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Switched exponential state estimation of neural networks based on passivity theory

Choon Ki Ahn

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Abstract In this paper, a new exponential state estimation method is proposed for switched Hopfield neural networks based on passivity theory. Through available output measurements, the main purpose is to estimate the neuron states such that the estimation error system is exponentially stable and passive from the control input to the output error. Based on augmented Lyapunov–Krasovskii functional, Jensen's inequality, and linear matrix inequality (LMI), a new delay-dependent state estimator for switched Hopfield neural networks can be achieved by solving LMIs, which can be easily facilitated by using some standard numerical packages. The unknown gain matrix is determined by solving delay-dependent LMIs. Finally, a numerical example is provided to demonstrate the effectiveness of the proposed method.

Keywords Passivity theory · State estimation · Switched Hopfield neural networks · Linear matrix inequality (LMI) · Augmented Lyapunov–Krasovskii functional

C.K. Ahn (✉)
Department of Automotive Engineering, Seoul National University of Science & Technology, Seoul 139-743, South Korea
e-mail: hironaka@snut.ac.kr

1 Introduction

Studying artificial neural networks has been the central focus of intensive research activities during the last decades since these artificial networks have found wide applications in areas like associative memory, pattern classification, reconstruction of moving images, signal processing, solving optimization problems, etc.; see [1]. Different models of neural networks such as Hopfield neural networks, cellular neural networks, Cohen–Grossberg neural networks, and bidirectional associative memory neural networks have been extensively investigated in the literature. Among several neural networks, Hopfield neural networks [1, 2] are the most popular. They have been extensively studied and successfully applied in many areas such as combinatorial optimization, signal processing, and pattern recognition [1].

On the other hand, a class of hybrid systems has attracted significant attention because it can model several practical control problems that involve the integration of supervisory logic-based control schemes and feedback control algorithms [3]. As a special class of hybrid systems, switched systems are regarded as nonlinear systems, which are composed of a family of continuous-time or discrete-time subsystems and a rule that orchestrates the switching between the subsystems [4–8]. Recently, switched neural networks, whose individual subsystems are a set of neural networks, have found applications in fields of high speed signal processing, artificial intelligence, and gene selection in a DNA microarray analysis [9–11]. Therefore, some researchers have studied the stability issues for switched Hopfield neural networks [12–14]. However, up to now, the dynamical behavior of switched Hopfield neural networks has received very little research attention, despite its potential and practical importance.

The neuron states in relatively large scale neural networks are not often completely available in the network outputs. Thus, in many applications, one often needs to estimate the neuron states through available measurements and then utilizes the estimated neuron states to achieve certain design objectives. For example, in [15], a recurrent neural network was applied to model an unknown system and the neuron states of the designed neural network were then utilized by the control law. Therefore, from the point of view of control, the state estimation problem for neural networks is of significance for many applications. Some partial results for the neuron state estimation problem are available [16–18]. In [19, 20], the authors studied state estimation for Markovian jumping recurrent neural networks with time-delays by constructing Lyapunov–Krasovskii functionals and linear matrix inequalities (LMIs). In spite of these advances in neural network state estimation, the state estimation problem for switched neural networks has not been investigated in the literature and it is very important in both theories and applications.

The passivity theory [21, 22] is an effective and appealing tool to analyze the stability of nonlinear systems. It may deal with nonlinear systems using only the general characteristics of the input–output dynamics and offers elegant solutions for the proof of absolute stability. The passivity framework is a promising approach to the stability analysis of neural networks because it can lead to general conclusions on stability using only input–output characteristics. Natural questions arise: can we obtain a passivity based state estimator for switched neural networks? Moreover, under what condition can this state estimator guarantee the exponential state estimation? This paper gives answers to these interesting questions. To the best of our knowledge, for the passivity based exponential state estimation of switched neural networks, there is no result in the literature so far, which still remains open and challenging.

In this paper, we propose a new exponential state estimator for switched Hopfield neural networks based on passivity theory. This state estimator is a new contribution to the topic of state estimation for neural networks. By constructing a suitable augmented Lyapunov–Krasovskii functional and employing Jensen's inequality, a new sufficient condition is derived such that the estimation error system is exponentially stable and passive from the control input to the output error. It is shown that an existence criterion for the proposed state estimator is represented in terms of LMIs, which can be solved efficiently by using recently developed convex optimization algorithms [23].

This paper is organized as follows. In Sect. 2, we formulate the problem. In Sect. 3, an LMI problem for the passivity based exponential state estimation of switched Hopfield neural networks is proposed. In Sect. 4, a numerical example is given, and finally, conclusions are presented in Sect. 5.

2 Problem formulation

Consider the following Hopfield neural network with time-delay:

$$\dot{x}(t) = Ax(t) + W\phi\big(x(t-\tau)\big) + J(t), \tag{1}$$

$$y(t) = Cx(t) + Dx(t-\tau), \tag{2}$$

where $x(t) = [x_1(t)\dots x_n(t)]^T \in \Re^n$ is the state vector, $y(t) = [y_1(t)\dots y_m(t)]^T \in \Re^m$ is the output vector, $\tau \ge 0$ is the time-delay, $A = \mathrm{diag}\{-a_1, \dots, -a_n\} \in \Re^{n\times n}$ ($a_k > 0, k = 1, \dots, n$) is the self-feedback matrix, $W \in \Re^{n\times n}$ is the delayed connection weight matrix, $\phi(x(t)) = [\phi_1(x(t))\dots\phi_n(x(t))]^T : \Re^n \to \Re^n$ is the nonlinear function vector satisfying the global Lipschitz condition with Lipschitz constant $L_\phi > 0$, $G \in \Re^{n\times m}$ and $C \in \Re^{m\times n}$ are known constant matrices, and $J(t) \in \Re^n$ is an external input vector.

Switched systems are a special class of hybrid systems consisting of a family of subsystems and a switching rule. In this paper, we consider the following model of switched Hopfield neural networks [12]:

$$\dot{x}(t) = A_\alpha x(t) + W_\alpha\phi\big(x(t-\tau)\big) + J_\alpha(t), \tag{3}$$

$$y(t) = C_\alpha x(t) + D_\alpha x(t-\tau), \tag{4}$$

where α is a switching signal which takes its values in the finite set $\mathcal{I} = \{1, 2, \dots, N\}$. This means that the matrices $(A_\alpha, W_\alpha, J_\alpha(t), C_\alpha, D_\alpha)$ are allowed to take values, at an arbitrary time, in the finite set

$\{(A_1, W_1, J_1(t), C_1, D_1), \ldots, (A_N, W_N, J_N(t), C_N, D_N)\}$. Throughout this paper, we assume that the switching rule α is not known a priori and its instantaneous value is available in real time. Define the indicator function $\xi(t) = (\xi_1(t), \xi_2(t), \ldots, \xi_N(t))^T$, where

$$\xi_i(t) = \begin{cases} 1, & \text{when the switched system is described by the } i\text{th mode } (A_i, W_i, J_i(t), C_i, D_i), \\ 0, & \text{otherwise,} \end{cases}$$

with $i = 1, \ldots, N$. Therefore, the model of the switched Hopfield neural networks (3) can also be written as

$$\dot{x}(t) = \sum_{i=1}^{N} \xi_i(t)\big[A_i x(t) + W_i \phi\big(x(t-\tau)\big) + J_i(t)\big], \tag{5}$$

$$y(t) = \sum_{i=1}^{N} \xi_i(t)\big[C_i x(t) + D_i x(t-\tau)\big], \tag{6}$$

where $\sum_{i=1}^{N} \xi_i(t) = 1$ is satisfied under any switching rules. For the switched Hopfield neural network (5)–(6), we propose the following switched state estimator:

$$\begin{aligned} \dot{\hat{x}}(t) = \sum_{i=1}^{N} \xi_i(t)\big[&A_i \hat{x}(t) + W_i \phi\big(\hat{x}(t-\tau)\big) + J_i(t) \\ &+ L\big(y(t) - \hat{y}(t)\big) - G_i I(t)\big], \end{aligned} \tag{7}$$

$$\hat{y}(t) = \sum_{i=1}^{N} \xi_i(t)\big[C_i \hat{x}(t) + D_i \hat{x}(t-\tau) - F_i I(t)\big], \tag{8}$$

where $\hat{x}(t) = [\hat{x}_1(t) \ldots \hat{x}_n(t)]^T \in \Re^n$ is the state vector of the state estimator, $\hat{y}(t) = [\hat{y}_1(t) \ldots \hat{y}_m(t)]^T \in \Re^m$ is the output vector of the state estimator, $I(t) \in \Re^m$ is the control input vector, $L \in \Re^{n \times m}$ is the gain matrix of the state estimator, and $G_i \in \Re^{n \times m}$ and $F_i \in \Re^{m \times m}$ are known constant matrices. Define the estimation error $e(t) = x(t) - \hat{x}(t)$ and the output error $\tilde{y}(t) = y(t) - \hat{y}(t)$. Then the estimation error system is represented as follows:

$$\begin{aligned} \dot{e}(t) = \sum_{i=1}^{N} \xi_i(t)\big\{&(A_i - LC_i)e(t) - LD_i e(t-\tau) \\ &+ W_i \phi\big(x(t-\tau)\big) - W_i \phi\big(\hat{x}(t-\tau)\big) \\ &+ (G_i - LF_i) I(t)\big\}, \end{aligned} \tag{9}$$

$$\tilde{y}(t) = \sum_{i=1}^{N} \xi_i(t)\big[C_i e(t) + D_i e(t-\tau) + F_i I(t)\big]. \tag{10}$$

