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ORIGINAL PAPER

Codimension-two bifurcation analysis in two-dimensional Hindmarsh–Rose model

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Abstract In this paper, we analyze the codimension-2 bifurcations of equilibria of a two-dimensional Hindmarsh–Rose model. By using the bifurcation methods and techniques, we give a rigorous mathematical analysis of Bautin bifurcation. The main result is that no more than two limit cycles can be bifurcated from the equilibrium via Hopf bifurcation; sufficient conditions for the existence of one or two limit cycles are obtained. This paper also shows that the model undergoes a Bogdanov–Takens bifurcation which includes a saddle-node bifurcation, an Andronov–Hopf bifurcation, and a homoclinic bifurcation. In some case, the globally asymptotical stability is discussed.

Keywords Hindmarsh–Rose model · Bifurcation · Limit cycle · Homoclinic orbit

1 Introduction

Bifurcation theory studies the qualitative change under the variation of the parameters on which the system depends. It is one of the main concerns in the study of nonlinear dynamical systems. Beginning with the fundamental work of Poincaré and Andronov, the literature on the bifurcation theory is enormous. In the recent decades, a number of new methods and techniques have been developed. For example, bifurcations in a generic one-parameter system on the plane near an equilibrium with purely imaginary eigenvalues was studied first by Andronov and Leontovich [1]; Hopf [2] proved the appearance of a family of periodic solutions of increasing amplitude for n-dimensional systems having an equilibrium with a pair of purely imaginary eigenvalues. One good approach is to use the so-called Lyapunov coefficients: Bautin [3] obtained an explicit expression for the first Lyapunov coefficient in terms of Taylor coefficients of a general planar system. He first studied generic two-parameter bifurcation diagrams near a point where the first Lyapunov coefficient vanishes; therefore, we call this bifurcation the Bautin bifurcation. The formulas for the first and the second Lyapunov coefficients can be found in many books and papers, such as [3–8]. For the research of the higher degeneracies at the Hopf bifurcation, see [8–10]. The classification and unfolding of the planar system having an equilibrium with two zero eigenvalues was done simultaneously (and independently) by Bogdanov [11] and Takens [12, 13], i.e., the Bogdanov–Takens bifurcation. The degenerate codimension-3 Bogdanov–Takens bifurcations have been studied in [14, 15].

Bautin bifurcation and Bogdanov–Takens bifurcation are frequently occurring in applied mathematical models. We will consider these bifurcations in a neuron model. As is known, one of the most important models in computational neuroscience is the Hodgkin–Huxley model [16]. Hodgkin and Huxley

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gave an explanation of action potential generation in the axon of the giant squid in terms of time- and voltage-dependent sodium and potassium conductances, respectively. This model consists of four coupled nonlinear differential equations, six functions and seven constants. Because of the complexity of these equations, FitzHugh [17] and Nagumo et al. [18] gave a simplification of the Hodgkin–Huxley equations and introduced a model of the following form:

$$\begin{cases} \dot{x} = \alpha(y - f(x) + z), \\ \dot{y} = \beta(g(x) - y), \end{cases} \tag{1.1}$$

where x represents the membrane potential and y is a recovery variable. The function f is cubic, the function g is linear, α, β are time constants, and z is stimulus intensity, a variable corresponding to membrane current I in the Hodgkin–Huxley model. This model does not provide a very realistic description of the rapid firing of the neuron compared to the relatively long interval between firings. In order to achieve a more realistic description of firing, Hindmarsh and Rose [19, 20] replaced the linear function $g(x)$ in the FitzHugh–Nagumo model (1.1) with a quadratic function. This two-dimensional Hindmarsh–Rose model can have more than one equilibrium. In order to terminate firing, to the model was added the third equation with an adaptation variable z. These two-dimensional and three-dimensional Hindmarsh–Rose models have been studied by many papers, see e.g. [19–25] and references therein. These papers discussed the bifurcations of Hindmarsh–Rose models mostly by computer simulations, but the upper bound of the maximal number of limit cycles bifurcated from the equilibrium via Hopf bifurcation has not been obtained.

In this paper, we consider the two-dimensional Hindmarsh–Rose type model

$$\begin{cases} \frac{dx}{dt} = y - ax^3 + bx^2, \\ \frac{dy}{dt} = -c - dx^2 - y, \end{cases} \tag{1.2}$$

where a, b, c, d are positive parameters. By using the bifurcation theory and methods [7, 26–28], we give the analytical study for codimension-2 bifurcations of equilibria of system (1.2). The paper is organized as follows: In Sect. 2, we discuss the existences of equilibria, and analyze the local or global stability of equilibria. In Sect. 3, we will show that the system undergoes a Bogdanov–Takens bifurcation which includes a saddle-node bifurcation, an Andronov–Hopf bifurcation, and a homoclinic bifurcation. In Sect. 4, we study the Andronov–Hopf and Bautin bifurcation, and obtain that the maximal number of limit cycles bifurcated from the equilibrium is two, and the sufficient conditions for the existence of one or two limit cycles near the equilibrium are given. Remarks and conclusions are drawn in Sect. 5.

