheduled controllers, and as noted in [13], the Type 3 controller is a less conservative approach than Types 1 and 2.

## 3 $\mathcal{H}_2$ state-feedback controllers

In this section, let us describe the design of the $\mathcal{H}_2$ state-feedback controller for system (1). On the basis of the three control methods, the state-feedback $\mathcal{H}_2$ controller is designed via a variable structure control (VSC) technique, where $f(x(t), \theta(t), t)$ in (10) is designed to minimize the upper bound of the following linear quadratic (LQ) cost:

$$\min \max_{\theta_i(t) \in \Theta} \left\{ \mathcal{J}(t) = \int_0^\infty x^{\mathrm{T}}(t) C^{\mathrm{T}} Q C x(t)\, dt \right\}, \tag{14}$$

where $Q$ is a positive definite matrix, $\Theta$ is a set of all possible parameters, and $\bar{u}(x(t), \theta(t), t)$ is designed to eliminate a matched disturbance.

First, the PLMI conditions for representing the saturation in Lemma 1 and achieving the $\mathcal{H}_2$ performance is derived. Then let us convert these PLMI conditions into LMI conditions using a parameter relaxation technique [13]. Since the design procedures for a Type 3 controller is similar to the design procedures for Types 1 and 2 controllers, we introduce the design procedure only for the Type 3 controller.

### 3.1 PLMI condition

Determine the input $u(t)$ and the auxiliary input $v(t)$, such that

$$u(t) = K(\theta(t))x(t) + \bar{u}(x(t), \theta(t), t), \tag{15}$$

$$v(t) = H(\theta(t))x(t) + \bar{u}(x(t), \theta(t), t), \tag{16}$$

where $v(t)$ is used to handle the input saturation in (9). Each element of the secondary control part, $\bar{u}(x(t), \theta(t), t)$, is defined for all $k \in [1, m]$ as

$$\begin{aligned} &[\bar{u}(x(t), \theta(t), t)]_k \\ &\quad = -\varepsilon\, \mathrm{sgn}(e_k^{\mathrm{T}} B^{\mathrm{T}}(\theta(t)) P^{-1}(\theta(t)) x(t)), \end{aligned} \tag{17}$$

where $\mathrm{sgn}(\psi)$ is the sign of $\psi$. To use the representation method in Lemma 1, the following condition should be satisfied: $\mu \geq |e_k^{\mathrm{T}} v(t)|$, for all $k \in [1, m]$,

$$\mu \geq \left| e_k^{\mathrm{T}} H\big(\theta(t)\big) x(t) + e_k^{\mathrm{T}} \bar{u}\big(x(t), \theta(t), t\big) \right|. \tag{18}$$

Using the secondary control part in (17), the right side of (18) can be derived as follows:

$$\begin{aligned} &\left| e_k^{\mathrm{T}} H\big(\theta(t)\big) x(t) + e_k^{\mathrm{T}} \bar{u}\big(x(t), \theta(t), t\big) \right| \\ &\quad \leq \left| e_k^{\mathrm{T}} H\big(\theta(t)\big) x(t) \right| + \left| e_k^{\mathrm{T}} \bar{u}\big(x(t), \theta(t), t\big) \right| \\ &\quad = \left| e_k^{\mathrm{T}} H\big(\theta(t)\big) x(t) \right| + \varepsilon. \end{aligned} \tag{19}$$

The following condition is then a sufficient condition of (18)

$$\mu \geq \left| e_k^{\mathrm{T}} H\big(\theta(t)\big) x(t) \right| + \varepsilon. \tag{20}$$

Therefore, the representation method in Lemma 1 can be used if it holds that for $k \in [1, m]$,

$$1 \geq x^{\mathrm{T}}(t) H^{\mathrm{T}}\big(\theta(t)\big) e_k \frac{1}{(\mu - \varepsilon)^2} e_k^{\mathrm{T}} H\big(\theta(t)\big) x(t). \tag{21}$$

It is then ensured that a weighting factor $\xi_s$ exists such that

$$\begin{aligned} sat\big(u(t)\big) = {} & \sum_{s=1}^{2^m} \xi_s \big\{ E_s K\big(\theta(t)\big) + E_s^- H\big(\theta(t)\big) \big\} x(t) \\ & + \bar{u}\big(x(t), \theta(t), t\big). \end{aligned} \tag{22}$$

If the upper bound of the PDLF in (13) is a positive scalar $\gamma$, then

$$V\big(x(t)\big) = x^{\mathrm{T}}(t) P^{-1}\big(\theta(t)\big) x(t) < \gamma. \tag{23}$$

Let $F \in \mathcal{R}^{n \times n}$ be a positive definite matrix. Then the following ellipsoid is defined:

$$\mho(F) \triangleq \big\{ x(t) \in \mathcal{R}^{n \times n} | x^{\mathrm{T}}(t) F x(t) < 1 \big\}. \tag{24}$$

To establish a set invariance condition [12], the ellipsoid $\mho(\bar{P}^{-1}(\theta(t)))$ is in the linear region of (21), i.e., for all $k \in [1, m]$,

$$\begin{aligned} & x^{\mathrm{T}}(t) \bar{P}^{-1}\big(\theta(t)\big) x(t) \\ &\quad > x^{\mathrm{T}}(t) H^{\mathrm{T}}\big(\theta(t)\big) e_k \frac{1}{(\mu - \varepsilon)^2} e_k^{\mathrm{T}} H\big(\theta(t)\big) x(t), \end{aligned} \tag{25}$$

or equivalently,

$$\begin{bmatrix} (\mu - \varepsilon)^2 & e_k^{\mathrm{T}} H(\theta(t)) \\ H^{\mathrm{T}}(\theta(t)) e_k & \bar{P}^{-1}(\theta(t)) \end{bmatrix} > 0, \tag{26}$$

where $\bar{P}(\theta(t)) \triangleq P(\theta(t))\gamma$. Then multiplying both sides of (26) by $\mathrm{diag}\{I, \bar{P}(\theta(t))\}$ for all $k \in [1, m]$ yields

$$\begin{bmatrix} (\mu - \varepsilon)^2 & e_k^{\mathrm{T}} \bar{H}(\theta(t)) \\ \bar{H}^{\mathrm{T}}(\theta(t)) e_k & \bar{P}(\theta(t)) \end{bmatrix} > 0, \tag{27}$$

where $\bar{H}(\theta(t)) \triangleq H(\theta(t)) \bar{P}(\theta(t))$.

Next, the $\mathcal{H}_2$ performance conditions is derived. Using (22), the derivative of $V(x(t))$ is

$$\begin{aligned} & \dot{V}\big(x(t)\big) \\ &\quad = 2x^{\mathrm{T}}(t) P^{-1}\big(\theta(t)\big) \dot{x}(t) + x^{\mathrm{T}}(t) \dot{P}^{-1}\big(\theta(t)\big) x(t) \\ &\quad = 2x^{\mathrm{T}}(t) P^{-1}\big(\theta(t)\big) \\ &\qquad \times \Bigg[ A\big(\theta(t)\big) + B\big(\theta(t)\big) \sum_{s=1}^{2^m} \xi_s \big\{ E_s K\big(\theta(t)\big) \\ &\qquad + E_s^- H\big(\theta(t)\big) \big\} \Bigg] x(t) \\ &\qquad + 2x^{\mathrm{T}}(t) P^{-1} B\big(\theta(t)\big) \big\{ \bar{u}\big(x(t), \theta(t), t\big) + d(t) \big\} \\ &\qquad + x^{\mathrm{T}}(t) \dot{P}^{-1}\big(\theta(t)\big) x(t). \end{aligned} \tag{28}$$