The main purpose of this paper is to design a switched state estimator (7)–(8) for the estimation of the state vector $x(t)$ based on passivity theory. Specifically, find a proper switched state estimator such that the estimation error system (9)–(10) satisfies the following passivity inequality:

$$\int_0^t I^T(\sigma)\bar{y}(\sigma)\, d\sigma + \beta \geq \int_0^t \Phi\big(e(\sigma)\big)\, d\sigma, \quad \forall t \geq 0, \tag{11}$$

where β is a nonnegative constant, $\bar{y}(t) = \exp(\kappa t)\tilde{y}(t)$, κ is an enough small positive constant, and $\Phi(e(t))$ is a positive semi-definite storage function.

$$\left[\begin{matrix} [1,1]_i & -MD_i & U_1 \\ -D_i^T M^T & L_\phi^2 I - \exp(-\kappa\tau)R_1 - \frac{1}{\tau}S_1 & -U_1 \\ U_1 & -U_1 & \kappa U_1 - \frac{1}{\tau}Q_1 \\ \frac{\exp(\kappa\tau)-1}{\kappa} Q_2^T + R_2^T + (PG_i - MF_i)^T - \frac{1}{2}C_i & -\frac{1}{2}D_i & U_2^T \\ 0 & -\exp(-\kappa\tau)R_2^T & -U_2^T \\ U_2^T & -U_2^T & \kappa U_2^T - \frac{1}{\tau}Q_2^T \\ W_i^T P & 0 & 0 \end{matrix}\right.$$

$$\begin{bmatrix} \frac{\exp(\kappa\tau)-1}{\kappa}Q_2+R_2+PG_i-MF_i-\frac{1}{2}C_i^T & 0 & U_2 & PW_i \\ -\frac{1}{2}D_i^T & -\exp(-\kappa\tau)R_2 & -U_2 & 0 \\ U_2 & -U_2 & \kappa U_2-\frac{1}{\tau}Q_2 & 0 \\ \frac{\exp(\kappa\tau)-1}{\kappa}Q_3+R_3-F_i & 0 & U_3 & 0 \\ 0 & -\exp(-\kappa\tau)R_3 & -U_3 & 0 \\ U_3 & -U_3 & \kappa U_3-\frac{1}{\tau}Q_3 & 0 \\ 0 & 0 & 0 & -I \end{bmatrix} < 0, \tag{12}$$

$$\begin{bmatrix} Q_1 & Q_2 \\ Q_2^T & Q_3 \end{bmatrix} > 0, \qquad \begin{bmatrix} R_1 & R_2 \\ R_2^T & R_3 \end{bmatrix} > 0,$$
$$\begin{bmatrix} U_1 & U_2 \\ U_2^T & U_3 \end{bmatrix} > 0. \tag{13}$$

3 Switched exponential state estimation based on passivity theory

This section is devoted to developing a passivity based approach to dealing with the exponential state estimation problem for switched Hopfield neural networks. A delay-dependent condition is derived such that the resulting estimation error system (9) is passive and exponentially stable. The design of an appropriate state estimator can be achieved by solving a corresponding set of LMIs.

Theorem 1 *Assume that there exist common matrices $P=P^T>0$, $Q_1=Q_1^T>0$, Q_2, $Q_3=Q_3^T>0$, $R_1=R_1^T>0$, R_2, $R_3=R_3^T>0$, $S_1=S_1^T>0$, $S_2=S_2^T>0$, $U_1=U_1^T>0$, U_2, $U_3=U_3^T>0$, and M such that LMIs (12) and (13) are satisfied, for $i=1,\dots,N$, where*

$$[1,1]_i=(PA_i-MC_i)^T+PA_i-MC_i+\kappa P + \frac{\exp(\kappa\tau)-1}{\kappa}Q_1+S_2+R_1-\frac{1}{\tau}S_1.$$

Then the estimation error system (9)–(10) is passive from the control input $I(t)$ to the output error $\bar{y}(t)$ and the gain matrix of the switched state estimator (7)–(8) is given by $L=P^{-1}M$.

Proof We consider the following Lyapunov–Krasovskii functional:

$$\begin{aligned} V(t) &= \exp(\kappa t)e^T(t)Pe(t) \\ &+\int_{-\tau}^{0}\exp(-\kappa\beta)\int_{t+\beta}^{t}\exp(\kappa\alpha)\begin{bmatrix} e(\alpha) \\ I(\alpha)\end{bmatrix}^T \times \begin{bmatrix} Q_1 & Q_2 \\ Q_2^T & Q_3 \end{bmatrix}\begin{bmatrix} e(\alpha) \\ I(\alpha)\end{bmatrix} d\alpha\, d\beta \\ &+\int_{-\tau}^{0}\exp\big(\kappa(t+\sigma)\big)\begin{bmatrix} e(t+\sigma) \\ I(t+\sigma)\end{bmatrix}^T \times \begin{bmatrix} R_1 & R_2 \\ R_2^T & R_3 \end{bmatrix}\begin{bmatrix} e(t+\sigma) \\ I(t+\sigma)\end{bmatrix} d\sigma \\ &+\exp(\kappa t)\begin{bmatrix} \int_{-\tau}^{0} e(t+\sigma)\,d\sigma \\ \int_{-\tau}^{0} I(t+\sigma)\,d\sigma \end{bmatrix}^T \begin{bmatrix} U_1 & U_2 \\ U_2^T & U_3 \end{bmatrix} \times \begin{bmatrix} \int_{-\tau}^{0} e(t+\sigma)\,d\sigma \\ \int_{-\tau}^{0} I(t+\sigma)\,d\sigma \end{bmatrix} \\ &+\int_{-\tau}^{0}\exp(-\kappa\beta) \times \int_{t+\beta}^{t}\exp(\kappa\alpha)\dot{e}^T(\alpha)S_1\dot{e}(\alpha)\,d\alpha\,d\beta. \end{aligned} \tag{14}$$

Calculating the time derivative of $V(t)$ along the trajectory of the estimation error system (9)–(10), we have

$$\begin{aligned} \dot{V}(t) &= \exp(\kappa t)\dot{e}(t)^TPe(t)+\exp(\kappa t)e^T(t)P\dot{e}(t) \\ &+\kappa\exp(\kappa t)e^T(t)Pe(t) \\ &+\frac{\exp(\kappa\tau)-1}{\kappa}\exp(\kappa t) \\ &\times\begin{bmatrix} e(t) \\ I(t)\end{bmatrix}^T\begin{bmatrix} Q_1 & Q_2 \\ Q_2^T & Q_3 \end{bmatrix}\begin{bmatrix} e(t) \\ I(t)\end{bmatrix} \\ &-\exp(\kappa t)\int_{t-\tau}^{t}\begin{bmatrix} e(\sigma) \\ I(\sigma)\end{bmatrix}^T\begin{bmatrix} Q_1 & Q_2 \\ Q_2^T & Q_3 \end{bmatrix} \\ &\times\begin{bmatrix} e(\sigma) \\ I(\sigma)\end{bmatrix} d\sigma \end{aligned}$$