2 Equilibria and stability

If $M_j(x_j, y_j)$ is one of the equilibria of system (1.2), then x_j is a root of the equation

$$ax^3 + (d - b)x^2 + c = 0, \tag{2.1}$$

and $y_j = -c - dx_j^2$. The Jacobian matrix of the system (1.2) evaluated at equilibrium M_j is

$$J(x_j, y_j) = \begin{pmatrix} -3ax_j^2 + 2bx_j & 1 \\ -2dx_j & -1 \end{pmatrix}.$$

By analyzing the sign of real parts of the eigenvalues of $J(x_j, y_j)$ and using the Routh–Hurwitz theorem, we have

Theorem 2.1 (1) *If $27a^2c - 4(b - d)^3 > 0$, then system (1.2) has a unique equilibrium $M_1(x_1, y_1)$, where $x_1 < \min\{0, \frac{2(b-d)}{3a}\}$. M_1 is a stable focus or a node.*

(2) *If $27a^2c - 4(b - d)^3 = 0$, then system (1.2) has exactly two equilibria, $M_1(x_1, y_1)$ and $M_2(x_2, y_2)$, where $x_1 < 0 < x_2 = \frac{2(b-d)}{3a}$. M_1 is a stable focus or node, M_2 is a higher-order equilibrium.*

(3) *If $27a^2c - 4(b - d)^3 < 0$, then system (1.2) has exactly three equilibria, $M_j(x_j, y_j)$, $j = 1, 2, 3$, where $x_1 < 0 < x_2 < x_3$. M_1 is a stable focus or node, M_2 is a saddle, M_3 is a focus or a node.*

Moreover, the following theorem holds.

Theorem 2.2 *If $27a^2c - 4(b - d)^3 > 0$, then M_1 is globally asymptotically stable.*

Proof If $27a^2c - 4(b-d)^3 > 0$, then system (1.2) has a unique equilibrium $M_1(x_1, -c - dx_1^2)$, where M_1 lies in the third quadrant. Denote $D_k = \{(x, y) : -k \le x \le k, -c - dk^2 \le y \le k\}$, where $k > 0$ is to be defined suitably. We can choose k big enough, such that

$$\dot{x}|_{(x=k, -c-dk^2 \le y \le k)} \le k - ak^3 + bk^2 < 0,$$

$$\begin{aligned}
&\dot{x}|_{(x=-k,-c-dk^2\le y\le k)}\\
&\quad\ge -c-dk^2+ak^3+bk^2>0,\\
&\dot{y}|_{y=k}=-c-dx^2-k<0,\\
&\dot{y}|_{(-k\le x\le k,y=-c-dk^2)}\\
&\quad=d\big(k^2-x^2\big)\big|_{-k\le x\le k}\ge 0,
\end{aligned}$$

hence D_k is a positive invariant set of system (1.2), and every solution of system (1.2) is bounded.

Since $\dot{y}|_{y\ge 0}<0, \dot{x}|_{(x=0,y<0)}<0$, it follows that if there exist closed orbits of system (1.2), then the closed orbits must be located in the third quadrant, but $P_x(x,y)+Q_y(x,y)=-3ax^2+2bx-1<0$ for $x<0$, and applying the Dulac theorem we see that system (1.2) has no closed orbits in the third quadrant, therefore system (1.2) has no closed orbits in R^2, which means that M_1 is a globally asymptotically stable equilibrium of system (1.2). ☐

3 Bogdanov–Takens bifurcation

In this section, by using the methods in [29], we discuss the Bogdanov–Takens bifurcation of system (1.2).

We rewrite system (1.2) as

$$\frac{dX}{dt}=F(X,\mu),$$

where $X=(x,y)^T$, $\mu=(a,b,c,d)^T$ and

$$F(X,\mu)=\begin{pmatrix} y-ax^3+bx^2\\ -c-dx^2-y\end{pmatrix}.$$

If $27a^2c=4(b-d)^3$, then $M_2(x_2,y_2)$ is a higher-order equilibrium of system (1.2). In order to discuss the Bogdanov–Takens bifurcation near M_2, we assume further that the trace of the Jacobian matrix of system (1.2) evaluated at M_2 vanishes, i.e., $trJ(x_2,y_2)=-3ax_2^2+2bx_2-1=0$; substituting $x_2=\frac{2(b-d)}{3a}$ into $3ax_2^2-2bx_2+1=0$ we get $a=\frac{4}{3}d(b-d)$, hence $c=\frac{4(b-d)^3}{27a^2}=\frac{b-d}{12d^2}$.

Now, if $a=\frac{4}{3}d(b-d)$, $c=\frac{b-d}{12d^2}$, then system (1.2) has the equilibrium $M_2(x_2,y_2)$ with two zero eigenvalues, where $x_2=\frac{1}{2d}$, $y_2=-\frac{b+2d}{12d^2}$. Hence

$$\begin{aligned}
&(X_0,\mu_0)\\
&\quad\equiv\left((x_2,y_2)^T,\left(\frac{4}{3}d(b-d),b,\frac{b-d}{12d^2},d\right)^T\right)\\
&\quad=\left(\left(\frac{1}{2d},-\frac{b+2d}{12d^2}\right)^T,\left(\frac{4}{3}d(b-d),b,\frac{b-d}{12d^2},d\right)^T\right)
\end{aligned}$$

is a family of equilibrium points whose linearization has a double-zero eigenvalue, and

$$p_1=\begin{pmatrix}1\\-1\end{pmatrix},\quad p_2=\begin{pmatrix}1\\0\end{pmatrix},$$

$$q_1=\begin{pmatrix}0\\-1\end{pmatrix},\quad q_2=\begin{pmatrix}1\\1\end{pmatrix}$$

are the right and left (generalized) eigenvectors, respectively, associated with the eigenvalue zero.