Substituting (17) into (28), the second term of (28) is

$$\begin{aligned} & 2x^{\mathrm{T}}(t) P^{-1}\big(\theta(t)\big) B\big(\theta(t)\big) \big\{ \bar{u}\big(x(t), \theta(t), t\big) + d(t) \big\} \\ &\quad \leq 0. \end{aligned} \tag{29}$$

Then, (28) can be rewritten as

$$\begin{aligned} & \dot{V}\big(x(t)\big) \\ &\quad \leq 2x^{\mathrm{T}}(t) P^{-1}\big(\theta(t)\big) \Bigg[ A\big(\theta(t)\big) + B\big(\theta(t)\big) \\ &\qquad \times \sum_{s=1}^{2^m} \xi_s \big\{ E_s K\big(\theta(t)\big) + E_s^- H\big(\theta(t)\big) \big\} \Bigg] x(t) \\ &\qquad + x^{\mathrm{T}}(t) \dot{P}^{-1}\big(\theta(t)\big) x(t). \end{aligned} \tag{30}$$

Here, if the upper bound of the LQ cost (14) is a PDLF, then

$$\mathcal{J}(t) = \int_t^{\infty} x^{\mathrm{T}}(t) C^{\mathrm{T}} Q C x(t)\, dt < V\big(x(t)\big). \tag{31}$$

Relation (31) is ensured if it holds that

$$\dot{V}\big(x(t)\big) < \dot{\mathcal{J}}(t), \qquad \mathcal{J}(\infty) = V(\infty) = 0,$$

which leads to the following condition:

$$\begin{aligned} 0 > {} & P^{-1}\big(\theta(t)\big)\big\{A\big(\theta(t)\big) + B\big(\theta(t)\big)E_s K\big(\theta(t)\big) \\ & + B\big(\theta(t)\big)E_s^- H\big(\theta(t)\big)\big\} \\ & + \big\{A^{\mathrm{T}}\big(\theta(t)\big) + K^{\mathrm{T}}\big(\theta(t)\big)E_s^{\mathrm{T}} B^{\mathrm{T}}\big(\theta(t)\big) \\ & + H^{\mathrm{T}}\big(\theta(t)\big)E_s^{-\mathrm{T}} B^{\mathrm{T}}\big(\theta(t)\big)\big\} P^{-1}\big(\theta(t)\big) \\ & + \dot{P}^{-1}\big(\theta(t)\big) + C^{\mathrm{T}} Q C. \end{aligned} \tag{32}$$

Multiplying both sides of (32) by $P(\theta(t))$ and replacing $\gamma P(\theta(t))$ with $\bar{P}(\theta(t))$ yield

$$\begin{aligned} 0 > {} & A\big(\theta(t)\big)\bar{P}\big(\theta(t)\big) + B\big(\theta(t)\big)E_s \bar{K}\big(\theta(t)\big) \\ & + B\big(\theta(t)\big)E_s^- \bar{H}\big(\theta(t)\big) + \bar{P}\big(\theta(t)\big)A^{\mathrm{T}}\big(\theta(t)\big) \\ & + \bar{K}^{\mathrm{T}}\big(\theta(t)\big)E_s^{\mathrm{T}} B^{\mathrm{T}}\big(\theta(t)\big) \\ & + \bar{H}^{\mathrm{T}}\big(\theta(t)\big)E_s^{-\mathrm{T}} B^{\mathrm{T}}\big(\theta(t)\big) \\ & + \bar{P}\big(\theta(t)\big)\dot{\bar{P}}^{-1}\big(\theta(t)\big)\bar{P}\big(\theta(t)\big) \\ & + \frac{\bar{P}(\theta(t))C^{\mathrm{T}} Q C \bar{P}(\theta(t))}{\gamma}, \end{aligned} \tag{33}$$

where

$$\bar{K}\big(\theta(t)\big) \triangleq K\big(\theta(t)\big)\bar{P}\big(\theta(t)\big).$$

Remark that

$$\frac{d\bar{P}^{-1}(\theta(t))}{dt} = -\bar{P}^{-1}\big(\theta(t)\big)\frac{d\bar{P}(\theta(t))}{dt}\bar{P}^{-1}\big(\theta(t)\big).$$

Relation (33) then provides the following PLMI condition:

$$\begin{bmatrix} (1,1) & \bar{P}(\theta(t))C^{\mathrm{T}} \\ C\bar{P}(\theta(t)) & -\gamma Q^{-1} \end{bmatrix} < 0, \tag{34}$$

where

$$\begin{aligned} (1,1) \triangleq {} & A\big(\theta(t)\big)\bar{P}\big(\theta(t)\big) + B\big(\theta(t)\big)E_s \bar{K}\big(\theta(t)\big) \\ & + B\big(\theta(t)\big)E_s^- \bar{H}\big(\theta(t)\big) + \bar{P}\big(\theta(t)\big)A^{\mathrm{T}}\big(\theta(t)\big) \\ & + \bar{K}^{\mathrm{T}}\big(\theta(t)\big)E_s^{\mathrm{T}} B^{\mathrm{T}}\big(\theta(t)\big) \\ & + \bar{H}^{\mathrm{T}}\big(\theta(t)\big)E_s^{-\mathrm{T}} B^{\mathrm{T}}\big(\theta(t)\big) \\ & - \dot{\bar{P}}\big(\theta(t)\big). \end{aligned}$$

Furthermore, since the upper bound of $V(x(t))$ is a positive scalar $\gamma$, the following condition is developed:

$$\mathcal{J}(0) < V\big(x(0)\big) = x^{\mathrm{T}}(0)P^{-1}\big(\theta(0)\big)x(0) < \gamma,$$

or equivalently,

$$\begin{bmatrix} 1 & x^{\mathrm{T}}(0) \\ x(0) & \bar{P}(\theta(0)) \end{bmatrix} > 0. \tag{35}$$

Consequently, three PLMI conditions are derived, i.e., (27), (34), and (35), for a state-feedback Type 3 controller. Based on the above conditions, the following theorems present the PLMI conditions for the three types of controllers.

**Theorem 1** (Type 3 controller: PLMI conditions) *For all the states $x(t)$ in $\mho(\bar{P}^{-1}(\theta(t)))$, $k \in [1, m]$, and $s \in [1, 2^m]$, the $\mathcal{H}_2$ control problem with the LQ cost in (14) can be solved using the following linear programming (LP) problem: minimize $\gamma$ over a positive definite matrix $\bar{P}(\theta(t))$, matrices $\dot{\bar{P}}(\theta(t))$, $\bar{K}(\theta(t))$, and $\bar{H}(\theta(t))$ with appropriate dimensions subject to (27), (34), and (35). In this case, the LQ cost in (14) is guaranteed by $\gamma$ and the controller is constructed as (15), where $K(\theta(t)) \triangleq \bar{K}(\theta(t))\bar{P}^{-1}(\theta(t))$ and each component of $\bar{u}(x(t), \theta(t), t)$ is given in (17).*

*Proof* The proof has been previously provided. □

**Theorem 2** (Type 2 controller: PLMI conditions) *For all the states $x(t)$ in $\mho(\bar{P}^{-1})$, $k \in [1, m]$, and $s \in [1, 2^m]$, the $\mathcal{H}_2$ control problem with the LQ cost in (14) can be solved using the following linear programming (LP) problem: minimize $\gamma$ over a positive definite matrix $\bar{P}$, matrices $\bar{K}(\theta(t))$ and $\bar{H}(\theta(t))$ with appropriate dimensions subject to*

$$\begin{bmatrix} (\mu - \varepsilon)^2 & e_k^{\mathrm{T}} \bar{H}(\theta(t)) \\ \bar{H}^{\mathrm{T}}(\theta(t))e_k & \bar{P} \end{bmatrix} > 0, \tag{36}$$

$$\begin{bmatrix} (1,1) & \bar{P}C^{\mathrm{T}} \\ C\bar{P} & -\gamma Q^{-1} \end{bmatrix} < 0, \tag{37}$$

$$\begin{bmatrix} 1 & x^{\mathrm{T}}(0) \\ x(0) & \bar{P} \end{bmatrix} > 0, \tag{38}$$