$$
\begin{aligned}
&+\exp(\kappa t)\begin{bmatrix} e(t) \\ I(t) \end{bmatrix}^T \begin{bmatrix} R_1 & R_2 \\ R_2^T & R_3 \end{bmatrix}\begin{bmatrix} e(t) \\ I(t) \end{bmatrix} \\
&-\exp\big(\kappa(t-\tau)\big)\begin{bmatrix} e(t-\tau) \\ I(t-\tau) \end{bmatrix}^T \begin{bmatrix} R_1 & R_2 \\ R_2^T & R_3 \end{bmatrix} \\
&\times\begin{bmatrix} e(t-\tau) \\ I(t-\tau) \end{bmatrix} + \kappa\exp(\kappa t)\begin{bmatrix} \int_{t-\tau}^t e(\sigma)\,d\sigma \\ \int_{t-\tau}^t I(\sigma)\,d\sigma \end{bmatrix}^T \\
&\times\begin{bmatrix} U_1 & U_2 \\ U_2^T & U_3 \end{bmatrix}\begin{bmatrix} \int_{t-\tau}^t e(\sigma)\,d\sigma \\ \int_{t-\tau}^t I(\sigma)\,d\sigma \end{bmatrix} \\
&+\exp(\kappa t)\begin{bmatrix} e(t)-e(t-\tau) \\ I(t)-I(t-\tau) \end{bmatrix}^T \begin{bmatrix} U_1 & U_2 \\ U_2^T & U_3 \end{bmatrix} \\
&\times\begin{bmatrix} \int_{t-\tau}^t e(\sigma)\,d\sigma \\ \int_{t-\tau}^t I(\sigma)\,d\sigma \end{bmatrix} \\
&+\exp(\kappa t)\begin{bmatrix} \int_{t-\tau}^t e(\sigma)\,d\sigma \\ \int_{t-\tau}^t I(\sigma)\,d\sigma \end{bmatrix}^T \\
&\times\begin{bmatrix} U_1 & U_2 \\ U_2^T & U_3 \end{bmatrix}\begin{bmatrix} e(t)-e(t-\tau) \\ I(t)-I(t-\tau) \end{bmatrix} \\
&+\frac{\exp(\kappa\tau)-1}{\kappa}\exp(\kappa t)\dot{e}^T(t)S_1\dot{e}(t) \\
&-\exp(\kappa t)\int_{t-\tau}^t \dot{e}^T(\sigma)S_1\dot{e}(\sigma)\,d\sigma \\
=&\sum_{i=1}^N \xi_i(t)\big\{\exp(\kappa t)e^T(t)\big[(A_i-LC_i)^T P \\
&+P(A_i-LC_i)+\kappa P\big]e(t) \\
&-\exp(\kappa t)e^T(t)PLD_i e(t-\tau) \\
&-\exp(\kappa t)e^T(t-\tau)D_i^T L^T P e(t) \\
&+\exp(\kappa t)e^T(t)PW_i\big(\phi\big(x(t-\tau)\big) \\
&-\phi\big(\hat{x}(t-\tau)\big)\big) \\
&+\exp(\kappa t)\big(\phi\big(x(t-\tau)\big) \\
&-\phi\big(\hat{x}(t-\tau)\big)\big)^T W_i^T P e(t) \\
&+\exp(\kappa t)e(t)^T P(G_i-LF_i)I(t) \\
&+\exp(\kappa t)I^T(t)(G_i-LF_i)^T Pe(t)\big\} \\
&+\frac{\exp(\kappa\tau)-1}{\kappa}\exp(\kappa t)\begin{bmatrix} e(t) \\ I(t) \end{bmatrix}^T \\
&\times\begin{bmatrix} Q_1 & Q_2 \\ Q_2^T & Q_3 \end{bmatrix}\begin{bmatrix} e(t) \\ I(t) \end{bmatrix} \\
&-\exp(\kappa t)\int_{t-\tau}^t \begin{bmatrix} e(\sigma) \\ I(\sigma) \end{bmatrix}^T \\
&\times\begin{bmatrix} Q_1 & Q_2 \\ Q_2^T & Q_3 \end{bmatrix}\begin{bmatrix} e(\sigma) \\ I(\sigma) \end{bmatrix} d\sigma \\
&+\exp(\kappa t)\begin{bmatrix} e(t) \\ I(t) \end{bmatrix}^T \begin{bmatrix} R_1 & R_2 \\ R_2^T & R_3 \end{bmatrix}\begin{bmatrix} e(t) \\ I(t) \end{bmatrix} \\
&-\exp\big(\kappa(t-\tau)\big)\begin{bmatrix} e(t-\tau) \\ I(t-\tau) \end{bmatrix}^T \begin{bmatrix} R_1 & R_2 \\ R_2^T & R_3 \end{bmatrix} \\
&\times\begin{bmatrix} e(t-\tau) \\ I(t-\tau) \end{bmatrix} + \kappa\exp(\kappa t)\begin{bmatrix} \int_{t-\tau}^t e(\sigma)\,d\sigma \\ \int_{t-\tau}^t I(\sigma)\,d\sigma \end{bmatrix}^T \\
&\times\begin{bmatrix} U_1 & U_2 \\ U_2^T & U_3 \end{bmatrix}\begin{bmatrix} \int_{t-\tau}^t e(\sigma)\,d\sigma \\ \int_{t-\tau}^t I(\sigma)\,d\sigma \end{bmatrix} \\
&+\exp(\kappa t)\begin{bmatrix} e(t)-e(t-\tau) \\ I(t)-I(t-\tau) \end{bmatrix}^T \begin{bmatrix} U_1 & U_2 \\ U_2^T & U_3 \end{bmatrix} \\
&\times\begin{bmatrix} \int_{t-\tau}^t e(\sigma)\,d\sigma \\ \int_{t-\tau}^t I(\sigma)\,d\sigma \end{bmatrix} \\
&+\exp(\kappa t)\begin{bmatrix} \int_{t-\tau}^t e(\sigma)\,d\sigma \\ \int_{t-\tau}^t I(\sigma)\,d\sigma \end{bmatrix}^T \\
&\times\begin{bmatrix} U_1 & U_2 \\ U_2^T & U_3 \end{bmatrix}\begin{bmatrix} e(t)-e(t-\tau) \\ I(t)-I(t-\tau) \end{bmatrix} \\
&+\frac{\exp(\kappa\tau)-1}{\kappa}\exp(\kappa t)\dot{e}^T(t)S_1\dot{e}(t) \\
&-\exp(\kappa t)\int_{t-\tau}^t \dot{e}^T(\sigma)S_1\dot{e}(\sigma)\,d\sigma. \qquad (15)
\end{aligned}
$$

Adding and subtracting $\exp(\kappa t)I^T(t)[C_i e(t) + D_i e(t-\tau) + F_i I(t)]$, we obtain

$$
\begin{aligned}
\dot{V}(t) = &\sum_{i=1}^N \xi_i(t)\Big\{\exp(\kappa t)e^T(t)\big[(A_i-LC_i)^T P \\
&+P(A_i-LC_i)+\kappa P\big]e(t) \\
&-\exp(\kappa t)e^T(t)PLD_i e(t-\tau) \\
&-\exp(\kappa t)e^T(t-\tau)D_i^T L^T P e(t) \\
&+\exp(\kappa t)e^T(t)PW_i \\
&\times\big(\phi\big(x(t-\tau)\big)-\phi\big(\hat{x}(t-\tau)\big)\big) \\
&+\exp(\kappa t)\big(\phi\big(x(t-\tau)\big) \\
&-\phi\big(\hat{x}(t-\tau)\big)\big)^T W_i^T P e(t) \\
&+\exp(\kappa t)e(t)^T
\end{aligned}
$$