Let $b\ne 2d$ and

$$\begin{aligned}
\bar{a}&\equiv\frac{1}{2}p_1^T\big(q_2\bullet D^2F(X_0,\mu_0)\big)p_1=d-b,\\
\bar{b}&\equiv p_1^T\big(q_1\bullet D^2F(X_0,\mu_0)\big)p_1\\
&\quad+p_1^T\big(q_2\bullet D^2F(X_0,\mu_0)\big)p_2=2(2d-b),\\
S_1^T&\equiv q_2^TF_\mu(X_0,\mu_0)\\
&=\left(-\frac{1}{8d^3},\frac{1}{4d^2},-1,-\frac{1}{4d^2}\right),\\
S_2&\equiv\left[\frac{2\bar{a}}{\bar{b}}\big(p_1^T\big(q_1\bullet D^2F(X_0,\mu_0)\big)p_2\right.\\
&\quad+p_2^T\big(q_2\bullet D^2F(X_0,\mu_0)\big)p2\big)\\
&\quad\left.-p_1^T\big(q_2\bullet D^2F(X_0,\mu_0)\big)p_2\right]\\
&\quad\times F_\mu^T(X_0,\mu_0)q_1\\
&\quad-\frac{2\bar{a}}{\bar{b}}\sum_{i=1}^{2}\big(q_i\bullet F_{\mu X}(X_0,\mu_0)\big)p_i\\
&\quad+\big(q_2\bullet F_{\mu X}(X_0,\mu_0)\big)p_1\\
&=\left(\frac{3}{4d(b-2d)}\,\frac{1}{2d-b}0-\frac{1}{d}\right)^T,\\
\beta_1&\equiv S_1^T(\mu-\mu_0),\\
\beta_2&\equiv S_2^T(\mu-\mu_0).
\end{aligned}$$

Using the theorem in [29], we have that system (1.2) is locally topologically equivalent to

$$\begin{cases} \dot{z}_1 = z_2, \\ \dot{z}_2 = \beta_1 + \beta_2 z_1 + \bar{a} z_1^2 + \bar{b} z_1 z_2. \end{cases} \tag{3.1}$$

If we choose λ_1 and λ_2 as bifurcation parameters, where

$$\lambda_1 = a - \frac{4}{3}d(b-d), \quad \lambda_2 = c - \frac{b-d}{12d^2},$$

then

$$\beta_1 = \left(-\frac{1}{8d^3}, \frac{1}{4d^2}, -1, -\frac{1}{4d^2}\right)(\lambda_1, 0, \lambda_2, 0)^T$$
$$= -\frac{1}{8d^3}\lambda_1 - \lambda_2,$$
$$\beta_2 = \left(\frac{3}{4d(b-2d)}\,\frac{1}{2d-b}0 - \frac{1}{d}\right)(\lambda_1, 0, \lambda_2, 0)^T$$
$$= \frac{3}{4d(b-2d)}\lambda_1.$$

System (3.1) becomes

$$\begin{cases} \dot{z}_1 = z_2, \\ \dot{z}_2 = -\frac{1}{8d^3}\lambda_1 - \lambda_2 + \frac{3}{4d(b-2d)}\lambda_1 z_1 \\ \qquad + (d-b)z_1^2 + 2(2d-b)z_1 z_2. \end{cases} \tag{3.2}$$

Making the change of variables by

$$t = \frac{2(2d-b)}{b-d}t_1, \quad z_1 = \frac{d-b}{4(2d-b)^2}\eta_1,$$
$$z_2 = -\frac{(d-b)^2}{8(2d-b)^3}\eta_2,$$

we obtain

$$\begin{cases} \frac{d\eta_1}{dt_1} = \eta_2, \\ \frac{d\eta_2}{dt_1} = \bar{\beta}_1(\lambda_1, \lambda_2) + \bar{\beta}_2(\lambda_1)\eta_1 + \eta_1^2 - \eta_1\eta_2, \end{cases} \tag{3.3}$$

where

$$\bar{\beta}_1(\lambda_1, \lambda_2) = \frac{16(2d-b)^4}{(b-d)^3}\left(\frac{1}{8d^3}\lambda_1 + \lambda_2\right),$$
$$\bar{\beta}_2(\lambda_1) = \frac{3(b-2d)}{d(d-b)^2}\lambda_1.$$

Since

$$4\bar{\beta}_1 - \bar{\beta}_2^2 = 0 \Leftrightarrow \lambda_2 + \frac{1}{8d^3}\lambda_1$$
$$- \frac{9}{64(b-d)d^2(2d-b)^2}\lambda_1^2 = 0,$$
$$\bar{\beta}_1 = 0 \Leftrightarrow \lambda_2 + \frac{1}{8d^3}\lambda_1 = 0,$$
$$\bar{\beta}_1 + \frac{6}{25}\bar{\beta}_2^2$$
$$= o\big(\bar{\beta}_2^2\big) \Leftrightarrow \lambda_2 + \frac{1}{8d^3}\lambda_1$$
$$+ \frac{27}{200(b-d)d^2(2d-b)^2}\lambda_1^2$$
$$= o\big(\lambda_1^2\big),$$
$$\bar{\beta}_2 < 0 \Leftrightarrow \lambda_1(b-2d) < 0.$$

By using the theorem in [7] and the analysis above, we have

Theorem 3.1 *Let* $a = \frac{4}{3}d(b-d) + \lambda_1$, $c = \frac{b-d}{12d^2} + \lambda_2$ *and* $b \neq 2d$. *Then system* (1.2) *is locally topologically equivalent to the following system*:

$$\begin{cases} \frac{d\eta_1}{dt_1} = \eta_2, \\ \frac{d\eta_2}{dt_1} = \frac{16(2d-b)^4}{(b-d)^3}\big(\frac{1}{8d^3}\lambda_1 + \lambda_2\big) \\ \qquad + \frac{3(b-2d)}{d(d-b)^2}\lambda_1\eta_1 + \eta_1^2 - \eta_1\eta_2, \end{cases} \tag{3.4}$$

which has the following local representations of the bifurcation curves in a small neighborhood of the origin:

(i) *there is a saddle-node bifurcation curve*

$$SN = \left\{(\lambda_1, \lambda_2) : \lambda_2 = -\frac{1}{8d^3}\lambda_1 + \frac{9}{64(b-d)d^2(2d-b)^2}\lambda_1^2\right\};$$

(ii) *there is an Andronov–Hopf bifurcation curve*

$$H = \left\{(\lambda_1, \lambda_2) : \lambda_2 = -\frac{1}{8d^3}\lambda_1,\ (b-2d)\lambda_1 < 0\right\};$$

(iii) *there is a homoclinic bifurcation curve*

$$HL = \left\{(\lambda_1, \lambda_2) : \lambda_2 = -\frac{1}{8d^3}\lambda_1 - \frac{27}{200d^2(2d-b)^2(b-d)}\lambda_1^2 + o\big(\lambda_1^2\big), (b-2d)\lambda_1 < 0\right\}.$$

Fig. 1 The Bogdanov–Takens bifurcation diagram for $d < b < 2d$

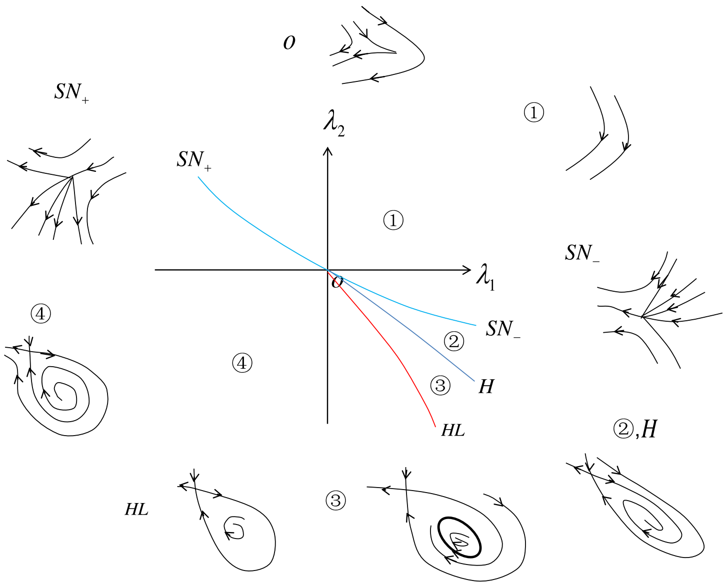
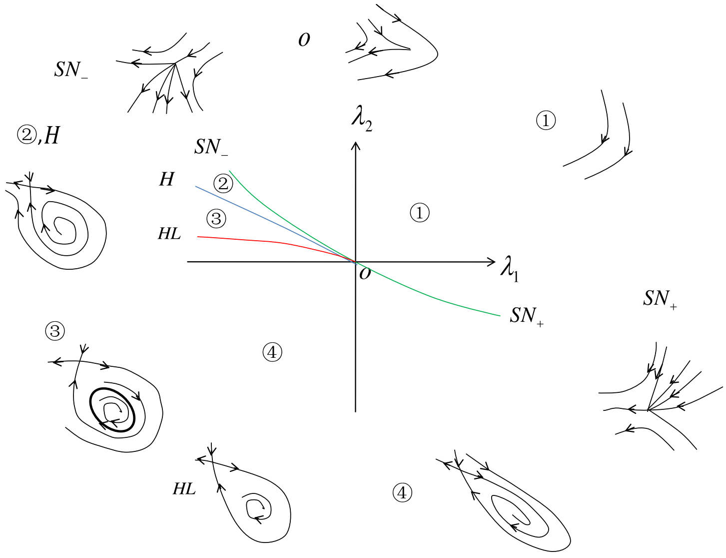


Fig. 2 The Bogdanov–Takens bifurcation diagram for $b > 2d$



Denote SN in $\lambda_1(b-2d) < 0$ (resp., $\lambda_1(b-2d) > 0$) by SN_-(resp., SN_+). The bifurcation diagram of system (1.2) near M_2 is presented in Fig. 1 (resp. Fig. 2) when $d < b < 2d$ (resp. $b > 2d$).

For example, when $(b, d, \lambda_1, \lambda_2) = (0.5, 0.4, 0.005, -0.012)$, numerical simulation of system (1.2) is depicted in Fig. 3: there is a stable limit cycle, which corresponds to the case (3) in Fig. 1. When $(b, d, \lambda_1, \lambda_2) = (0.5, 0.4, 0.005, -0.01317749024)$, numerical simulation of system (1.2) is depicted in Fig. 4: a homoclinic orbit occurs, which corresponds to the case HL in Fig. 1.

4 Andronov–Hopf bifurcation and Bautin bifurcation

In this section, we discuss the Andronov–Hopf bifurcation and Bautin bifurcation near equilibrium M_3 of system (1.2) when $27a^2c - 4(b-d)^3 < 0$.