*where*

$$\begin{aligned} (1,1) \triangleq {} & A\big(\theta(t)\big)\bar{P} + B\big(\theta(t)\big)E_s \bar{K}\big(\theta(t)\big) \\ & + B\big(\theta(t)\big)E_s^- \bar{H}\big(\theta(t)\big) + \bar{P}A^{\mathrm{T}}\big(\theta(t)\big) \end{aligned}$$

$$+ \bar{K}^{\mathrm{T}}\big(\theta(t)\big)E_s^{\mathrm{T}}B^{\mathrm{T}}\big(\theta(t)\big) + \bar{H}^{\mathrm{T}}\big(\theta(t)\big)E_s^{-\mathrm{T}}B^{\mathrm{T}}\big(\theta(t)\big).$$

*In this case, the LQ cost in* (14) *is guaranteed by* $\gamma$ *and the controller is constructed as* $u(t) = K(\theta(t))x(t) + \bar{u}(x(t), \theta(t), t)$, *where* $K(\theta(t)) \triangleq \bar{K}(\theta(t))\bar{P}^{-1}$ *and each component of* $\bar{u}(x(t), \theta(t), t)$ *is defined as*

$$\big[\bar{u}\big(x(t), \theta(t), t\big)\big]_k = -\varepsilon \operatorname{sgn}\big(e_k^{\mathrm{T}}B^{\mathrm{T}}\big(\theta(t)\big)P^{-1}x(t)\big). \tag{39}$$

*Proof* This proof is similar to that of Theorem 1, except $P$ is used instead of $P(\theta(t))$. Thus, it is left for the reader. □

**Theorem 3** (Type 1 controller: PLMI conditions) *For all the states* $x(t)$ *in* $\mho(\bar{P}^{-1})$, $k \in [1, m]$, *and* $s \in [1, 2^m]$, *the* $\mathcal{H}_2$ *control problem with the LQ cost in* (14) *can be solved using the following linear programming* (*LP*) *problem*: *minimize* $\gamma$ *over a positive definite matrix* $\bar{P}$, *matrices* $\bar{K}$ *and* $\bar{H}$ *with appropriate dimensions subject to* (37), (38), *and*

$$\begin{bmatrix} (\mu - \varepsilon)^2 & e_k^{\mathrm{T}}\bar{H} \\ \bar{H}^{\mathrm{T}}e_k & \bar{P} \end{bmatrix} > 0, \tag{40}$$

*where*

$$(1,1) \triangleq A\big(\theta(t)\big)\bar{P} + B\big(\theta(t)\big)E_s\bar{K} + B\big(\theta(t)\big)E_s^{-}\bar{H} + \bar{P}A^{\mathrm{T}}\big(\theta(t)\big) + \bar{K}^{\mathrm{T}}E_s^{\mathrm{T}}B^{\mathrm{T}}\big(\theta(t)\big) + \bar{H}^{\mathrm{T}}E_s^{-\mathrm{T}}B^{\mathrm{T}}\big(\theta(t)\big).$$

*In this case, the LQ cost in* (14) *is guaranteed by* $\gamma$ *and the controller is constructed as* $u(t) = Kx(t) + \bar{u}(x(t), \theta(t), t)$, *where* $K \triangleq \bar{K}\bar{P}^{-1}$ *and each component of* $\bar{u}(x(t), \theta(t), t)$ *is the same as that in* (39).

*Proof* This proof is similar to that of Theorem 1, except $P$, $K$, and $H$ are used instead of $P(\theta(t))$, $K(\theta(t))$, and $H(\theta(t))$. Thus, it is left for the reader. □

### 3.2 LMI condition

The above PLMI conditions involve an infinite number of LMI conditions, and thus, the task of determining the controller is numerically intractable. To overcome it, derived PLMI conditions in Sect. 3.1 are converted into LMI conditions using the parameter relaxation technique [13] for a time-varying parameter $\theta(t)$. Thus, to obtain a finite number of LMI conditions from the derived PLMI conditions, we select the special structure of matrices $\bar{P}(\theta(t))$, $\dot{\bar{P}}(\theta(t))$, $\bar{K}(\theta(t))$, and $\bar{H}(\theta(t))$ as follows:

$$\bar{P}\big(\theta(t)\big) \triangleq \bar{P}_0 + \sum_{i=1}^{r}\theta_i(t)\bar{P}_i,$$

$$\dot{\bar{P}}\big(\theta(t)\big) \triangleq \sum_{i=1}^{r}\dot{\theta}_i(t)\bar{P}_i,$$

$$\bar{K}\big(\theta(t)\big) \triangleq \bar{K}_0 + \sum_{i=1}^{r}\theta_i(t)\bar{K}_i,$$

$$\bar{H}\big(\theta(t)\big) \triangleq \bar{H}_0 + \sum_{i=1}^{r}\theta_i(t)\bar{H}_i,$$

where $\bar{P}_0$ and $\bar{P}_i$ are positive matrices, and $\bar{K}_0$, $\bar{K}_i$, $\bar{H}_0$, and $\bar{H}_i$ are real constant matrices with appropriate dimensions. Thus, the convexity of PLMI conditions is fully exploited.

The Schur complement technique is used to convert the PLMI condition in (27) to

$$(\mu - \varepsilon)^2\bar{P}\big(\theta(t)\big) - \bar{H}^{\mathrm{T}}\big(\theta(t)\big)e_k e_k^{\mathrm{T}}\bar{H}\big(\theta(t)\big) > 0. \tag{41}$$

Further, consider the two constraints in (3) and (4) such that for all $i \in [1, r]$,

$$\left(\sum_{i=1}^{r}\theta_i(t) - \bar{\theta}_{\min}\right)\left(\sum_{i=1}^{r}\theta_i(t) - \bar{\theta}_{\max}\right)\big(\bar{\Lambda} + \bar{\Lambda}^{\mathrm{T}}\big) \le 0, \tag{42}$$

$$\big(\theta_i(t) - \alpha_i\big)\big(\theta_i(t) - \beta_i\big)\big(\bar{\Lambda}_i + \bar{\Lambda}_i^{\mathrm{T}}\big) \le 0, \tag{43}$$

where matrices $\bar{\Lambda} \in \mathcal{R}^{n\times n}$ and $\bar{\Lambda}_i \in \mathcal{R}^{n\times n}$ satisfy

$$\big(\bar{\Lambda} + \bar{\Lambda}^{\mathrm{T}}\big) \ge 0, \qquad \big(\bar{\Lambda}_i + \bar{\Lambda}_i^{\mathrm{T}}\big) \ge 0. \tag{44}$$