$$
\begin{aligned}
&\times\left[P(G_i-LF_i)-\frac{1}{2}C_i^T\right]I(t)\\
&+\exp(\kappa t)I^T(t)\\
&\times\left[(G_i-LF_i)^TP-\frac{1}{2}C_i\right]e(t)\\
&-\frac{1}{2}\exp(\kappa t)I^T(t)\\
&\times D_ie(t-\tau)-\frac{1}{2}\exp(\kappa t)e^T(t-\tau)D_i^TI(t)\\
&+\exp(\kappa t)I^T(t)\left[\frac{\exp(\kappa\tau)-1}{\kappa}Q_3\right.\\
&\left.+R_3-F_i\right]I(t)\\
&+\exp(\kappa t)I^T(t)\big[C_ie(t)+D_ie(t-\tau)\\
&\left.+F_iI(t)\big]\right\}\\
&+\frac{\exp(\kappa\tau)-1}{\kappa}\exp(\kappa t)\begin{bmatrix}e(t)\\ I(t)\end{bmatrix}^T\\
&\times\begin{bmatrix}Q_1 & Q_2\\ Q_2^T & 0\end{bmatrix}\begin{bmatrix}e(t)\\ I(t)\end{bmatrix}\\
&-\exp(\kappa t)\int_{t-\tau}^{t}\begin{bmatrix}e(\sigma)\\ I(\sigma)\end{bmatrix}^T\begin{bmatrix}Q_1 & Q_2\\ Q_2^T & Q_3\end{bmatrix}\\
&\times\begin{bmatrix}e(\sigma)\\ I(\sigma)\end{bmatrix}d\sigma\\
&+\exp(\kappa t)\begin{bmatrix}e(t)\\ I(t)\end{bmatrix}^T\begin{bmatrix}R_1 & R_2\\ R_2^T & 0\end{bmatrix}\\
&\times\begin{bmatrix}e(t)\\ I(t)\end{bmatrix}-\exp\big(\kappa(t-\tau)\big)\begin{bmatrix}e(t-\tau)\\ I(t-\tau)\end{bmatrix}^T\\
&\times\begin{bmatrix}R_1 & R_2\\ R_2^T & R_3\end{bmatrix}\begin{bmatrix}e(t-\tau)\\ I(t-\tau)\end{bmatrix}\\
&+\kappa\exp(\kappa t)\begin{bmatrix}\int_{t-\tau}^{t}e(\sigma)\,d\sigma\\ \int_{t-\tau}^{t}I(\sigma)\,d\sigma\end{bmatrix}^T\begin{bmatrix}U_1 & U_2\\ U_2^T & U_3\end{bmatrix}\\
&\times\begin{bmatrix}\int_{t-\tau}^{t}e(\sigma)\,d\sigma\\ \int_{t-\tau}^{t}I(\sigma)\,d\sigma\end{bmatrix}\\
&+\exp(\kappa t)\begin{bmatrix}e(t)-e(t-\tau)\\ I(t)-I(t-\tau)\end{bmatrix}^T\begin{bmatrix}U_1 & U_2\\ U_2^T & U_3\end{bmatrix}\\
&\times\begin{bmatrix}\int_{t-\tau}^{t}e(\sigma)\,d\sigma\\ \int_{t-\tau}^{t}I(\sigma)\,d\sigma\end{bmatrix}\\
&+\exp(\kappa t)\begin{bmatrix}\int_{t-\tau}^{t}e(\sigma)\,d\sigma\\ \int_{t-\tau}^{t}I(\sigma)\,d\sigma\end{bmatrix}^T\begin{bmatrix}U_1 & U_2\\ U_2^T & U_3\end{bmatrix}\\
&\times\begin{bmatrix}e(t)-e(t-\tau)\\ I(t)-I(t-\tau)\end{bmatrix}\\
&+\frac{\exp(\kappa\tau)-1}{\kappa}\exp(\kappa t)\dot{e}^T(t)S_1\dot{e}(t)\\
&-\exp(\kappa t)\int_{t-\tau}^{t}\dot{e}^T(\sigma)S_1\dot{e}(\sigma)\,d\sigma. \qquad (16)
\end{aligned}
$$

If we use the inequality $X^TY+Y^TX\le X^T\Lambda X+Y^T\Lambda^{-1}Y$, which is valid for any matrices $X\in R^{n\times m}$, $Y\in R^{n\times m}$, $\Lambda=\Lambda^T>0$, $\Lambda\in R^{n\times n}$, we have

$$
\begin{aligned}
&e^T(t)PW_i\big(\phi\big(x(t-\tau)\big)-\phi\big(\hat{x}(t-\tau)\big)\big)\\
&\quad+\big(\phi\big(x(t-\tau)\big)-\phi\big(\hat{x}(t-\tau)\big)\big)^TW_i^TPe(t)\\
&\le\big(\phi\big(x(t-\tau)\big)-\phi\big(\hat{x}(t-\tau)\big)\big)^T\\
&\quad\times\big(\phi\big(x(t-\tau)\big)-\phi\big(\hat{x}(t-\tau)\big)\big)\\
&\quad+e^T(t)PW_iW_i^TPe(t)\\
&\le L_\phi^2\big(x(t-\tau)-\hat{x}(t-\tau)\big)^T\big(x(t-\tau)-\hat{x}(t-\tau)\big)\\
&\quad+e^T(t)PW_iW_i^TPe(t)\\
&=L_\phi^2e^T(t-\tau)e(t-\tau)\\
&\quad+e^T(t)PW_iW_i^TPe(t). \qquad (17)
\end{aligned}
$$

Using the Jesen's inequality [24], we have

$$
\begin{aligned}
&-\exp(\kappa t)\int_{t-\tau}^{t}\begin{bmatrix}e(\sigma)\\ I(\sigma)\end{bmatrix}^T\begin{bmatrix}Q_1 & Q_2\\ Q_2^T & Q_3\end{bmatrix}\begin{bmatrix}e(\sigma)\\ I(\sigma)\end{bmatrix}d\sigma\\
&\quad\le-\frac{\exp(\kappa t)}{\tau}\left\{\int_{t-\tau}^{t}\begin{bmatrix}e(\sigma)\\ I(\sigma)\end{bmatrix}d\sigma\right\}^T\begin{bmatrix}Q_1 & Q_2\\ Q_2^T & Q_3\end{bmatrix}\\
&\qquad\times\left\{\int_{t-\tau}^{t}\begin{bmatrix}e(\sigma)\\ I(\sigma)\end{bmatrix}d\sigma\right\}\\
&\quad=-\frac{\exp(\kappa t)}{\tau}\begin{bmatrix}\int_{t-\tau}^{t}e(\sigma)\,d\sigma\\ \int_{t-\tau}^{t}I(\sigma)\,d\sigma\end{bmatrix}^T\begin{bmatrix}Q_1 & Q_2\\ Q_2^T & Q_3\end{bmatrix}\\
&\qquad\times\begin{bmatrix}\int_{t-\tau}^{t}e(\sigma)\,d\sigma\\ \int_{t-\tau}^{t}I(\sigma)\,d\sigma\end{bmatrix} \qquad (18)
\end{aligned}
$$

and

$$
\begin{aligned}
&-\exp(\kappa t)\int_{t-\tau}^{t}\dot{e}^T(\sigma)S_1\dot{e}(\sigma)\,d\sigma\\
&\quad\le-\frac{\exp(\kappa t)}{\tau}\left[\int_{t-\tau}^{t}\dot{e}(\sigma)\,d\sigma\right]^TS_1\left[\int_{t-\tau}^{t}\dot{e}^T(\sigma)\,d\sigma\right]
\end{aligned}
$$

$$
= -\frac{\exp(\kappa t)}{\tau}\big[e(t)-e(t-\tau)\big]^T \times S_1\big[e(t)-e(t-\tau)\big]. \tag{19}
$$

Finally, using (17), (18), and (19), the time derivative of $V(t)$ can be obtained as

