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ORIGINAL PAPER

Soliton-shape-preserving and soliton-complex interactions for a (1 + 1)-dimensional nonlinear dispersive-wave system in shallow water

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Abstract Under investigation in this paper is a $(1+1)$-dimensional nonlinear dispersive-wave system for the long gravity waves in shallow water. With symbolic computation, we derive the multi-soliton solutions for the system. Four sorts of interactions for the system are discussed: (1) Soliton shape preserving, in which two solitons undergo the fusion behavior while the amplitudes and velocities of the other two remain unchanged during the interaction process; (2) Head-on collisions between the two-soliton complexes; (3) Overtaking collisions between the two-soliton complexes; (4) Two-soliton complexes formed by the inelastic collisions. Such soliton structures might be of certain value in fluid dynamics.

Keywords $(1+1)$-dimensional nonlinear dispersive-wave system · Shallow water · Soliton shape preserving · Soliton complex · Darboux transformation · Symbolic computation

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1 Introduction

Nonlinear evolution equations (NLEEs) have been frequently seen in fluid mechanics, plasma physics and other fields [1–15]. Those NLEEs have been reported to possess the soliton solutions, i.e., some stationary pulses or wave packets propagating, e.g., in the nonlinear dispersive media [1–15]. Solitons preserve their stable wave forms because of a dynamical balance among the nonlinear, dissipative and dispersive effects [1–15]. On a uniform layer of water, Korteweg–de Vries equation describes the interaction of unidirectional (overtaking collisions) solitary waves, N-soliton solution of which has been presented by the inverse scattering method [16]. Other NLEEs in fluid dynamics have also been found with the corresponding solitons obtained, such as the Kadomtsev–Petviashvili [17], Whitham–Broer–Kaup [18–20] and Camassa–Holm [21] equations.

With different fluid velocities such as the depth-mean, bottom and surface ones considered, the following set of NLEEs [22–24]:

$$\begin{aligned} &u_t + uu_x + vu_y + \zeta_x = 0, \\ &v_t + uv_x + vv_y + \zeta_y = 0, \\ &\zeta_t + \big[(1+\zeta)u\big]_x + \big[(1+\zeta)v\big]_y \\ &\quad + \frac{1}{3}(u_{xxx} + u_{xyy} + v_{xxy} + v_{yyy}) = 0, \end{aligned} \tag{1}$$

has been seen, which models certain nonlinear and dispersive long gravity waves traveling in two horizontal directions on shallow waters of uniform depth, with the higher-order terms ignored [22–24], where x and y represent the space, t denotes the time, $u(x, y, t)$ is the surface velocity of water along the x-direction, $v(x, y, t)$ is the surface velocity of water along the y-direction and $\zeta(x, y, t)$ is the elevation of the water wave [23, 24]. Model (1) can also be used to describe the evolution of the solitary waves on a uniform layer of the water including the oblique interaction, oblique reflection from a vertical wall and turning in a curved channel [23, 24]. Solutions of Model (1) are said by [23, 24] to be valuable for the coastal and civil engineers to study the nonlinear water wave during a harbor and coastal design. Model (1) cannot pass the Painlevé test [23]; however, via the scaling transformation [24] and symmetry reduction [25], Model (1) reduces to the (1 + 1)-dimensional integrable nonlinear dispersive-wave system as follows [24]:

$$\xi_t + \left[(1+\xi)u\right]_x = -\frac{1}{4}u_{xxx}, \qquad u_t + uu_x + \xi_x = 0. \tag{2}$$

In System (2), $u(x, t)$ denotes the surface velocity of the water wave along the x-direction, $\xi(x, t)$ is the wave elevation [24].

System (2) is integrable and has three Hamiltonian structures [26, 27]. Bidirectional one-, two-, three- and four-soliton solutions of System (2) have been presented via the Darboux transformations (DTs) of a Broer–Kaup (BK) system [28, 29], while the multi-soliton solutions have also be given through the N-fold DT method [30, 31]. System (2) is equivalent to a member of the Ablowitz–Kaup–Newell–Segur (AKNS) system and given the multi-soliton solutions in terms of the Vandermonde determinant via the DTs of the latter [24, 32, 33]. Two types of DTs have been constructed based on the Lax pair of System (2) with the one- and two-soliton solutions obtained [34]. Relevant issues on the bidirectional solitonic solutions of the variable-coefficient NLEEs are seen, e.g., in [35, 36].