Combining (41), (42), and (43) using the $\mathcal{S}$-procedure provides the following PLMI condition:

$$\Bigg[(\mu - \varepsilon)^2\bar{P}_0 + \bar{\theta}_{\min}\bar{\theta}_{\max}\big(\bar{\Lambda} + \bar{\Lambda}^{\mathrm{T}}\big) + \sum_{i=1}^{r}\alpha_i\beta_i\big(\bar{\Lambda}_i + \bar{\Lambda}_i^{\mathrm{T}}\big)\Bigg] + \sum_{i=1}^{r}\theta_i(t)\big[(\mu - \varepsilon)^2\bar{P}_i - \big(\bar{\theta}_{\min} + \bar{\theta}_{\max}\big)\big(\bar{\Lambda} + \bar{\Lambda}^{\mathrm{T}}\big) - (\alpha_i + \beta_i)\big(\bar{\Lambda}_i + \bar{\Lambda}_i^{\mathrm{T}}\big)\big]$$

$$+\sum_{i=1}^{r}\sum_{j\neq i}^{r}\theta_i(t)\theta_j(t)\big(\bar{\Lambda}+\bar{\Lambda}^{\mathrm{T}}\big)$$

$$+\sum_{i=1}^{r}\theta_i^2(t)\big[\big(\bar{\Lambda}+\bar{\Lambda}^{\mathrm{T}}\big)+\big(\bar{\Lambda}_i+\bar{\Lambda}_i^{\mathrm{T}}\big)\big]$$

$$-\Bigg(\bar{H}_0^{\mathrm{T}}e_ke_k^{\mathrm{T}}\bar{H}_0+\sum_{i=1}^{r}\theta_i(t)\bar{H}_i^{\mathrm{T}}e_ke_k^{\mathrm{T}}\bar{H}_0$$

$$+\sum_{i=1}^{r}\theta_i(t)\bar{H}_0^{\mathrm{T}}e_ke_k^{\mathrm{T}}\bar{H}_i$$

$$+\sum_{i=1}^{r}\sum_{j=1}^{r}\theta_i(t)\theta_j(t)\bar{H}_i^{\mathrm{T}}e_ke_k^{\mathrm{T}}\bar{H}_j\Bigg)>0. \tag{45}$$

Condition (45) can be reduced to

$$\Theta_a^{\mathrm{T}}(t)\big\{U_a-\mathcal{R}_a\mathrm{I}\mathcal{R}_a^{\mathrm{T}}\big\}\Theta_a(t)>0, \tag{46}$$

where

$$U_a\triangleq\left[\begin{array}{c|cccc} M & N_1 & N_2 & \cdots & N_r \\ \hline N_1^{\mathrm{T}} & O_1 & L_{12} & & L_{1r} \\ N_2^{\mathrm{T}} & L_{12}^{\mathrm{T}} & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & L_{(r-1)r} \\ N_r^{\mathrm{T}} & L_{1r}^{\mathrm{T}} & \cdots & L_{(r-1)r}^{\mathrm{T}} & O_r \end{array}\right],$$

$$M\triangleq(\mu-\varepsilon)^2\bar{P}_0+\bar{\theta}_{\min}\bar{\theta}_{\max}\big(\bar{\Lambda}+\bar{\Lambda}^{\mathrm{T}}\big)+\sum_{i=1}^{r}\alpha_i\beta_i\big(\bar{\Lambda}_i+\bar{\Lambda}_i^{\mathrm{T}}\big),$$

$$N_i\triangleq 0.5(\mu-\varepsilon)^2\bar{P}_i-(\bar{\theta}_{\min}+\bar{\theta}_{\max})\bar{\Lambda}-(\alpha_i+\beta_i)\bar{\Lambda}_i,$$

$$L_{ij}\triangleq\bar{\Lambda}+\bar{\Lambda}^{\mathrm{T}},$$

$$O_i\triangleq\bar{\Lambda}+\bar{\Lambda}^{\mathrm{T}}+\bar{\Lambda}_i+\bar{\Lambda}_i^{\mathrm{T}},$$

$$\Theta_a(t)\triangleq\big[I\quad\theta_1(t)I\quad\theta_2(t)I\quad\cdots\quad\theta_r(t)I\big]^{\mathrm{T}},$$

$$\mathcal{R}_a\triangleq\big[e_k^{\mathrm{T}}\bar{H}_0\quad e_k^{\mathrm{T}}\bar{H}_1\quad\cdots\quad e_k^{\mathrm{T}}\bar{H}_r\big]^{\mathrm{T}}.$$

Condition (46) is ensured if the following LMI condition holds:

$$\begin{bmatrix} U_a & \mathcal{R}_a \\ \mathcal{R}_a^{\mathrm{T}} & \mathrm{I}\end{bmatrix}>0. \tag{47}$$

Therefore, we have the LMI condition in (47) for the PLMI condition in (27).

To derive the LMI-based $\mathcal{H}_2$ performance conditions from (34), (33) can be written as follows:

$$T_{(0,0)}+R+\sum_{i=1}^{r}\theta_i(t)\big\{T_{(0,i)}+T_{(0,i)}^{\mathrm{T}}\big\}+\sum_{i=1}^{r-1}\sum_{j=i+1}^{r}\theta_i(t)\theta_j(t)\big\{T_{(i,j)}+T_{(i,j)}^{\mathrm{T}}\big\}+\sum_{i=1}^{r}\theta_i^2(t)\{T_{(i,i)}\}+\sum_{i=1}^{r}\dot{\theta}_i(t)\big\{T_{(1,i)}+T_{(1,i)}^{\mathrm{T}}\big\}<0, \tag{48}$$

where

$$T_{(0,0)}\triangleq A_0\bar{P}_0+\bar{P}_0A_0^{\mathrm{T}}+B_0E_s\bar{K}_0+B_0E_s^-\bar{H}_0+\bar{K}_0^{\mathrm{T}}E_s^{\mathrm{T}}B_0^{\mathrm{T}}+\bar{H}_0^{\mathrm{T}}E_s^{-\mathrm{T}}B_0^{\mathrm{T}},$$

$$T_{(0,i)}\triangleq A_0\bar{P}_i+A_i\bar{P}_0+B_0E_s\bar{K}_i+B_iE_s\bar{K}_0+B_0E_s^-\bar{H}_i+B_iE_s^-\bar{H}_0,$$

$$T_{(i,j)}\triangleq A_i\bar{P}_j+\bar{P}_iA_j^{\mathrm{T}}+B_iE_s\bar{K}_j+B_iE_s^-\bar{H}_j+\bar{K}_i^{\mathrm{T}}E_s^{\mathrm{T}}B_j^{\mathrm{T}}+\bar{H}_i^{\mathrm{T}}E_s^{-\mathrm{T}}B_j^{\mathrm{T}},$$

$$T_{(1,i)}\triangleq-0.5\bar{P}_i,$$

$$R\triangleq\Bigg\{\bar{P}_0C^{\mathrm{T}}QC\bar{P}_0+\sum_{i=1}^{r}\theta_i(t)\big(\bar{P}_0C^{\mathrm{T}}QC\bar{P}_i+\bar{P}_iC^{\mathrm{T}}QC\bar{P}_0\big)+\sum_{i=1}^{r}\sum_{j=1}^{r}\theta_i(t)\theta_j(t)\bar{P}_iC^{\mathrm{T}}QC\bar{P}_j\Bigg\}/\gamma.$$

Moreover, using the parameter relaxation technique, let us convert constraints (3)–(5) for all $i\in[1,r]$, respectively, into

$$-\Bigg(\sum_{i=1}^{r}\theta_i(t)-\bar{\theta}_{\min}\Bigg)\Bigg(\sum_{i=1}^{r}\theta_i(t)-\bar{\theta}_{\max}\Bigg)\big(\Lambda+\Lambda^{\mathrm{T}}\big)\geq 0, \tag{49}$$

$$-\big(\theta_i(t)-\alpha_i\big)\big(\theta_i(t)-\beta_i\big)\big(\Lambda_i+\Lambda_i^{\mathrm{T}}\big)\geq 0, \tag{50}$$

$$-\big(\dot{\theta}_i(t)-\eta_i\big)\big(\dot{\theta}_i(t)-\nu_i\big)\big(\Sigma_i+\Sigma_i^{\mathrm{T}}\big)\geq 0, \tag{51}$$

where matrices $\varLambda \in \mathcal{R}^{n\times n}$, $\varLambda_i \in \mathcal{R}^{n\times n}$, and $\varSigma_i \in \mathcal{R}^{n\times n}$ satisfy

$$\begin{aligned}&\big(\varLambda+\varLambda^{\mathrm{T}}\big)>0, \qquad \big(\varLambda_i+\varLambda_i^{\mathrm{T}}\big)>0,\\&\big(\varSigma_i+\varSigma_i^{\mathrm{T}}\big)>0.\end{aligned} \tag{52}$$