Suppose that $p \equiv trJ(x_3, y_3) = -3ax_3^2 + 2bx_3 - 1 = 0$. Then M_3 is a weak focus of system (1.2). It is easy to obtain that equations

$$\begin{cases} -3ax_3^2 + 2bx_3 - 1 = 0, \\ ax_3^3 + (d-b)x_3^2 + c = 0 \end{cases} \tag{4.1}$$

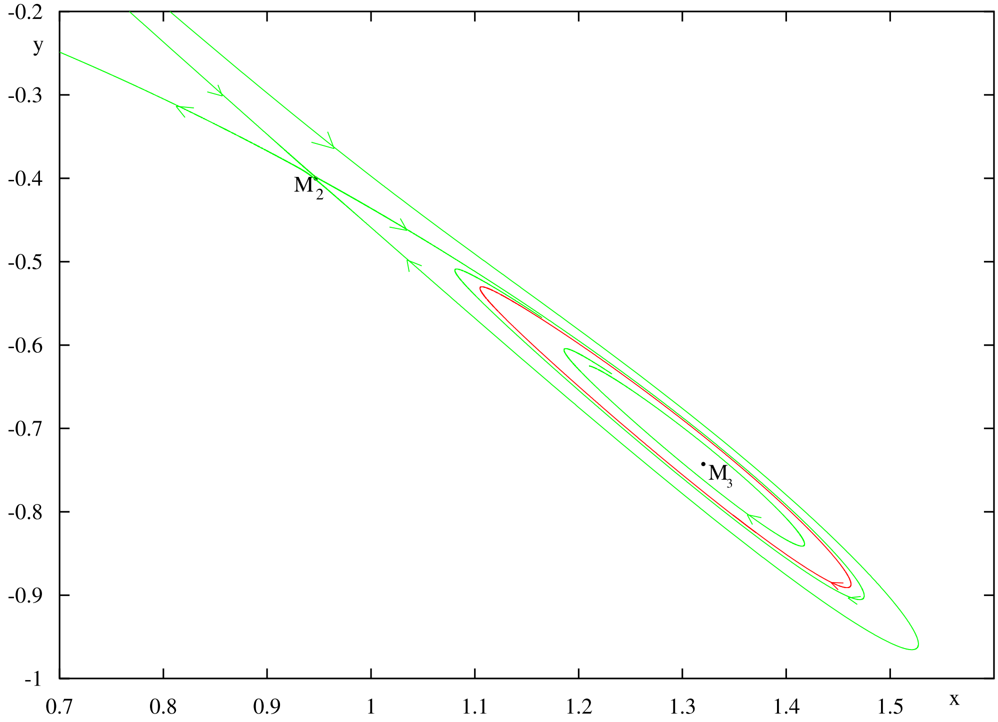


Fig. 3 A stable limit cycle when $b = 0.5$, $d = 0.4$, $\lambda_1 = 0.005$, $\lambda_2 = -0.012$

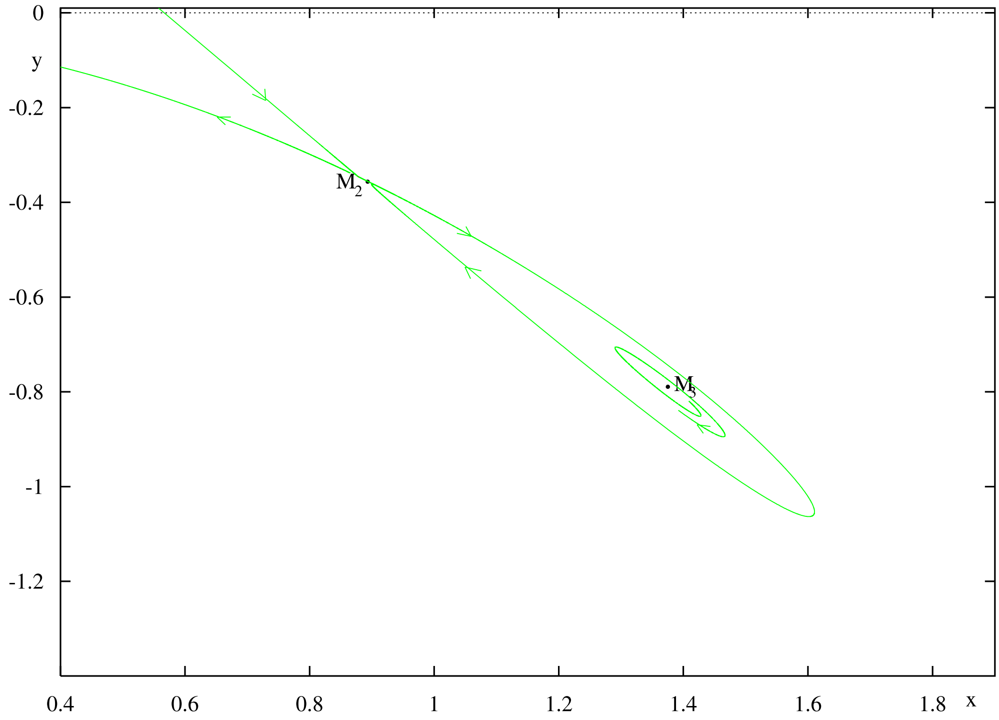


Fig. 4 A homoclinic orbit when $b = 0.5$, $d = 0.4$, $\lambda_1 = 0.005$, $\lambda_2 = -0.01317749024$

have solutions if and only if

$$\begin{aligned} &27a^2c^2 - 18acd + 12b^2cd + 3d^2 - 4bd \\ &\quad - 4b^3c + b^2 + a = 0. \end{aligned} \tag{4.2}$$

Moreover, $x_3 = \frac{3d-b-9ac}{6bd-2b^2-3a}$ is the unique solution of (4.1).