$$
\begin{aligned}
\dot{V}(t) \le \sum_{i=1}^{N} \xi_i(t)\exp(\kappa t)\Big\{ & e^T(t)\Big[(A_i - LC_i)^T P + P(A_i - LC_i) + \kappa P + P W_i W_i^T P - \frac{1}{\tau}S_1\Big]e(t) \\
& - e^T(t)\Big[PLD_i - \frac{1}{\tau}S_1\Big]e(t-\tau) - e^T(t-\tau)\Big[D_i^T L^T P - \frac{1}{\tau}S_1\Big]e(t) \\
& - \frac{1}{2}I^T(t)D_i e(t-\tau) - \frac{1}{2}e^T(t-\tau)D_i^T I(t) + I^T(t)\big[C_i e(t) + D_i e(t-\tau) + F_i I(t)\big] \\
& + \begin{bmatrix} e(t) \\ I(t) \end{bmatrix}^T \times \begin{bmatrix} \frac{\exp(\kappa\tau)-1}{\kappa}Q_1 + R_1 & \frac{\exp(\kappa\tau)-1}{\kappa}Q_2 + R_2 + P(G_i - LF_i) - \frac{1}{2}C_i^T \\ \frac{\exp(\kappa\tau)-1}{\kappa}Q_2^T + R_2^T + (G_i - LF_i)^T P - \frac{1}{2}C_i & \frac{\exp(\kappa\tau)-1}{\kappa}Q_3 + R_3 - F_i \end{bmatrix} \times \begin{bmatrix} e(t) \\ I(t) \end{bmatrix} \\
& - \begin{bmatrix} e(t-\tau) \\ I(t-\tau) \end{bmatrix}^T \times \begin{bmatrix} \exp(-\kappa\tau)R_1 - L_\phi^2 I + \frac{1}{\tau}S_1 & \exp(-\kappa\tau)R_2 \\ \exp(-\kappa\tau)R_2^T & \exp(-\kappa\tau)R_3 \end{bmatrix} \times \begin{bmatrix} e(t-\tau) \\ I(t-\tau) \end{bmatrix} \\
& + \begin{bmatrix} \int_{t-\tau}^{t} e(\sigma)\,d\sigma \\ \int_{t-\tau}^{t} I(\sigma)\,d\sigma \end{bmatrix}^T \times \begin{bmatrix} \kappa U_1 - \frac{1}{\tau}Q_1 & \kappa U_2 - \frac{1}{\tau}Q_2 \\ \kappa U_2^T - \frac{1}{\tau}Q_2^T & \kappa U_3 - \frac{1}{\tau}Q_3 \end{bmatrix} \times \begin{bmatrix} \int_{t-\tau}^{t} e(\sigma)\,d\sigma \\ \int_{t-\tau}^{t} I(\sigma)\,d\sigma \end{bmatrix} \\
& + \begin{bmatrix} e(t)-e(t-\tau) \\ I(t)-I(t-\tau) \end{bmatrix}^T \times \begin{bmatrix} U_1 & U_2 \\ U_2^T & U_3 \end{bmatrix}\begin{bmatrix} \int_{t-\tau}^{t} e(\sigma)\,d\sigma \\ \int_{t-\tau}^{t} I(\sigma)\,d\sigma \end{bmatrix} \\
& + \begin{bmatrix} \int_{t-\tau}^{t} e(\sigma)\,d\sigma \\ \int_{t-\tau}^{t} I(\sigma)\,d\sigma \end{bmatrix}^T \begin{bmatrix} U_1 & U_2 \\ U_2^T & U_3 \end{bmatrix} \times \begin{bmatrix} e(t)-e(t-\tau) \\ I(t)-I(t-\tau) \end{bmatrix} \\
& + I^T(t)\big[C_i e(t) + D_i e(t-\tau) + F_i I(t)\big] + \frac{\exp(\kappa\tau)-1}{\kappa}\dot{e}^T(t)S_1\dot{e}(t)\Big\}
\end{aligned}
$$

$$
\begin{aligned}
= \sum_{i=1}^{N}\xi_i(t)\exp(\kappa t) & \left\{ \begin{bmatrix} e(t) \\ e(t-\tau) \\ \int_{t-\tau}^{t} e(\sigma)\,d\sigma \\ I(t) \\ I(t-\tau) \\ \int_{t-\tau}^{t} I(\sigma)\,d\sigma \end{bmatrix}^T \right. \\
& \times \begin{bmatrix} (1,1)_i & \frac{1}{\tau}S_1 - PLD_i & U_1 & (1,4)_i & 0 & U_2 \\ \frac{1}{\tau}S_1 - D_i^T L^T P & (2,2)_i & -U_1 & -\frac{1}{2}D_i^T & -\exp(-\kappa\tau)R_2 & -U_2 \\ U_1 & -U_1 & \kappa U_1 - \frac{1}{\tau}Q_1 & U_2 & -U_2 & \kappa U_2 - \frac{1}{\tau}Q_2 \\ (1,4)_i^T & -\frac{1}{2}D_i & U_2^T & (4,4)_i & 0 & U_3 \\ 0 & -\exp(-\kappa\tau)R_2^T & -U_2^T & 0 & -\exp(-\kappa\tau)R_3 & -U_3 \\ U_2^T & -U_2^T & \kappa U_2^T - \frac{1}{\tau}Q_2^T & U_3 & -U_3 & \kappa U_3 - \frac{1}{\tau}Q_3 \end{bmatrix} \\
& \left. \times \begin{bmatrix} e(t) \\ e(t-\tau) \\ \int_{t-\tau}^{t} e(\sigma)\,d\sigma \\ I(t) \\ I(t-\tau) \\ \int_{t-\tau}^{t} I(\sigma)\,d\sigma \end{bmatrix} \right\} + \sum_{i=1}^{N}\xi_i(t)\exp(\kappa t) \\
& \times \big\{-e^T(t)S_2 e(t) + I^T(t)\big[C_i e(t) + D_i e(t-\tau) + F_i I(t)\big]\big\},
\end{aligned} \tag{20}
$$

where

$$
\begin{aligned}
(1,1)_i &= (A_i - LC_i)^T P + P(A_i - LC_i) + \kappa P + P W_i W_i^T P + \frac{\exp(\kappa\tau)-1}{\kappa}Q_1 + S_2 + R_1 - \frac{1}{\tau}S_1, \\
(2,2)_i &= L_\phi^2 I - \exp(-\kappa\tau)R_1 - \frac{1}{\tau}S_1, \\
(4,4)_i &= \frac{\exp(\kappa\tau)-1}{\kappa}Q_3 + R_3 - F_i,
\end{aligned}
$$

$$(1,4)_i = \frac{\exp(\kappa\tau)-1}{\kappa} Q_2 + R_2 + P(G_i - LF_i) - \frac{1}{2} C_i^T.$$

If the following matrix inequality is satisfied

$$\begin{bmatrix} (1,1)_i & \frac{1}{\tau}S_1 - PLD_i & U_1 & (1,4)_i & 0 & U_2 \\ \frac{1}{\tau}S_1 - D_i^T L^T P & (2,2)_i & -U_1 & -\frac{1}{2}D_i^T & -\exp(-\kappa\tau)R_2 & -U_2 \\ U_1 & -U_1 & \kappa U_1 - \frac{1}{\tau}Q_1 & U_2 & -U_2 & \kappa U_2 - \frac{1}{\tau}Q_2 \\ (1,4)_i^T & -\frac{1}{2}D_i & U_2^T & (4,4)_i & 0 & U_3 \\ 0 & -\exp(-\kappa\tau)R_2^T & -U_2^T & 0 & -\exp(-\kappa\tau)R_3 & -U_3 \\ U_2^T & -U_2^T & \kappa U_2^T - \frac{1}{\tau}Q_2^T & U_3 & -U_3 & \kappa U_3 - \frac{1}{\tau}Q_3 \end{bmatrix} < 0, \tag{21}$$

for $i = 1, \ldots, N$, we have

$$\begin{aligned} \dot{V}(t) &< \sum_{i=1}^{N} \xi_i(t) \exp(\kappa t)\{-e^T(t)S_2 e(t) \\ &\quad + I^T(t)[C_i e(t) + D_i e(t-\tau) + F_i I(t)]\} \\ &= -\exp(\kappa t)e^T(t)S_2 e(t) + I^T(t)\bar{y}(t). \end{aligned} \tag{22}$$

Integrating both sides of (22) from 0 to t gives

$$V(t) - V(0) < -\int_0^t \exp(\kappa\sigma)e^T(\sigma)S_2 e(\sigma)\,d\sigma + \int_0^t I^T(\sigma)\bar{y}(\sigma)\,d\sigma. \tag{23}$$

Let $\beta = V(0)$. Since $V(t) \geq 0$,

$$\begin{aligned} &\int_0^t I^T(\sigma)\bar{y}(\sigma)\,d\sigma + \beta \\ &\quad > \int_0^t \exp(\kappa\sigma)e^T(\sigma)S_2 e(\sigma)\,d\sigma + V(t) \\ &\quad \geq \int_0^t \exp(\kappa\sigma)e^T(\sigma)S_2 e(\sigma)\,d\sigma. \end{aligned} \tag{24}$$

The relation (24) satisfies the passivity inequality (11). Therefore, the estimation error system (9)–(10) is rendered to be passive from the control input $I(t)$ to the output error $\bar{y}(t)$ under the switched state estimator (7)–(8). From the Schur complement, the matrix inequality (21) is equivalent to

$$\begin{bmatrix} \{1,1\}_i & \frac{1}{\tau}S_1 - PLD_i & U_1 & (1,4)_i & 0 & U_2 & PW_i \\ \frac{1}{\tau}S_1 - D_i^T L^T P & (2,2)_i & -U_1 & -\frac{1}{2}D_i^T & -\exp(-\kappa\tau)R_2 & -U_2 & 0 \\ U_1 & -U_1 & \kappa U_1 - \frac{1}{\tau}Q_1 & U_2 & -U_2 & \kappa U_2 - \frac{1}{\tau}Q_2 & 0 \\ (1,4)_i^T & -\frac{1}{2}D_i & U_2^T & (4,4)_i & 0 & U_3 & 0 \\ 0 & -\exp(-\kappa\tau)R_2^T & -U_2^T & 0 & -\exp(-\kappa\tau)R_3 & -U_3 & 0 \\ U_2^T & -U_2^T & \kappa U_2^T - \frac{1}{\tau}Q_2^T & U_3 & -U_3 & \kappa U_3 - \frac{1}{\tau}Q_3 & 0 \\ W_i^T P & 0 & 0 & 0 & 0 & 0 & -I \end{bmatrix} < 0, \tag{25}$$

where

$$\{1,1\}_i = (A_i - LC_i)^T P + P(A_i - LC_i) + \kappa P + \frac{\exp(\kappa\tau)-1}{\kappa} Q_1 + S_2 + R_1 - \frac{1}{\tau} S_1.$$