Then, combining conditions (49)–(51) gives

$$\begin{aligned}&V_{(0,0)}+\sum_{i=1}^{r}\theta_i(t)\big\{V_{(0,i)}+V_{(0,i)}^{\mathrm{T}}\big\}\\&\quad+\sum_{i=1}^{r-1}\sum_{j=i+1}^{r}\theta_i(t)\theta_j(t)\big\{V_{(i,j)}+V_{(i,j)}^{\mathrm{T}}\big\}\\&\quad+\sum_{i=1}^{r}\theta_i^2(t)\{V_{(1,i)}\}\\&\quad+\sum_{i=1}^{r}\dot{\theta}_i(t)\big\{V_{(2,i)}+V_{(2,i)}^{\mathrm{T}}\big\}+\sum_{i=1}^{r}\dot{\theta}_i^2(t)\{V_{(3,i)}\}\\&\geq 0,\end{aligned} \tag{53}$$

where

$$\begin{aligned}V_{(0,0)} &\triangleq -\bar{\theta}_{\min}\bar{\theta}_{\max}\big(\varLambda+\varLambda^{\mathrm{T}}\big)-\sum_{i=1}^{r}\alpha_i\beta_i\big(\varLambda_i+\varLambda_i^{\mathrm{T}}\big)\\&\quad-\sum_{i=1}^{r}\eta_i\nu_i\big(\varSigma_i+\varSigma_i^{\mathrm{T}}\big),\\V_{(0,i)} &\triangleq \big(\bar{\theta}_{\min}+\bar{\theta}_{\max}\big)\varLambda+(\alpha_i+\beta_i)\varLambda_i,\\V_{(1,i)} &\triangleq -\big(\varLambda+\varLambda^{\mathrm{T}}\big)-\big(\varLambda_i+\varLambda_i^{\mathrm{T}}\big),\\V_{(2,i)} &\triangleq (\eta_i+\nu_i)\varSigma_i,\\V_{(3,i)} &\triangleq -\big(\varSigma_i+\varSigma_i^{\mathrm{T}}\big), \qquad V_{(i,j)}\triangleq -\big(\varLambda+\varLambda^{\mathrm{T}}\big).\end{aligned}$$

Using the $\mathcal{S}$-procedure, conditions (48) and (53) can be formulated as

$$\begin{aligned}&T_{(0,0)}+V_{(0,0)}+R\\&\quad+\sum_{i=1}^{r}\theta_i(t)\big\{T_{(0,i)}+T_{(0,i)}^{\mathrm{T}}+V_{(0,i)}+V_{(0,i)}^{\mathrm{T}}\big\}\\&\quad+\sum_{i=1}^{r-1}\sum_{j=i+1}^{r}\theta_i(t)\theta_j(t)\big\{T_{(i,j)}+T_{(i,j)}^{\mathrm{T}}+V_{(i,j)}\\&\quad+V_{(i,j)}^{\mathrm{T}}\big\}+\sum_{i=1}^{r}\theta_i^2(t)\big\{T_{(i,i)}+V_{(1,i)}\big\}\\&\quad+\sum_{i=1}^{r}\dot{\theta}_i(t)\big\{T_{(1,i)}+T_{(1,i)}^{\mathrm{T}}+V_{(2,i)}+V_{(2,i)}^{\mathrm{T}}\big\}\\&\quad+\sum_{i=1}^{r}\dot{\theta}_i^2(t)\{V_{(3,i)}\}<0,\end{aligned} \tag{54}$$

which can be converted into

$$\varTheta_b^{\mathrm{T}}(t)\big\{J_a+\mathcal{P}_a\big(\gamma^{-1}Q\big)\mathcal{P}_a^{\mathrm{T}}\big\}\varTheta_b(t)<0, \tag{55}$$

where

$$J_a \triangleq \left[\begin{array}{c|cccc|ccc} X & \varGamma_1 & \varGamma_2 & \cdots & \varGamma_r & \varOmega_1 & \cdots & \varOmega_r\\ \hline \varGamma_1^{\mathrm{T}} & D_1 & \varDelta_{12} & \cdots & \varDelta_{1r} & 0 & \cdots & 0\\ \varGamma_2^{\mathrm{T}} & \varDelta_{12}^{\mathrm{T}} & D_2 & \ddots & \vdots & \vdots & & \vdots\\ \vdots & \vdots & \ddots & \ddots & \varDelta_{(r-1)r} & \vdots & & \vdots\\ \varGamma_r^{\mathrm{T}} & \varDelta_{1r}^{\mathrm{T}} & & \varDelta_{(r-1)r}^{\mathrm{T}} & D_r & 0 & \cdots & 0\\ \hline \varOmega_1^{\mathrm{T}} & 0 & \cdots & \cdots & 0 & W_1 & 0 & 0\\ \vdots & \vdots & & & \vdots & 0 & \ddots & 0\\ \varOmega_r^{\mathrm{T}} & 0 & \cdots & \cdots & 0 & 0 & 0 & W_r\end{array}\right],$$

$$\begin{aligned}&X \triangleq T_{(0,0)}+V_{(0,0)}, \qquad \varOmega_i \triangleq T_{(1,i)}+V_{(2,i)},\\&\varGamma_i \triangleq T_{(0,i)}+V_{(0,i)}, \qquad W_i \triangleq V_{(3,i)},\\&\varDelta_{ij} \triangleq T_{(i,j)}+V_{(i,j)}, \qquad D_i \triangleq T_{(i,i)}+V_{(1,i)},\\&\varTheta_b(t) \triangleq\\&\big[\mathrm{I}\quad \theta_1(t)\mathrm{I}\quad \theta_2(t)\mathrm{I}\quad \cdots\quad \theta_r(t)\mathrm{I}\quad \dot{\theta}_1(t)\mathrm{I}\quad \cdots\quad \dot{\theta}_r(t)\mathrm{I}\big]^{\mathrm{T}},\\&\mathcal{P}_a \triangleq \big[C\bar{P}_0\quad C\bar{P}_1\quad \cdots\quad C\bar{P}_r\big]^{\mathrm{T}}.\end{aligned}$$

Then (55) is ensured if the following LMI condition holds:

$$\begin{bmatrix} J_a & \mathcal{P}_a\\ \mathcal{P}_a^{\mathrm{T}} & -\gamma Q^{-1}\end{bmatrix}<0. \tag{56}$$

The PLMI condition in (35) can be written as

$$\begin{bmatrix} 1 & x^{\mathrm{T}}(0)\\ x(0) & \bar{P}_0+\sum_{i=1}^{r}\theta_i(0)\bar{P}_i\end{bmatrix}>0. \tag{57}$$

Consequently, the LMI conditions are derived in (47), (56), and (57) for the design of the Type 3 state-feedback controller. The following theorems provide the LMI conditions for the three types of controllers.