Let $w_1 = x - x_3$, $w_2 = y - y_3$. Then (1.2) becomes

$$\begin{cases} \dot{w}_1 = w_1 + w_2 + (b - 3ax_3)w_1^2 - aw_1^3, \\ \dot{w}_2 = -2dx_3w_1 - w_2 - dw_1^2. \end{cases} \tag{4.3}$$

Setting $\omega_0 = \sqrt{2dx_3 - 1}$, the transformation

$$\xi = 2dx_3w_1 + w_2, \eta = -\omega_0 w_2$$

transforms (4.3) into

$$\begin{cases} \dot{\xi} = -\omega_0\eta - \frac{2bx_3-1}{4dx_3^2}(\xi + \frac{1}{\omega_0}\eta)^2 - \frac{a}{4d^2x_3^2}(\xi + \frac{1}{\omega_0}\eta)^3, \\ \dot{\eta} = \omega_0\xi + \frac{\omega_0}{4dx_3^2}(\xi + \frac{1}{\omega_0}\eta)^2. \end{cases} \tag{4.4}$$

It is convenient to rewrite (4.4) in complex form by introducing $z = \xi + i\eta$:

$$\dot{z} = i\omega_0 z + \sum_{2\le k+l\le 5} \frac{1}{k!l!} g_{kl} z^k \bar{z}^l,$$

where

$$g_{20} = \frac{(1 - 2bx_3 + \omega_0 i)(\omega_0 - i)^2}{8d\omega_0^2x_3^2},$$

$$g_{11} = \frac{(1 - 2bx_3 + \omega_0 i)(\omega_0^2 + 1)}{8d\omega_0^2x_3^2},$$

$$g_{02} = \frac{(1 - 2bx_3 + \omega_0 i)(\omega_0 + i)^2}{8d\omega_0^2x_3^2},$$

$$g_{30} = \frac{-3a(\omega_0 - i)^3}{16d^2\omega_0^3x_3^2},$$

$$g_{21} = \frac{-3a(\omega_0 - i)(\omega_0^2 + 1)}{16d^2\omega_0^3x_3^2},$$

$$g_{12} = \bar{g}_{21}, \quad g_{03} = \bar{g}_{30}.$$

and $g_{kl} = 0$ for $4 \le k + l \le 5$.

By using the formula of the first Lyapunov coefficient in [7], and $\omega_0^2 = 2dx_3 - 1$ and (4.1), we get

$$l_1(0) = \frac{1}{2\omega_0^2}\mathrm{Re}\,(ig_{20}g_{11} + w_0g_{21}) = \frac{2b(b-d) - 3a}{16\omega_0^5dx_3}.$$

If $a > \frac{2}{3}b(b-d)$, then $l_1(0) < 0$, the Andronov–Hopf bifurcation is supercritical; if $a < \frac{2}{3}b(b-d)$, then $l_1(0) > 0$, the Andronov–Hopf bifurcation is subcritical; if $a = \frac{2}{3}b(b-d)$, then $l_1(0) = 0$, a Bautin bifurcation occurs. Applying the formula of the second Lyapunov coefficient in [7],

$$\begin{aligned} 12l_2(0) = {} & \frac{1}{w_0}\mathrm{Re}\,g_{32} + \frac{1}{w_0^2}\mathrm{Im}\Big[g_{20}\bar{g}_{31} - g_{11}(4g_{31} \\ & + 3\bar{g}_{22}) - \frac{1}{3}g_{02}(g_{40} + \bar{g}_{13}) - g_{30}g_{12}\Big] \\ & + \frac{1}{w_0^3}\Big\{\mathrm{Re}\Big[g_{20}\Big(\bar{g}_{11}(3g_{12} - \bar{g}_{30}) \\ & + g_{02}\Big(\bar{g}_{12} - \frac{1}{3}g_{30}\Big) + \frac{1}{3}\bar{g}_{02}g_{03}\Big) \\ & + g_{11}\Big(\bar{g}_{02}\Big(\frac{5}{3}\bar{g}_{30} + 3g_{12}\Big) \\ & + \frac{1}{3}g_{02}\bar{g}_{03} - 4g_{11}g_{30}\Big)\Big] \\ & + 3\,\mathrm{Im}(g_{20}g_{11})\,\mathrm{Im}g_{21}\Big\} \\ & + \frac{1}{w_0^4}\{\mathrm{Im}[g_{11}\bar{g}_{02}(\bar{g}_{20}^2 \\ & - 3\bar{g}_{20}g_{11} - 4g_{11}^2)] \\ & + \mathrm{Im}(g_{20}g_{11})[3\,\mathrm{Re}(g_{20}g_{11}) - 2|g_{02}|^2]\}, \end{aligned}$$

notice that $\omega_0^2 = 2dx_3 - 1$, $x_3 = \frac{3d-b-9ac}{6bd-2b^2-3a}$ and $a = \frac{2}{3}b(b-d)$. By a direct computation with MAPLE, we have

$$l_2(0) = \frac{5}{576\omega_0^{11}x_3^6bd^2(d-b)^2}f(c), \tag{4.5}$$

where $f(c) = 12bd(d^2 + b(2d - b))c + 4d^3 - 3bd^2 - 2b^2d + b^3$. Since $a = \frac{2}{3}b(b-d)$, we have

$$27a^2c - 4(b-d)^3 < 0 \Leftrightarrow c < \frac{b-d}{3b^2}.$$

Equation (4.2) becomes

$$12b^2(b-d)^2c^2-4b(b^2-3d^2)c + \frac{1}{3}(b-d)(5b-9d)=0, \tag{4.6}$$

and we can solve (4.6) in c:

$$c=c_j=\frac{b^2-3d^2+(-1)^j 2\sqrt{b(2d-b)^3}}{6b(b-d)^2}, \quad j=1,2.$$

If $b>2d$, then (4.6) in c has no solutions; if $b=2d$, then $c=c_1=c_2=\frac{1}{12d}$, but it is contrary to $c<\frac{b-d}{3b^2}=\frac{1}{12d}$, hence the necessary condition for the Bautin bifurcation to occur is $d<b<2d$.