If we let $M = PL$, (25) is equivalently changed into the LMI (12). Then the gain matrix of the switched state estimator is given by $L = P^{-1}M$. This completes the proof. □

Corollary 1 (Zero-input error response) *If the control input $I(t)$ is zero, the estimation error system (9)–(10) is exponentially stable.*

Proof When $I(t) = 0$, we obtain

$$\dot{V}(t) < -\exp(\kappa t)e^T(t)Se(t) \tag{26}$$

from (22). That is, $\dot{V}(t) < 0$ for all $e(t) \neq 0$. Thus, it implies that $V(t) < V(0)$ for any $t \geq 0$. In addition, from (14), one has

$$
\begin{aligned}
V(t) &< V(0) \\
&= e^T(0)Pe(0) + \int_{-\tau}^{0} \exp(-\kappa\beta) \int_{\beta}^{0} \exp(\kappa\alpha) \\
&\quad \times e^T(\alpha)Q_1 e(\alpha)\, d\alpha\, d\beta \\
&\quad + \int_{-\tau}^{0} \exp(\kappa\sigma) e^T(\sigma) R_1 e(\sigma)\, d\sigma \\
&\quad + \left[\int_{-\tau}^{0} e(\sigma)\, d\sigma\right]^T U_1 \left[\int_{-\tau}^{0} e(\sigma)\, d\sigma\right].
\end{aligned} \tag{27}
$$

Also, we have

$$
V(t) \geq \lambda_{\min}(P) \exp(\kappa t) \|e(t)\|^2, \tag{28}
$$

where $\lambda_{\min}(P)$ is the minimum eigenvalue of the matrix P. It follows immediately from (27) and (33) that

$$
\begin{aligned}
\|e(t)\| &< \frac{1}{\sqrt{\lambda_{\min}(P)\exp(\kappa t)}} \bigg\{ e^T(0)Pe(0) \\
&\quad + \int_{-\tau}^{0} \exp(-\kappa\beta) \\
&\quad \times \int_{\beta}^{0} \exp(\kappa\alpha) e^T(\alpha) Q_1 e(\alpha)\, d\alpha\, d\beta \\
&\quad + \int_{-\tau}^{0} \exp(\kappa\sigma) e^T(\sigma) R_1 e(\sigma)\, d\sigma \\
&\quad + \left[\int_{-\tau}^{0} e(\sigma)\, d\sigma\right]^T U_1 \\
&\quad \times \left[\int_{-\tau}^{0} e(\sigma)\, d\sigma\right] \bigg\}^{\frac{1}{2}} \\
&= \frac{1}{\sqrt{\lambda_{\min}(P)}} \bigg\{ e^T(0)Pe(0) + \int_{-\tau}^{0} \exp(-\kappa\beta) \\
&\quad \times \int_{\beta}^{0} \exp(\kappa\alpha) e^T(\alpha) Q_1 e(\alpha)\, d\alpha\, d\beta \\
&\quad + \int_{-\tau}^{0} \exp(\kappa\sigma) e^T(\sigma) R_1 e(\sigma)\, d\sigma \\
&\quad + \left[\int_{-\tau}^{0} e(\sigma)\, d\sigma\right]^T \\
&\quad \times U_1 \left[\int_{-\tau}^{0} e(\sigma)\, d\sigma\right] \bigg\}^{\frac{1}{2}} \exp\left(-\frac{\kappa}{2} t\right).
\end{aligned} \tag{29}
$$

Let

$$
\begin{aligned}
M_{c1} &= \frac{1}{\sqrt{\lambda_{\min}(P)}} \bigg\{ e^T(0)Pe(0) + \int_{-\tau}^{0} \exp(-\kappa\beta) \\
&\quad \times \int_{\beta}^{0} \exp(\kappa\alpha) e^T(\alpha) Q_1 e(\alpha)\, d\alpha\, d\beta \\
&\quad + \int_{-\tau}^{0} \exp(\kappa\sigma) e^T(\sigma) R_1 e(\sigma)\, d\sigma \\
&\quad + \left[\int_{-\tau}^{0} e(\sigma)\, d\sigma\right]^T U_1 \left[\int_{-\tau}^{0} e(\sigma)\, d\sigma\right] \bigg\}^{\frac{1}{2}} > 0, \\
N_{c1} &= \frac{\kappa}{2} > 0.
\end{aligned}
$$

Then (29) is represented by

$$
\|e(t)\| < M_{c1} \exp(-N_{c1} t). \tag{30}
$$

This implies the exponential stability of the error system (9)–(10). This completes the proof. □

According to Theorem 3.2 in [22], once the estimation error system (9)–(10) has been rendered passive, the control input $I(t) = -\Lambda(\bar{y}(t))$ satisfying $\Lambda(0) = 0$ and $\bar{y}^T(t)\Lambda(\bar{y}(t)) > 0$ for each nonzero $\bar{y}(t)$ stabilizes the estimation error system (9)–(10). For example, a pure gain output feedback $I(t) = -\nu\bar{y}(t)$ $(\nu > 0)$ can stabilize the estimation error system (9)–(10).

Corollary 2 (Nonzero-input error response) *If the control input $I(t)$ is selected as*

$$
I(t) = -\nu\bar{y}(t) = -\nu \exp(\kappa t)\big(y(t) - \hat{y}(t)\big), \quad \nu > 0, \tag{31}
$$

the estimation error system (9)–(10) is exponentially stable.

Proof For $I(t) = -\nu\bar{y}(t)$, the time derivative of $V(t)$ satisfies

$$
\dot{V}(t) < -e^T(t)Se(t) - \nu\bar{y}^T(t)\bar{y}(t) \tag{32}
$$

from (22). That is, $\dot{V}(t) < 0$ for all $e(t) \neq 0$. Thus, it implies that $V(t) < V(0)$ for any $t \geq 0$. From (14), we have

$$
\begin{aligned}
V(t) &< V(0) \\
&= e^T(0)Pe(0) + \int_{-\tau}^{0} \exp(-\kappa\beta) \int_{\beta}^{0} \exp(\kappa\alpha) \\
&\quad \times \begin{bmatrix} e(\alpha) \\ -\nu\exp(\kappa\alpha)(y(\alpha)-\hat{y}(\alpha)) \end{bmatrix}^T \\
&\quad \times \begin{bmatrix} Q_1 & Q_2 \\ Q_2^T & Q_3 \end{bmatrix} \\
&\quad \times \begin{bmatrix} e(\alpha) \\ -\nu\exp(\kappa\alpha)(y(\alpha)-\hat{y}(\alpha)) \end{bmatrix} d\alpha\, d\beta \\
&\quad + \int_{-\tau}^{0} \exp(\kappa\sigma) \\
&\quad \times \begin{bmatrix} e(\sigma) \\ -\nu\exp(\kappa\sigma)(y(\sigma)-\hat{y}(\sigma)) \end{bmatrix}^T \\
&\quad \times \begin{bmatrix} R_1 & R_2 \\ R_2^T & R_3 \end{bmatrix} \\
&\quad \times \begin{bmatrix} e(\sigma) \\ -\nu\exp(\kappa\sigma)(y(\sigma)-\hat{y}(\sigma)) \end{bmatrix} d\sigma \\
&\quad + \begin{bmatrix} \int_{-\tau}^{0} e(\sigma)\, d\sigma \\ -\nu\int_{-\tau}^{0} \exp(\kappa\sigma)(y(\sigma)-\hat{y}(\sigma))\, d\sigma \end{bmatrix}^T \\
&\quad \times \begin{bmatrix} U_1 & U_2 \\ U_2^T & U_3 \end{bmatrix} \\
&\quad \times \begin{bmatrix} \int_{-\tau}^{0} e(\sigma)\, d\sigma \\ -\nu\int_{-\tau}^{0} \exp(\kappa\sigma)(y(\sigma)-\hat{y}(\sigma))\, d\sigma \end{bmatrix}.
\end{aligned} \tag{33}
$$