**Theorem 4** (Type 3: LMI conditions) *For all the states $x(t)$ in $\mho(\bar{P}^{-1}(\theta(t)))$, $k \in [1, m]$, $i \in [1, r]$, and $s \in [1, 2^m]$, the $\mathcal{H}_2$ control problem with the LQ cost in (14) can be solved using the following LP problem: minimize $\gamma$ over positive definite matrices $\bar{P}_0$ and $\bar{P}_i$, matrices $\bar{K}_0$, $\bar{K}_i$, $\bar{H}_0$, $\bar{H}_i$, $\Lambda$, $\Lambda_i$, $\bar{\Lambda}$, $\bar{\Lambda}_i$, and $\Sigma_i$ with appropriate dimensions subject to (44), (47), (52), (56), and (57). In this case, the LQ cost in (14) is guaranteed by $\gamma$ and the $\mathcal{H}_2$ state-feedback controller is constructed as*

$$u(t) = K\big(\theta(t)\big)x(t) + \bar{u}\big(x(t), \theta(t), t\big),$$

*where $K(\theta(t)) \triangleq \bar{K}(\theta(t))\bar{P}^{-1}(\theta(t))$ and each component of $\bar{u}(x(t), \theta(t), t)$ is given by*

$$\big[\bar{u}\big(x(t), \theta(t), t\big)\big]_k = -\varepsilon \operatorname{sgn}\big(e_k^{\mathrm{T}} B^{\mathrm{T}}\big(\theta(t)\big) P^{-1}\big(\theta(t)\big) x(t)\big).$$

*Proof* The proof has been previously provided. □

**Theorem 5** (Type 2: LMI conditions) *For all the states $x(t)$ in $\mho(\bar{P}^{-1})$, $k \in [1, m]$, $i \in [1, r]$, and $s \in [1, 2^m]$, the $\mathcal{H}_2$ control problem with the LQ cost in (14) can be solved using the following LP problem: minimize $\gamma$ over a positive definite matrix $\bar{P}$, matrices $\bar{K}_0$, $\bar{K}_i$, $\bar{H}_0$, $\bar{H}_i$, $\Lambda$, $\Lambda_i$, $\bar{\Lambda}$, and $\bar{\Lambda}_i$ with appropriate dimensions subject to (44), (47), and the following equations:*

$$\big(\Lambda + \Lambda^{\mathrm{T}}\big) > 0, \qquad \big(\Lambda_i + \Lambda_i^{\mathrm{T}}\big) > 0, \tag{58}$$

$$\begin{bmatrix} J_b & \mathcal{P}_b \\ \mathcal{P}_b^T & -\gamma Q^{-1} \end{bmatrix} < 0, \tag{59}$$

$$\begin{bmatrix} 1 & x^{\mathrm{T}}(0) \\ x(0) & \bar{P} \end{bmatrix} > 0, \tag{60}$$

*where*

$$M \triangleq (\mu - \varepsilon)^2 \bar{P} + \bar{\theta}_{\min}\bar{\theta}_{\max}\big(\bar{\Lambda} + \bar{\Lambda}^{\mathrm{T}}\big) + \sum_{i=1}^{r} \alpha_i \beta_i \big(\bar{\Lambda}_i + \bar{\Lambda}_i^{\mathrm{T}}\big),$$

$$N_i \triangleq -(\bar{\theta}_{\min} + \bar{\theta}_{\max})\bar{\Lambda} - (\alpha_i + \beta_i)\bar{\Lambda}_i,$$

$$L_{ij} \triangleq \bar{\Lambda} + \bar{\Lambda}^{\mathrm{T}},$$

$$O_i \triangleq \bar{\Lambda} + \bar{\Lambda}^{\mathrm{T}} + \bar{\Lambda}_i + \bar{\Lambda}_i^{\mathrm{T}},$$

$$J_b \triangleq \left[\begin{array}{c|cccc} X & \Gamma_1 & \Gamma_2 & \cdots & \Gamma_r \\ \hline \Gamma_1^{\mathrm{T}} & D_1 & \Delta_{12} & \cdots & \Delta_{1r} \\ \Gamma_2^{\mathrm{T}} & \Delta_{12}^{\mathrm{T}} & D_2 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \Delta_{(r-1)r} \\ \Gamma_r^{\mathrm{T}} & \Delta_{1r}^{\mathrm{T}} & \cdots & \Delta_{(r-1)r}^{\mathrm{T}} & D_r \end{array}\right],$$

$$X \triangleq A_0\bar{P} + \bar{P}A_0^{\mathrm{T}} + B_0 E_s \bar{K}_0 + B_0 E_s^- \bar{H}_0 + \bar{K}_0^{\mathrm{T}} E_s^{\mathrm{T}} B_0^{\mathrm{T}} + \bar{H}_0^{\mathrm{T}} E_s^{-\mathrm{T}} B_0^{\mathrm{T}} - \bar{\theta}_{\min}\bar{\theta}_{\max}\big(\Lambda + \Lambda^{\mathrm{T}}\big) - \sum_{i=1}^{r} \alpha_i \beta_i \big(\Lambda_i + \Lambda_i^{\mathrm{T}}\big),$$

$$\Gamma_i \triangleq A_i\bar{P} + B_0 E_s \bar{K}_i + B_i E_s \bar{K}_0 + B_0 E_s^- \bar{H}_i + B_i E_s^- \bar{H}_0 + (\bar{\theta}_{\min} + \bar{\theta}_{\max})\Lambda + (\alpha_i + \beta_i)\Lambda_i,$$

$$\Delta_{ij} \triangleq B_i E_s \bar{K}_j + B_i E_s^- \bar{H}_j + \bar{K}_i^{\mathrm{T}} E_s^{\mathrm{T}} B_j^{\mathrm{T}} + \bar{H}_i^{\mathrm{T}} E_s^{-\mathrm{T}} B_j^{\mathrm{T}} - \big(\Lambda + \Lambda^{\mathrm{T}}\big),$$

$$D_i \triangleq B_i E_s \bar{K}_i + B_i E_s^- \bar{H}_i + \bar{K}_i^{\mathrm{T}} E_s^{\mathrm{T}} B_i^{\mathrm{T}} + \bar{H}_i^{\mathrm{T}} E_s^{-\mathrm{T}} B_i^{\mathrm{T}} - \big(\Lambda + \Lambda^{\mathrm{T}}\big) - \big(\Lambda_i + \Lambda_i^{\mathrm{T}}\big),$$

$$\mathcal{P}_b \triangleq \begin{bmatrix} C\bar{P} & 0 & \cdots & 0 \end{bmatrix}^{\mathrm{T}}.$$

*In this case, the LQ cost in (14) is guaranteed by $\gamma$ and the $\mathcal{H}_2$ state-feedback controller is constructed as*

$$u(t) = K\big(\theta(t)\big)x(t) + \bar{u}\big(x(t), \theta(t), t\big),$$

*where $K(\theta(t)) \triangleq \bar{K}(\theta(t))\bar{P}^{-1}$ and each component of $\bar{u}(x(t), \theta(t), t)$ is given by*

$$\big[\bar{u}\big(x(t), \theta(t), t\big)\big]_k = -\varepsilon \operatorname{sgn}\big(e_k^{\mathrm{T}} B^{\mathrm{T}}\big(\theta(t)\big) P^{-1} x(t)\big).$$

*Proof* This proof is similar to that of Theorem 4, except $P$ is used instead of $P(\theta(t))$, and only the parameter conditions (3) and (4) are used. Thus, it is left for the reader. □

**Theorem 6** (Type 1: LMI conditions) *For all the states $x(t)$ in $\mho(\bar{P}^{-1})$, $k \in [1, m]$, $i \in [1, r]$, and $s \in [1, 2^m]$, the $\mathcal{H}_2$ control problem with cost (14) can be solved using the following LP problem: minimize $\gamma$ over a positive definite matrix $\bar{P}$, matrices $\bar{K}$, $\bar{H}$, $\Lambda$, and $\Lambda_i$ with appropriate dimensions subject to (58), (59), (60), and*

$$\begin{bmatrix} (\mu - \varepsilon)^2 & e_k^{\mathrm{T}}\bar{H} \\ \bar{H}^{\mathrm{T}} e_k & \bar{P} \end{bmatrix} > 0, \tag{61}$$