If $d<b<2d$, then $c=c_1$ or $c=c_2$. We have

$$f(c_2)=\frac{2d-b}{(d-b)^2}\Big(4d\big(b(2d-b)+d^2\big)\sqrt{b(2d-b)} - \big(b(2d-b)\big(6d^2+b(2d-b)\big)+d^4\big)\Big),$$

and by

$$\Big(4d\big(b(2d-b)+d^2\big)\sqrt{b(2d-b)}\Big)^2 - \Big(b(2d-b)\big(6d^2+b(2d-b)\big)+d^4\Big)^2 = -(b-d)^8<0$$

we have $f(c_2)<0$, hence $f(c_1)<f(c_2)<0$, which yields $l_2(0)<0$.

Therefore, the following theorem holds.

Theorem 4.1 *Suppose that* $27a^2c-4(b-d)^3<0$. *For sufficiently small* $\varepsilon>0$, *we have*:

(1) *If* $p=0$, *then the equilibrium* M_3 *is a weak focus of order at most two.*

(2) *If* $a>\frac{2}{3}b(b-d)$ *and* $0<p<\varepsilon$, *then system* (1.2) *has a stable limit cycle near* M_3.

(3) *If* $a<\frac{2}{3}b(b-d)$ *and* $-\varepsilon<p<0$, *then system* (1.2) *has an unstable limit cycle near* M_3.

(4) *If* $0<\frac{2}{3}b(b-d)-a<\varepsilon$ *and* $-\varepsilon<p<0$, *then system* (1.2) *has two limit cycles*: Γ_1 *and* Γ_2 *near* M_3, *where* $\Gamma_1\subset\Gamma_2$, Γ_1 *is unstable and* Γ_2 *is stable.*

Example 4.1 (i) Set $a=0.330127019$, $b=1.01$, $c=0.02445385326$, $d=0.669872981$, then the condition (2) of Theorem 4.1 holds. Numerical simulation of system (1.2) is depicted in Fig. 5, where a stable limit cycle near M_3 occurs.

(ii) Set $a=0.1993587371$, $b=1.018$, $c=0.3987174743$, $d=0.4019237886$, then condition (3) of Theorem 4.1 holds. Numerical simulation of system (1.2) is depicted in Fig. 6, where an unstable limit cycle near M_3 occurs.

5 Remarks and conclusions

For planar systems, the only codimension-2 bifurcations of equilibria that may occur are the cusp, Bogdanov–Takens and Bautin bifurcations. But the cusp bifurcation of equilibrium is not analyzed in this document because it cannot occur. We give the reason for this as follows:

If $27a^2c=4(b-d)^3$ and $trJ(x_2,y_2)=-3ax_2^2+2bx_2-1=\frac{4d(b-d)-3a}{3a}\neq 0$, then $M_2(x_2,y_2)$ may be a saddle-node or a triple equilibrium of system (1.2). Let $u_1=x-x_2$, $u_2=y-y_2$. Then system (1.2) becomes

$$\begin{cases}\frac{du_1}{dt}=\frac{4d(b-d)}{3a}u_1+u_2-(b-2d)u_1^2-au_1^3,\\ \frac{du_2}{dt}=\frac{4d(d-b)}{3a}u_1-u_2-du_1^2.\end{cases} \tag{5.1}$$

Making the transformation of variables by

$$u_1=3av_1+v_2,\quad u_2=4d(d-b)v_1-v_2,$$

we have

$$\begin{cases}\frac{dv_1}{dt}=-\frac{1}{3a+4d(d-b)}[(3av_1+v_2)^2(b-d) \\ \qquad +a(3av_1+v_2)^3],\\ \frac{dv_2}{dt}=\frac{4d(b-d)-3a}{3a}v_2 \\ \qquad +\frac{d(8d^2+4b^2-12bd+3a)}{3a+4d(d-b)}(3av_1+v_2)^2 \\ \qquad +\frac{4ad(b-d)}{3a+4d(d-b)}(3av_1+v_2)^3.\end{cases} \tag{5.2}$$

The center manifold of (5.2) near the origin has the representation $v_2=O(v_1^2)$, hence the restriction of (5.2) to its center manifold is:

$$\dot{v}_1=\frac{9a^2(d-b)}{3a+4d(d-b)}v_1^2+O\big(v_1^3\big).$$

Since $\frac{9a^2(d-b)}{3a+4d(d-b)}\neq 0$, it follows that M_2 is a saddle-node equilibrium, not a triple equilibrium of system (1.2). Therefore the cusp bifurcation of equilibrium cannot occur.