In this paper, with symbolic computation [1–15], we will present some solutions with mechanical properties for System (2), which involve the soliton shape preserving (when the fusion behavior and elastic collision occur simultaneously), head-on and overtaking collisions among the two-soliton complexes, two-soliton complexes formed by the inelastic collision (when the fission behavior and elastic collision occur simultaneously). To our knowledge, those results have not been reported in the literatures. Outline of this paper will be organized as follows: Two sorts of multi-soliton solutions for System (2) will be derived in Sect. 2; Sect. 3 will discuss the soliton interactions; Sect. 4 will be our conclusions.

2 Multi-soliton solutions

Via the variable transformations [32],

$$u = \frac{q_x}{q}, \tag{3a}$$

$$\xi = -1 - qr + \frac{1}{2}u_x, \tag{3b}$$

one sees the following conclusion [32]:

If q and r are the solutions of the second-order AKNS system

$$q_t - rq^2 + \frac{1}{2}q_{xx} - q = 0, \tag{4a}$$

$$r_t + qr^2 - \frac{1}{2}r_{xx} + r = 0, \tag{4b}$$

then u and v defined by transformations (3) satisfy System (2), where System (4) is associated with the following Lax pair [24, 32, 33, 44]:

$$\phi_x = U\phi, \qquad \phi_t = V\phi \tag{5}$$

with

$$U = \begin{pmatrix} \lambda & q \\ r & -\lambda \end{pmatrix}, \qquad V = \begin{pmatrix} -\lambda^2 + \frac{1}{2}qr + \frac{1}{2} & -\lambda q - \frac{1}{2}q_x \\ -\lambda r + \frac{1}{2}r_x & \lambda^2 - \frac{1}{2}qr - \frac{1}{2} \end{pmatrix}. \tag{6}$$

Next, we will use the DT of System (4) (see Appendix) to derive the multi-soliton solutions in terms of the double Wronskian for System (2).

Different from [32], substituting $q = r = 0$ into the Lax pair of System (4), we have two basic solutions:

$$\varphi(\lambda_j) = (\varphi_1, \varphi_2)^T = \left(e^{\xi_j}, 0\right)^T, \qquad \psi(\lambda_j) = (\psi_1, \psi_2)^T = \left(0, e^{-\xi_j}\right)^T, \tag{7}$$

with

$$\xi_j = \lambda_j(x - \lambda_j t) + \frac{1}{2}t \quad (1 \le j \le 2N). \tag{8}$$

According to (38) (see Appendix), we have

$$\sigma_j = -r_j e^{-2\xi_j} \quad (1 \le j \le 2N). \tag{9}$$

The double Wronskian is defined as [37–42]:

$$W^{N,M}(\varphi;\psi) = \det\left(\varphi, \partial_x\varphi, \ldots, \partial_x^{N-1}\varphi; \psi, \partial_x\psi, \ldots, \partial_x^{M-1}\psi\right), \tag{10}$$

where $\varphi = (\varphi_1, \varphi_2, \ldots, \varphi_{M+N})^T$, $\psi = (\psi_1, \psi_2, \ldots, \psi_{M+N})^T$.

With the help of the DT of System (4) (see Appendix), we can prove that System (4) has the double Wronskian-type solutions as follows:

$$\bar{r} = -2\frac{|\widehat{N-2}; \widehat{N}|}{|\widehat{N-1}; \widehat{N-1}|}, \qquad \bar{q} = 2\frac{|\widehat{N}; \widehat{N-2}|}{|\widehat{N-1}; \widehat{N-1}|}, \tag{11}$$

where

$$\begin{aligned} |\widehat{N-1}; \widehat{N-1}| &= W^{N,N}[\varphi_1(\lambda_j); -r_j\psi_2(\lambda_j)], \\ |\widehat{N}; \widehat{N-2}| &= W^{N+1,N-1}[\varphi_1(\lambda_j); -r_j\psi_2(\lambda_j)], \\ |\widehat{N-2}; \widehat{N}| &= W^{N-1,N+1}[\varphi_1(\lambda_j); -r_j\psi_2(\lambda_j)]. \end{aligned} \tag{12}$$

In fact, from the Lax pair of System (4), we obtain the relations

$$\varphi_{1x} = \lambda_j\varphi_1, \tag{13}$$

$$\psi_{2x} = -\lambda_j\psi_2, \tag{14}$$

$$j = 1, 2, \ldots, 2N. \tag{15}$$

Noticing (38) (see Appendix), we have

$$\sigma_j = -\frac{r_j\psi_2(\lambda_j)}{\varphi_1(\lambda_j)} \quad (j = 1, 2, \ldots, 2N). \tag{16}$$