Using the arguments in the proof of Corollary 1, we obtain

$$
\begin{aligned}
&\|e(t)\| \\
&\quad < \frac{1}{\sqrt{\lambda_{\min}(P)}} \Bigg\{ e^T(0)Pe(0) + \int_{-\tau}^{0} \exp(-\kappa\beta) \\
&\qquad \times \int_{\beta}^{0} \exp(\kappa\alpha) \begin{bmatrix} e(\alpha) \\ -\nu\exp(\kappa\alpha)(y(\alpha)-\hat{y}(\alpha)) \end{bmatrix}^T \\
&\qquad \times \begin{bmatrix} Q_1 & Q_2 \\ Q_2^T & Q_3 \end{bmatrix} \\
&\qquad \times \begin{bmatrix} e(\alpha) \\ -\nu\exp(\kappa\alpha)(y(\alpha)-\hat{y}(\alpha)) \end{bmatrix} d\alpha\, d\beta \\
&\qquad + \int_{-\tau}^{0} \exp(\kappa\sigma) \begin{bmatrix} e(\sigma) \\ -\nu\exp(\kappa\sigma)(y(\sigma)-\hat{y}(\sigma)) \end{bmatrix}^T \\
&\qquad \times \begin{bmatrix} R_1 & R_2 \\ R_2^T & R_3 \end{bmatrix} \begin{bmatrix} e(\sigma) \\ -\nu\exp(\kappa\sigma)(y(\sigma)-\hat{y}(\sigma)) \end{bmatrix} d\sigma \\
&\qquad + \begin{bmatrix} \int_{-\tau}^{0} e(\sigma)\, d\sigma \\ -\nu\int_{-\tau}^{0} \exp(\kappa\sigma)(y(\sigma)-\hat{y}(\sigma))\, d\sigma \end{bmatrix}^T \\
&\qquad \times \begin{bmatrix} U_1 & U_2 \\ U_2^T & U_3 \end{bmatrix} \\
&\qquad \times \begin{bmatrix} \int_{-\tau}^{0} e(\sigma)\, d\sigma \\ -\nu\int_{-\tau}^{0} \exp(\kappa\sigma)(y(\sigma)-\hat{y}(\sigma))\, d\sigma \end{bmatrix} \Bigg\}^{\frac{1}{2}} \\
&\qquad \times \exp\left(-\frac{\kappa}{2}t\right).
\end{aligned} \tag{34}
$$

Let

$$
\begin{aligned}
M_{c2} &= \frac{1}{\sqrt{\lambda_{\min}(P)}} \Bigg\{ e^T(0)Pe(0) + \int_{-\tau}^{0} \exp(-\kappa\beta) \\
&\quad \times \int_{\beta}^{0} \exp(\kappa\alpha) \begin{bmatrix} e(\alpha) \\ -\nu\exp(\kappa\alpha)(y(\alpha)-\hat{y}(\alpha)) \end{bmatrix}^T \\
&\quad \times \begin{bmatrix} Q_1 & Q_2 \\ Q_2^T & Q_3 \end{bmatrix} \\
&\quad \times \begin{bmatrix} e(\alpha) \\ -\nu\exp(\kappa\alpha)(y(\alpha)-\hat{y}(\alpha)) \end{bmatrix} d\alpha\, d\beta \\
&\quad + \int_{-\tau}^{0} \exp(\kappa\sigma) \begin{bmatrix} e(\sigma) \\ -\nu\exp(\kappa\sigma)(y(\sigma)-\hat{y}(\sigma)) \end{bmatrix}^T \\
&\quad \times \begin{bmatrix} R_1 & R_2 \\ R_2^T & R_3 \end{bmatrix} \\
&\quad \times \begin{bmatrix} e(\sigma) \\ -\nu\exp(\kappa\sigma)(y(\sigma)-\hat{y}(\sigma)) \end{bmatrix} d\sigma \\
&\quad + \begin{bmatrix} \int_{-\tau}^{0} e(\sigma)\, d\sigma \\ -\nu\int_{-\tau}^{0} \exp(\kappa\sigma)(y(\sigma)-\hat{y}(\sigma))\, d\sigma \end{bmatrix}^T \\
&\quad \times \begin{bmatrix} U_1 & U_2 \\ U_2^T & U_3 \end{bmatrix} \\
&\quad \times \begin{bmatrix} \int_{-\tau}^{0} e(\sigma)\, d\sigma \\ -\nu\int_{-\tau}^{0} \exp(\kappa\sigma)(y(\sigma)-\hat{y}(\sigma))\, d\sigma \end{bmatrix} \Bigg\}^{\frac{1}{2}} \\
&> 0, \\
N_{c2} &= \frac{\kappa}{2} > 0.
\end{aligned}
$$

Then (34) is represented by

$$
\|e(t)\| < M_{c2}\exp(-N_{c2}t). \tag{35}
$$

This implies the exponential stability of the error system (9)–(10). This completes the proof. □

Remark 1 Various efficient convex optimization algorithms can be used to check whether the LMI (12) is feasible. In this paper, in order to solve the LMI, we utilize MATLAB LMI Control Toolbox [25], which implements state-of-the-art interior-point algorithms.

4 Numerical example

Consider the following time-delayed switched Hopfield neural network:

$$\dot{x}(t) = \sum_{i=1}^{2} \xi_i(t)\big[A_i x(t) + W_i \phi\big(x(t-1)\big) + J_i(t)\big], \tag{36}$$

$$y(t) = \sum_{i=1}^{2} \xi_i(t)\big[C_i x(t) + D_i x(t-1)\big], \tag{37}$$

where

$$\begin{aligned}
x(t) &= \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \qquad \phi\big(x(t)\big) = \begin{bmatrix} \tanh(x_1(t)) \\ \tanh(x_2(t)) \end{bmatrix},\\
A_1 &= \begin{bmatrix} -2.2 & 0 \\ 0 & -3.5 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} -3.9 & 0 \\ 0 & -2.8 \end{bmatrix},\\
W_1 &= \begin{bmatrix} -1 & 0.4 \\ 0 & -0.1 \end{bmatrix}, \qquad W_2 = \begin{bmatrix} 0.2 & -0.8 \\ 0.4 & 0.5 \end{bmatrix},\\
G_1 &= G_2 = \begin{bmatrix} 1 \\ -0.5 \end{bmatrix}, \qquad F_1 = F_2 = 1,\\
J_1(t) &= \begin{bmatrix} \sin(1.8t) \\ \cos^2(t) \end{bmatrix}, \qquad J_2(t) = \begin{bmatrix} 3\cos^2(0.1t) \\ -\sin(t) \end{bmatrix},\\
C_1 &= [1 \quad 0], \qquad C_2 = [0 \quad 1],\\
D_1 &= [0.5 \quad 1], \qquad D_2 = [-1 \quad 0.3].
\end{aligned} \tag{38}$$

By solving the LMI (12) in Theorem 1, a feasible solution is obtained as

$$\begin{aligned}
P &= \begin{bmatrix} 0.6821 & 0.2624 \\ 0.2624 & 1.1716 \end{bmatrix}, \qquad M = \begin{bmatrix} 0.3022 \\ -0.2372 \end{bmatrix},\\
Q_1 &= \begin{bmatrix} 2.6196 & 0.0832 \\ 0.0832 & 2.8001 \end{bmatrix}, \qquad Q_2 = \begin{bmatrix} -0.0511 \\ 0.1885 \end{bmatrix},\\
Q_3 &= 0.3400,\\
R_1 &= \begin{bmatrix} 2.9370 & 0.0318 \\ 0.0318 & 2.7684 \end{bmatrix}, \qquad R_2 = \begin{bmatrix} -0.0096 \\ 0.1374 \end{bmatrix},\\
R_3 &= 0.3242,\\
U_1 &= \begin{bmatrix} 0.7772 & -0.0191 \\ -0.0191 & 0.7795 \end{bmatrix}, \qquad U_2 = \begin{bmatrix} 0.0192 \\ 0.0015 \end{bmatrix},\\
U_3 &= 0.0989,\\
S_1 &= \begin{bmatrix} 6.2081 & -1.3980 \\ -1.3980 & 4.0715 \end{bmatrix},\\
S_2 &= \begin{bmatrix} 1.6612 & 0.1803 \\ 0.1803 & 1.9236 \end{bmatrix}.
\end{aligned}$$