*where*

$$X \triangleq A_0 \bar{P} + B_0 E_s \bar{K} + B_0 E_s^- \bar{H} + \bar{P} A_0^{\mathrm{T}} + \bar{K}^{\mathrm{T}} E_s^{\mathrm{T}} \bar{B}_0^{\mathrm{T}} + \bar{H}^{\mathrm{T}} E_s^{-\mathrm{T}} \bar{B}_0^{\mathrm{T}} - \bar{\theta}_{\min}\bar{\theta}_{\max}\big(\Lambda + \Lambda^{\mathrm{T}}\big) - \sum_{i=1}^{r} \alpha_i \beta_i \big(\Lambda_i + \Lambda_i^{\mathrm{T}}\big),$$

$$\Gamma_i \triangleq A_i \bar{P} + B_i E_s \bar{K} + B_i E_s^- \bar{H} + (\bar{\theta}_{\min} + \bar{\theta}_{\max})\Lambda + (\alpha_i + \beta_i)\Lambda_i,$$

$$\Delta_{ij} \triangleq -\big(\Lambda + \Lambda^{\mathrm{T}}\big),$$

$$D_i \triangleq -\big(\Lambda + \Lambda^{\mathrm{T}}\big) - \big(\Lambda_i + \Lambda_i^{\mathrm{T}}\big).$$

*In this case, the LQ cost in (14) is guaranteed by $\gamma$ and the $\mathcal{H}_2$ state-feedback controller is constructed as*

$$u(t) = Kx(t) + \bar{u}\big(x(t), \theta(t), t\big),$$

*where $K \triangleq \bar{K}\bar{P}^{-1}$ and each component of $\bar{u}(x(t), \theta(t), t)$ is given by*

$$\big[\bar{u}\big(x(t), \theta(t), t\big)\big]_k = -\varepsilon \operatorname{sgn}\big(e_k^{\mathrm{T}} B^{\mathrm{T}}\big(\theta(t)\big) P^{-1} x(t)\big).$$

*Proof* This proof is similar to that of Theorem 4, except that $P$, $K$, and $H$ are used instead of $P(\theta(t))$, $K(\theta(t))$, and $H(\theta(t))$, and only the parameter conditions (3) and (4) are used. Thus, it is left for the reader. □

## 4 Simulation results

In this section, let us demonstrate the performance of the proposed three types of the controllers using examples of a two-mass-spring [15, 16] and a satellite system [6, 18]. Since intensive studies have not yet been carried out on an $\mathcal{H}_2$ controller design for LPV systems with input saturation and a matched disturbance, we are unable to compare the performances of the proposed controllers with those of the conventional controllers. Therefore, the performances of the proposed controllers will be compared with those of the conventional controllers (Theorem 8 in [12]) under no disturbance condition. Next, the performances of the three types of proposed controllers for LPV systems with input saturation and a matched disturbance will be demonstrated. The simulations are performed using the Matlab 7.9.0.529 LMI toolbox on a PC with a 2.80-GHz AMD Athlon 7850 dual-core processor and 3.25GB of RAM.

### 4.1 Example 1: two-mass-spring system

Consider the two-mass-spring system shown in Fig. 1, which is represented in terms of state-space equations as follows:

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \\ \dot{x}_4(t) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{k}{m_1} & \frac{k}{m_1} & 0 & 0 \\ \frac{k}{m_2} & -\frac{k}{m_2} & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{1}{m_1} \\ 0 \end{bmatrix} \times \big(sat\big(u(t)\big) + d(t)\big),$$

$$y(t) = x_2(t),$$

where $x_1(t)$ and $x_2(t)$ are the positions of the two masses; $x_3(t)$ and $x_4(t)$ are the velocities of the two masses, respectively; $u(t)$ is the control input of $m_1$; $d(t)$ is the matched disturbance; and $y(t)$ is the output. We choose $m_1 = m_2 = 1$ for the nominal system. Also, the spring constant $k$ is uncertain in $k_{\min} = 0.5 \le k \le 2 = k_{\max}$, and the matched disturbance $d(t)$ is chosen as $d(t) = 0.1 \sin t$. If the two parameters $\theta_1(t)$ and $\theta_2(t)$ is chosen as

$$\theta_1(t) = 0.5 + 0.5 \sin(0.01\pi t),$$

$$\theta_2(t) = 0.5 - 0.5 \sin(0.01\pi t),$$

then the LPV representation of the two-mass-spring system is as follows:

$$A_0 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

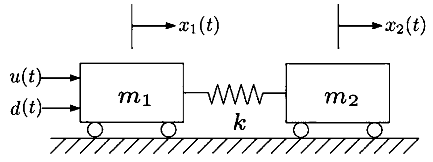


**Fig. 1** Two-mass-spring system with uncertain parameters

$$A_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{k_{\min}}{m_1} & \frac{k_{\min}}{m_1} & 0 & 0 \\ \frac{k_{\min}}{m_2} & -\frac{k_{\min}}{m_2} & 0 & 0 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{k_{\max}}{m_1} & \frac{k_{\max}}{m_1} & 0 & 0 \\ \frac{k_{\max}}{m_2} & -\frac{k_{\max}}{m_2} & 0 & 0 \end{bmatrix},$$

$$B_0 = \begin{bmatrix} 0 & 0 & \frac{1}{m_1} & 0 \end{bmatrix}^{\mathrm{T}},$$

$$B_1 = B_2 = [0 \quad 0 \quad 0 \quad 0]^{\mathrm{T}},$$

$$C = [0 \quad 1 \quad 0 \quad 0], \qquad \bar{\theta}_{\min} = \bar{\theta}_{\max} = 1,$$

$$\alpha_1 = \alpha_2 = 0,$$

$$\beta_1 = \beta_2 = 1, \qquad \eta_1 = \eta_2 = -0.005\pi,$$

$$\nu_1 = \nu_2 = 0.005\pi,$$

$$\mu = 1, \qquad \varepsilon = 0.1.$$

For Theorems 4, 5, and 6, the following parameters are additionally considered:

$$Q = I, \qquad x(0) = [0.2 \quad -0.3 \quad 0 \quad 0].$$

#### 4.1.1 *Performance of the proposed state-feedback $\mathcal{H}_2$ controllers under a no disturbance condition*

In this section, the performance of the proposed state-feedback $\mathcal{H}_2$ controllers is compared with that of the gain-scheduled controller of Theorem 8 in [12] under no disturbance condition. Table 1 and Fig. 2 show the minimum cost $\gamma$ and the trajectories of the states, respectively. Table 1 and Fig. 2 show that Type 1 controller exhibits a lower performance than the controller described in [12], but that Type 2 and Type 3 controllers exhibit a better performance.

#### 4.1.2 *Performance of the proposed state-feedback $\mathcal{H}_2$ controllers*

In this subsection, we demonstrate the performance of the three types of controllers for an LPV system with input saturation and a matched disturbance. Table 2 shows the minimum cost $\gamma$ of each of the three controllers, while Figs. 3 and 4 show their output $y(t)$ and $sat(u(t))$ trajectories. Table 2 and Fig. 3 show that Type 3 controller outperforms the other two types of controllers with respect to their minimum cost $\gamma$ and transient response. Further, to reduce the chattering phenomenon of $sat(u(t))$ shown in Fig. 4, each component of secondary control part in (17) can be used $\tanh(\cdot)$ instead of $\mathrm{sgn}(\cdot)$ [14].