Solving System (37) (see Appendix), we get

$$B_{N-1} = \frac{\triangle_{B_{N-1}}}{\triangle}, \qquad C_{N-1} = \frac{\triangle_{C_{N-1}}}{\triangle} \tag{17}$$

with

$$\triangle = \begin{vmatrix} 1 & \sigma_1 & \lambda_1 & \sigma_1\lambda_1 & \cdots & \lambda_1^{N-1} & \sigma_1\lambda_1^{N-1} \\ 1 & \sigma_2 & \lambda_2 & \sigma_2\lambda_2 & \cdots & \lambda_2^{N-1} & \sigma_2\lambda_2^{N-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \sigma_{2N} & \lambda_{2N} & \sigma_{2N}\lambda_{2N} & \cdots & \lambda_{2N}^{N-1} & \sigma_{2N}\lambda_{2N}^{N-1} \end{vmatrix}, \tag{18}$$

$$\triangle_{B_{N-1}} = \begin{vmatrix} 1 & \sigma_1 & \lambda_1 & \sigma_1\lambda_1 & \cdots & \lambda_1^{N-1} & -\lambda_1^N \\ 1 & \sigma_2 & \lambda_2 & \sigma_2\lambda_2 & \cdots & \lambda_2^{N-1} & -\lambda_2^N \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \sigma_{2N} & \lambda_{2N} & \sigma_{2N}\lambda_{2N} & \cdots & \lambda_{2N}^{N-1} & -\lambda_{2N}^N \end{vmatrix}, \tag{19}$$

$$\triangle_{C_{N-1}} = \begin{vmatrix} 1 & \sigma_1 & \lambda_1 & \sigma_1\lambda_1 & \cdots & -\sigma_1\lambda_1^N & \sigma_1\lambda_1^{N-1} \\ 1 & \sigma_2 & \lambda_2 & \sigma_2\lambda_2 & \cdots & -\sigma_2\lambda_2^N & \sigma_2\lambda_2^{N-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \sigma_{2N} & \lambda_{2N} & \sigma_{2N}\lambda_{2N} & \cdots & -\sigma_{2N}\lambda_{2N}^N & \sigma_{2N}\lambda_{2N}^{N-1} \end{vmatrix}. \tag{20}$$

Then, substituting (16) into (17), we obtain

$$B_{N-1} = -\frac{|\widehat{N}; \widehat{N-2}|}{|\widehat{N-1}; \widehat{N-1}|}, \tag{21}$$

$$C_{N-1} = -\frac{|\widehat{N-2}; \widehat{N}|}{|\widehat{N-1}; \widehat{N-1}|}. \tag{22}$$

Thus, the double Wronkian-type solution for System (4) is presented in the form of (11). So, we can obtain the double Wronskian-type solution for System (2):

$$u = \frac{\left(\frac{|\widehat{N}; \widehat{N-2}|}{|\widehat{N-1}; \widehat{N-1}|}\right)_x}{\left(\frac{|\widehat{N}; \widehat{N-2}|}{|\widehat{N-1}; \widehat{N-1}|}\right)}, \tag{23}$$

$$\xi = -1 + 4\left(\frac{|\widehat{N}; \widehat{N-2}|}{|\widehat{N-1}; \widehat{N-1}|}\right)\left(\frac{|\widehat{N-2}; \widehat{N}|}{|\widehat{N-1}; \widehat{N-1}|}\right) + \frac{1}{2}\left[\frac{\left(\frac{|\widehat{N}; \widehat{N-2}|}{|\widehat{N-1}; \widehat{N-1}|}\right)_x}{\left(\frac{|\widehat{N}; \widehat{N-2}|}{|\widehat{N-1}; \widehat{N-1}|}\right)}\right]_x. \tag{24}$$

As mentioned in [42, 57], expressions (23) and (24) could be also obtained by the Hirota method [58].

In [28, 29], with the transformations

$$u = -v, \tag{25a}$$

$$\xi = w - 1 - \frac{1}{2}v_x, \tag{25b}$$

System (2) can be changed to the BK system,

$$v_t = \frac{1}{2}\left(v^2 + 2w - v_x\right)_x, \tag{26a}$$

$$w_t = \left(vw + \frac{1}{2} w_x \right)_x . \tag{26b}$$

With the seeds $v = 0$ and $w = 1$ selected, the bidirectional soliton solutions of System (2) have been presented [28, 29].