Then the filter gain L can be designed as

$$L = \begin{bmatrix} 0.5700 \\ -0.3301 \end{bmatrix}.$$

The switching signal $\alpha \in \{1, 2\}$ is given by

$$\alpha = \begin{cases} 1, & 0 \le t \le 2, \\ 2, & \text{otherwise}, \end{cases}$$

which means that $\dot{x}(t) = A_1 x(t) + W_1\phi(x(t-1)) + J_1(t)$ and $y(t) = C_1 x(t) + D_1 x(t-\tau)$ are switched to $\dot{x}(t) = A_2 x(t) + W_2\phi(x(t-1)) + J_2(t)$ and $y(t) = C_2 x(t) + D_2 x(t-\tau)$, respectively, at $t = 2$. Network states on $t \in [0, 2]$ have zero impact for stability of systems because stability is an asymptotic behavior which does not concern with initial network states. When the initial conditions are given by

$$x(0) = \begin{bmatrix} 3 \\ -1 \end{bmatrix}, \qquad \hat{x}(0) = \begin{bmatrix} -2 \\ 2.3 \end{bmatrix}, \tag{39}$$

the simulation results for the exponential switched passive state estimator design are shown in Figs. 1–3. Figures 1 and 2 show the true states $x_1(t)$ and $x_2(t)$ and their estimations $\hat{x}_1(t)$ and $\hat{x}_2(t)$, respectively, and Fig. 3 shows the responses of the estimation error $e(t)$. These results demonstrate the effectiveness of the developed approach for the design of the exponential switched passive state estimator.

5 Conclusion

In this paper, a new exponential state estimator has been proposed for switched Hopfield neural networks based on passivity theory. By the proposed method, it was shown that the estimation error system is exponentially stable and passive from the control input

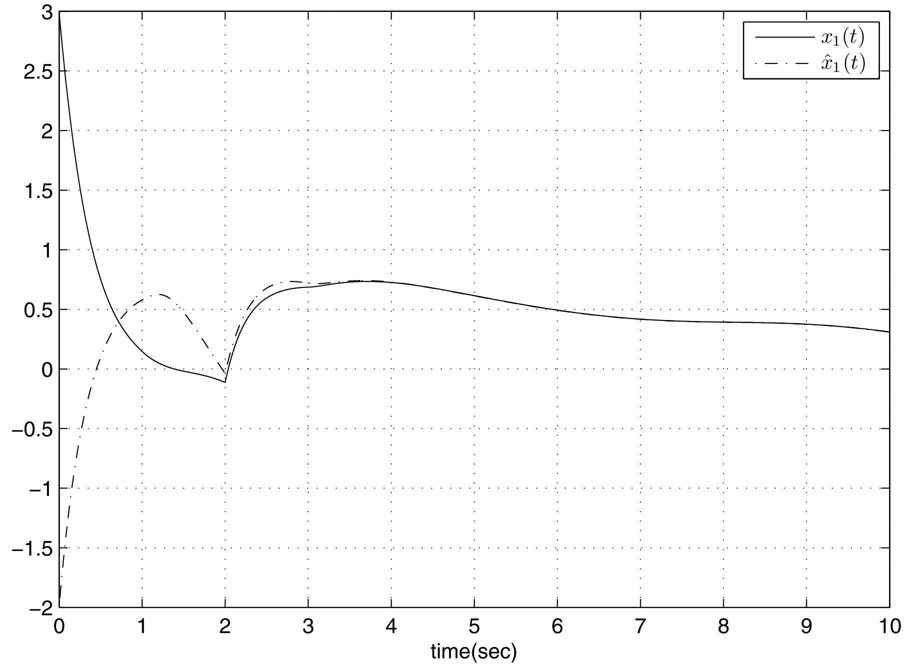

Fig. 1 Responses of the state $x_1(t)$ and its estimation $\hat{x}_1(t)$

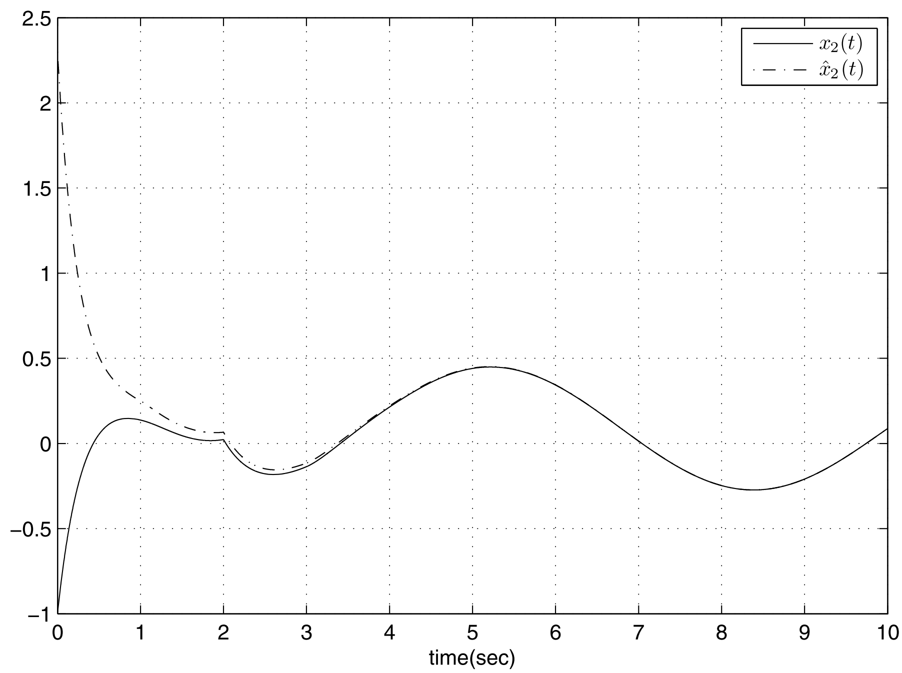

Fig. 2 Responses of the state $x_2(t)$ and its estimation $\hat{x}_2(t)$

to the output error. The gain matrix of the proposed state estimator was determined by solving the LMI problem. A simulation example was given to show the effectiveness of the proposed state estimator. The proposed switched state estimator can be used in output–feedback control applications. A switched neural network is applied to model an unknown switched nonlinear system and the states estimated by the proposed switched state estimator can be then utilized to achieve control design objectives. Therefore, the proposed switched state estimator is of significance for several control applications.

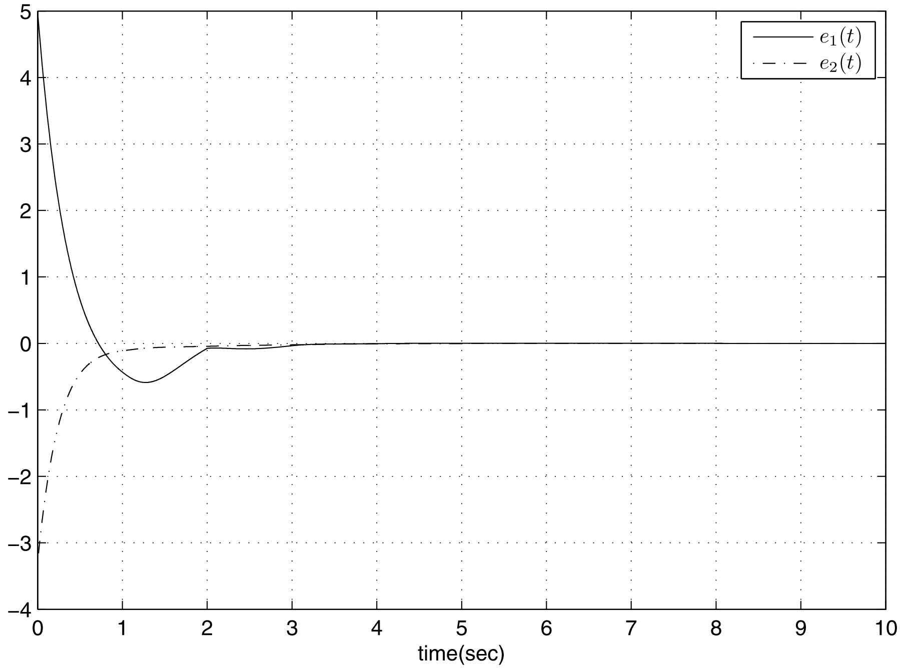

Fig. 3 Responses of the estimation error $e(t)$

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