### 4.2 Example 2: satellite system

Consider the problem of controlling the yaw angles of a satellite system that consists of two rigid bodies joined by a flexible link. It is represented in terms of state-space equations as follows:

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \\ \dot{x}_4(t) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -k & k & -f & f \\ k & -k & f & -f \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \times \big(sat(u(t)) + d(t)\big),$$

$$y(t) = x_1(t),$$

where $x_1(t)$ and $x_2(t)$ are the yaw angles for the main body and the sensor module; $x_3(t) = \dot{x}_1(t)$ and $x_4(t) = \dot{x}_2(t)$; $u(t)$ is the controller torque; $d(t)$ is

**Table 1** The minimum cost $\gamma$ under no disturbance

| | Theorem 8 in [12] | Theorem 4 | Theorem 5 | Theorem 6 |
|---|---|---|---|---|
| $\gamma$ | 14.4708 | 0.0769 | 11.4776 | 30.3071 |

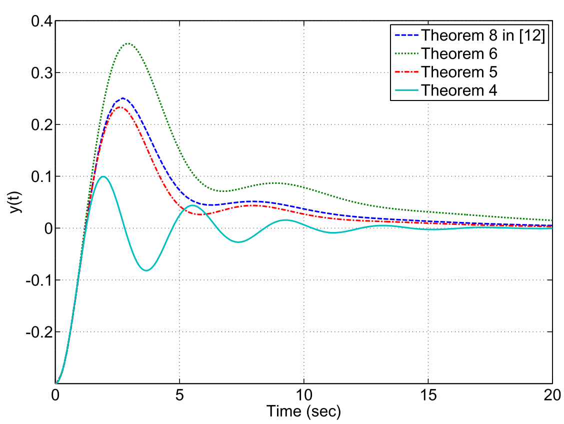


**Fig. 2** The trajectories of state $y(t)$ when no disturbance

**Table 2** The minimum cost $\gamma$ to be computed for the three types

| | Theorem 4 | Theorem 5 | Theorem 6 |
|---|---|---|---|
| $\gamma$ | 0.0785 | 21.4906 | 175.9121 |

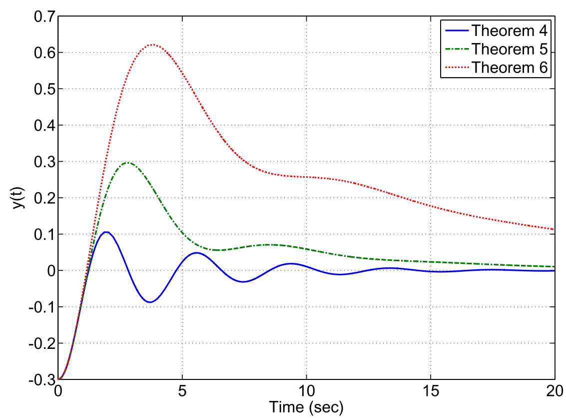


**Fig. 3** The trajectories of state $y(t)$ with matched disturbance

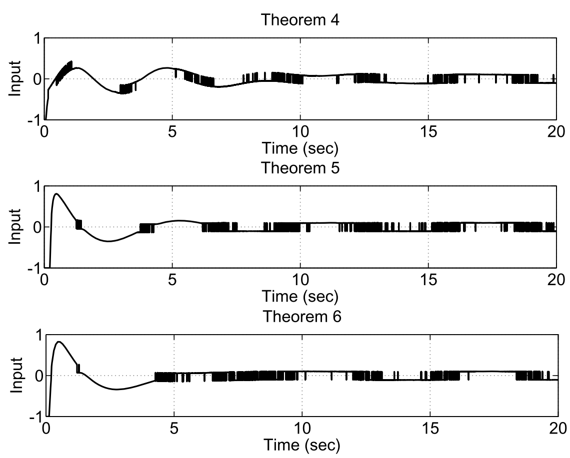


**Fig. 4** The trajectories of the saturated inputs with a function $\mathrm{sgn}(\cdot)$

the matched disturbance; and $y(t)$ is the output. The torque constant $k$ and viscous damping $f$ are uncertain in $0.09 \le k \le 0.4$ and $0.0038 \le f \le 0.04$, and the matched disturbance $d(t)$ is chosen as $d(t) = 0.2\cos t$. If the parameters $\theta_1(t)$, $\theta_2(t)$, $\theta_3(t)$, and $\theta_4(t)$ are chosen as

$$\theta_1(t) = 0.5 - 0.5\cos(0.05\pi t),$$
$$\theta_2(t) = 0.5 + 0.5\cos(0.05\pi t),$$
$$\theta_3(t) = 0.05 - 0.05\sin(0.05\pi t),$$
$$\theta_4(t) = 0.05 + 0.05\sin(0.05\pi t),$$

then the LPV representation of the satellite system is as follows:

$$A_0 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$A_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -k_{\min} & k_{\min} & 0 & 0 \\ k_{\min} & -k_{\min} & 0 & 0 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -k_{\max} & k_{\max} & 0 & 0 \\ k_{\max} & -k_{\max} & 0 & 0 \end{bmatrix},$$

$$A_3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -f_{\min} & f_{\min} \\ 0 & 0 & f_{\min} & -f_{\min} \end{bmatrix},$$

$$A_4 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -f_{\max} & f_{\max} \\ 0 & 0 & f_{\max} & -f_{\max} \end{bmatrix},$$

$$B_0 = [0 \quad 0 \quad 1 \quad 0]^{\mathrm{T}},$$
$$B_1 = B_2 = B_3 = B_4 = [0 \quad 0 \quad 0 \quad 0]^{\mathrm{T}},$$
$$C = [1 \quad 0 \quad 0 \quad 0], \qquad \bar{\theta}_{\min} = \bar{\theta}_{\max} = 1.1,$$
$$\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0, \qquad \beta_1 = \beta_2 = 1,$$
$$\beta_3 = \beta_4 = 0.1, \qquad \eta_1 = \eta_2 = -0.025\pi,$$
$$\eta_3 = \eta_4 = -0.0025\pi, \qquad \nu_1 = \nu_2 = 0.025\pi,$$
$$\nu_3 = \nu_4 = 0.0025\pi, \qquad \mu = 1.5, \qquad \varepsilon = 0.2,$$
$$k_{\min} = 0.09, \qquad k_{\max} = 0.4, \qquad f_{\min} = 0.038,$$
$$f_{\max} = 0.4,$$
$$Q = I, \qquad x(0) = [2 \quad -1 \quad 0 \quad 0].$$

Let us demonstrate the validity of the three types of controllers for an LPV systems with input saturation and a matched disturbance. Table 3 presents the minimum cost $\gamma$ of each of the three controllers. Figure 2 shows the trajectories of the yaw angle of the main body. Type 3 controller shows the fastest decay

**Table 3** The minimum cost $\gamma$ to be computed for the three types

| | Theorem 4 | Theorem 5 | Theorem 6 |
|---|---|---|---|
| $\gamma$ | 3.5137 | 625.2877 | $4.8535\times10^3$ |

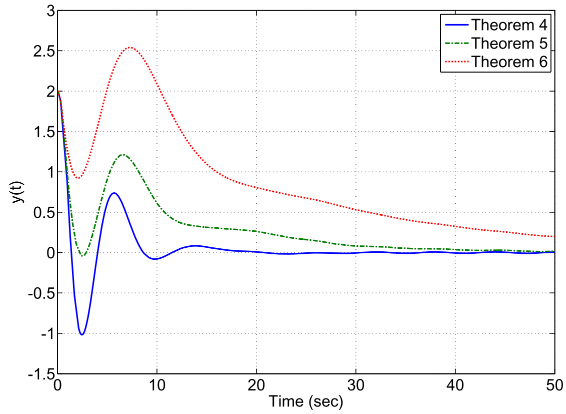


**Fig. 5** The trajectories of state $y(t)$ with matched disturbance

rate and the smallest $\gamma$ among the three types of controllers as shown in Fig. 5 and Table 3.

## 5 Conclusion

This paper proposed three types of $\mathcal{H}_2$ state-feedback controllers for LPV systems with input saturation and a matched disturbance. The proposed controllers comprised two control parts: a main control part used for achieving $\mathcal{H}_2$ performance, and secondary control part used for rejecting a matched disturbance. The three controller types were designed using a feedback gain matrix $K(\theta(t))$ and Lyapunov function $V(x(t))$. We first derived PLMI conditions for the controller designs. Then, using a parameter relaxation technique, the PLMI conditions were converted into LMI conditions. Simulation results under a no disturbance condition showed that the proposed controllers have a better performance than a conventional controller. Further, simulation results under a matched disturbance showed that Type 3 controller had a better performance than both Type 1 and Type 2 controllers.

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