Different from [28, 29], substituting $v = w = 0$ into the Lax pair of System (26), we have two basic solutions

$$\varphi(\lambda_j) = (\varphi_1, \varphi_2)^T = \left(e^{-\xi_j}, 0\right)^T,$$
$$\psi(\lambda_j) = (\psi_1, \psi_2)^T = \left(\frac{1}{2\lambda_j} e^{\xi_j}, e^{\xi_j} \right)^T, \tag{27}$$

with

$$\xi_j = \lambda_j (x + \lambda_j t) \quad (1 \le j \le 2N). \tag{28}$$

According to (19) in [43], we have

$$\sigma_j = \frac{2\lambda_j r_j}{r_j - 2\lambda_j e^{-2\xi_j}} \quad (1 \le j \le 2N). \tag{29}$$

Via transformations (25) and DT of System (26) [43], the multi-soliton solution of System (2) is presented as

$$u = \frac{2B_{N-1,x}}{1 + 2B_{N-1}}, \tag{30}$$

$$\xi = -1 + 2C_{N-1}(1 + 2B_{N-1}) + \left(\frac{B_{N-1,x}}{1 + 2B_{N-1}} \right)_x , \tag{31}$$

where B_{N-1} and C_{N-1} have the same forms as expression (17) except that σ_j is replaced with (29).

3 Soliton interactions

Figure 1 displays a type of nonlinear phenomenon for System (2), namely, the soliton shape preserving, in which the fusion behavior accompanies with the elastic collision. Hereby, we will take the wave elevation ξ depicted in Fig. 1(b) for example [the similar analysis for the surface velocity u can be seen from Fig. 1(a) which describes the shock wave interaction]. Waves S_1 and S_4 fuse into the wave S_{14} with the higher amplitude at $t > 0$, while the shapes, amplitudes and velocities of the solitons S_2 and S_3 remain unchanged after the collision except for their phase shifts. The elastic and fusion behaviors occur simultaneously without any affection on each other. Therefore, System (2) could not only exhibit the pure elastic interactions among bidirectional solitons [24, 28–34] but also admits the coupled ones (fusion behavior and elastic collision). Figure 2 describes the head-on collision between two soliton complexes [45], each of which is formed by two solitons. We also take Fig. 2(b) as example to explain such behavior [the shock-wave interaction for Fig. 2(a)]. It is observed that two parallel solitons R_1 and R_2 (or R_3 and R_4) propagate with the same velocity, which can be considered as a two-soliton complex, R_1–R_2 (or R_3–R_4). R_2 and R_3 firstly fuse into one solitary wave with the higher amplitude and then split, so do the waves R_1 and R_4. On the whole, the interaction between the two-soliton complexes R_1–R_2 and R_3–R_4 is elastic; that is to say, besides the amplitudes, velocities and shapes, the relative separation distance between R_1 and R_2 (or R_3 and R_4) is also unchanged before and after the collision. In contrast, Fig. 3 depicts the overtaking collision of two soliton complexes along same directions of

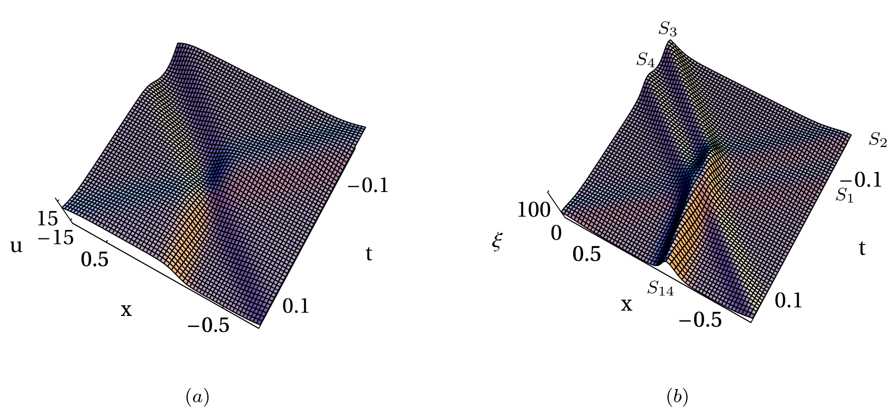


Fig. 1 Soliton shape preserving of System (2) via expressions (23) and (24) with $q = 0$, $r = 0$, $\lambda_1 = -7$, $\lambda_2 = -0.1$, $\lambda_3 = 1$, $\lambda_4 = 6$, $r_1 = -0.5$, $r_2 = 0.5$, $r_3 = -0.5$ and $r_4 = 0.5$