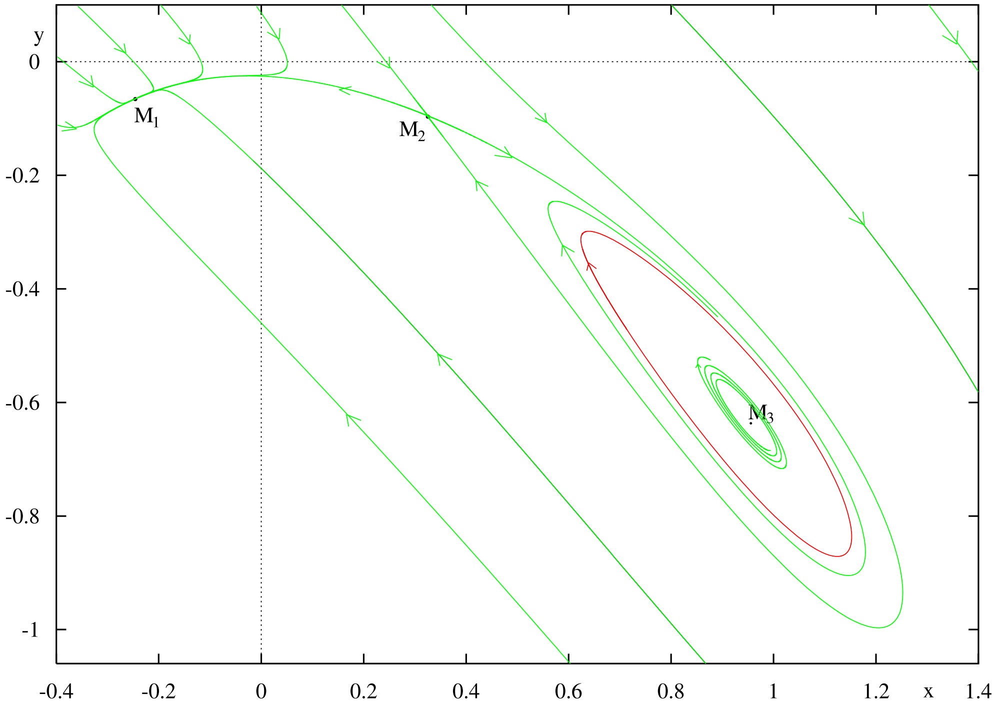


Fig. 5 When $a = 0.330127019$, $b = 1.01$, $c = 0.02445385326$, $d = 0.669872981$, there is a stable limit cycle

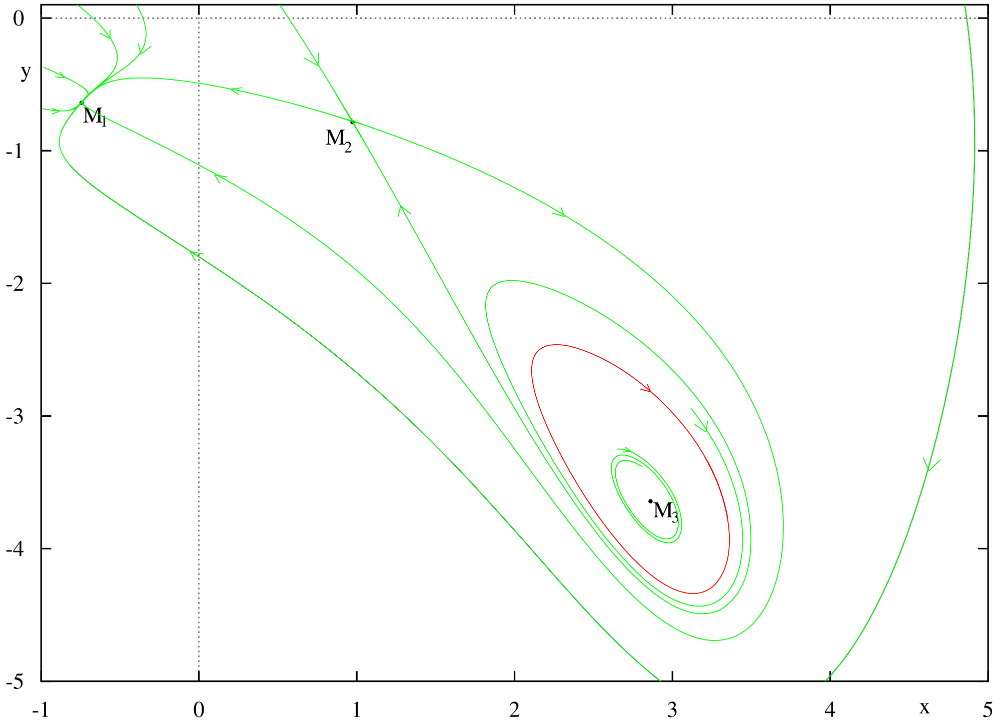


Fig. 6 When $a = 0.1993587371$, $b = 1.018$, $c = 0.3987174743$, $d = 0.4019237886$, there is an unstable limit cycle

In papers [30, 31], the authors studied the existence and number of limit cycles in the FitzHugh–Nagumo system: the main result of [31] is that the FitzHugh–Nagumo system has at most two limit cycles bifurcated from equilibrium via Hopf bifurcation. In this paper, we consider the two-dimensional Hindmarsh–Rose model, which is a modification of FitzHugh–Nagumo system, and give a rigorous mathematical analysis of codimension-2 bifurcations of this model. We determine the sign of the second Lyapunov coefficient at Bautin point, and obtain that the model has a weak focus of order at most two, therefore no more than two limit cycles can be bifurcated from the equilibrium via Hopf bifurcation. The Bogdanov–Takens bifurcations are also discussed, and we obtain the saddle-node bifurcation curve, the Andronov–Hopf bifurcation curve and Homoclinic bifurcation curve near the Bogdanov–Takens point. Some numerical simulation results are given to support the theoretical predictions.

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