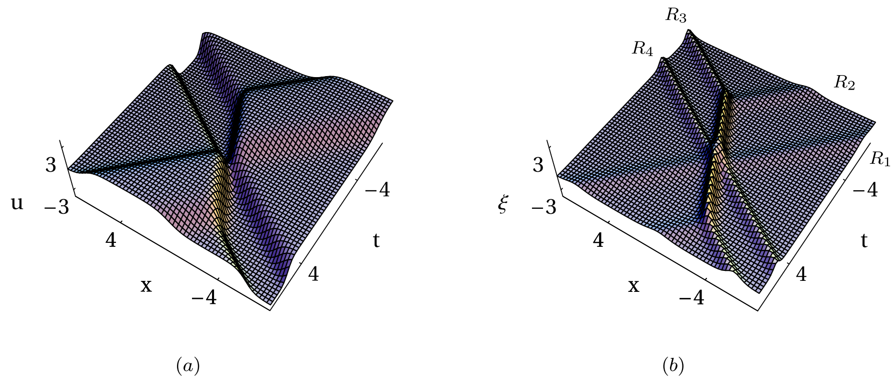
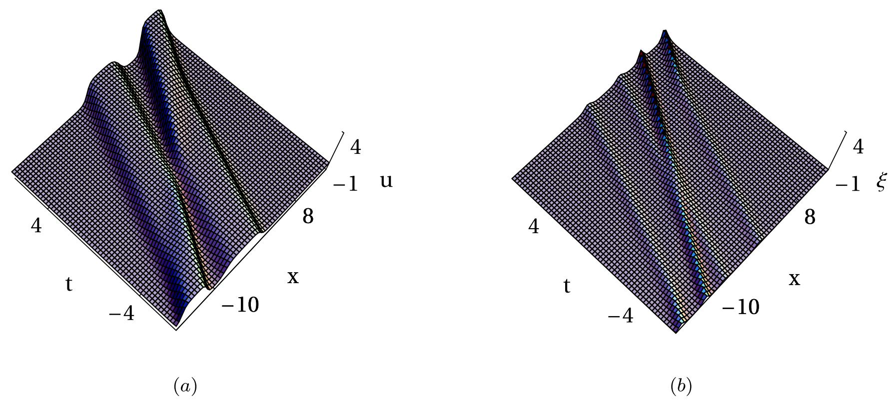


Fig. 2 Head-on collision between two soliton complexes for System (2) via expressions (30) and (31) with $v=0$, $w=0$, $\lambda_1=0.002$, $\lambda_2=-1$, $\lambda_3=0.005$, $\lambda_4=1.5$, $r_1=-0.5$, $r_2=1.5$, $r_3=0.5$ and $r_4=-1.5$

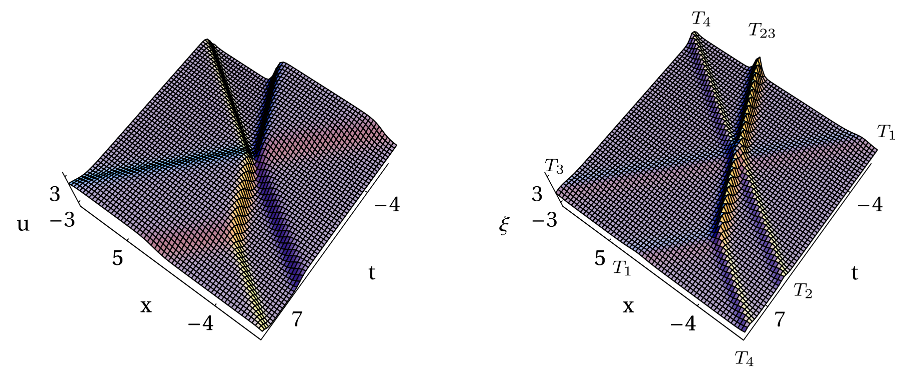
Fig. 3 Overtaking collision between two soliton complexes for System (2) via expressions (30) and (31) with $v=0$, $w=0$, $\lambda_1=0.002$, $\lambda_2=-1$, $\lambda_3=0.005$, $\lambda_4=-1.5$, $r_1=-0.5$, $r_2=1.5$, $r_3=0.5$ and $r_4=-1.5$



propagation, which can show the large-amplitude two-soliton complex with faster velocity overtaking the small-amplitude one, and after collision, the shorter being left behind. Through the above pictures, we can know that System (2) possesses bidirectional soliton-complex interactions including the head-on and overtaking collisions. Figure 4 exhibits the inelastic interaction among three waves where the two-soliton complexes are formed after the collision. For example, three waves shown in Fig. 4(b) firstly interact with each other and split into four ones later. Taken separately, T_{23} fissions into a left-going soliton T_2 and a right-going one T_3 at $t>0$, while the wave shapes, amplitudes and velocities of T_1 and T_4 do not change after the collision except for their phase shifts. Similarly to the soliton complexes in Figs. 2 and 3, T_1 and T_3 (or T_2 and T_4) also progress with the same velocity and constant separation distance. It should be pointed out that such structure can be also seen as a type of soliton shape preserving discussed above, while the differences include two aspects: (1) Figs. 1 and 4 involve the fusion and fission behaviors, respectively; (2) There are the soliton complexes in Fig. 4 (not in Fig. 1).

4 Conclusions

In this paper, with symbolic computation [1–15], our interest has been focused on System (2) for the long gravity waves in shallow water. Relevant topics can be found in [46–56]. Different from [32], substituting the seeds $q=r=0$ into System (4), we have derived the multi-soliton solutions with the determinant representations for System (2), i.e., expressions (23) and (24) {selecting $v=w=0$ into System (26), we have obtained expressions (30) and (31), which are different from the results in [28, 29]}. Via those solutions, we can conclude that System (2) possesses the following properties:



(a) (b)

Fig. 4 Soliton complexes formed by the inelastic collision via expressions (30) and (31) with $v = 0$, $w = 0$, $\lambda_1 = 0.002$, $\lambda_2 = -1$, $\lambda_3 = 0.005$, $\lambda_4 = 1.5$, $r_1 = -0.5$, $r_2 = 1.5$, $r_3 = 0$ and $r_4 = -1.5$

- Surface velocity of the water wave u has the interactions among the shock waves, as seen in parts (a) of Figs. 1–4.
- In addition to the bidirectional soliton solutions [24, 28–34], System (2) admits the coupled interactions for both the surface velocity of the water wave u and the wave elevation ξ, i.e., soliton shape preserving in which the fusion behavior accompanies with the elastic collision (Fig. 1).
- Head-on and overtaking collisions can occur between/among the two soliton complexes, each of which is formed by two solitons with the same velocity (Figs. 2 and 3).
- Inelastic collision can lead to the two-soliton complexes in which the fission behavior and elastic collision occur simultaneously (Fig. 4).

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Appendix

N-fold DT of a 2 × 2 AKNS system includes System (4) as a member of it [44]. A gauge transformation has been introduced in [44], i.e.,

$$\overline{\phi} = T^*\phi, \tag{32}$$

where T is defined by

$$T_x^* + T^*U = \overline{U}T^*, \tag{33a}$$

$$T_t^* + T^*V = \overline{V}T^*. \tag{33b}$$

Lax pair (5) can be transformed into

$$\overline{\phi}_x = \overline{U\phi}, \qquad \overline{\phi}_t = \overline{V\phi}, \tag{34}$$

where $\overline{U}$ and $\overline{V}$ have the same forms as U and V, respectively, except replacing r and q with $\overline{r}$ and $\overline{q}$. Let the matrix T^* in (32) be in the form of

$$T^* = T^*(\lambda) = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \tag{35}$$

with

$$A = \lambda^N + \sum_{k=0}^{N-1} \lambda^k A_k, \qquad B = \sum_{k=0}^{N-1} \lambda^k B_k, \tag{36a}$$

$$C = \sum_{k=0}^{N-1} \lambda^k C_k, \qquad D = \lambda^N + \sum_{k=0}^{N-1} \lambda^k D_k, \tag{36b}$$

where A_k, B_k, C_k and D_k are the functions of x and t. A_k, B_k, C_k and D_k can be determined by the following linear algebraic system:

$$\sum_{k=0}^{N-1} \lambda_j^k (A_k + B_k\sigma_j) = -\lambda_j^N, \tag{37a}$$

$$\sum_{k=0}^{N-1} \lambda_j^k (C_k + D_k\sigma_j) = -\sigma_j\lambda_j^N, \tag{37b}$$

with

$$\sigma_j = \frac{\varphi_2(\lambda_j) - r_j \psi_2(\lambda_j)}{\varphi_1(\lambda_j) - r_j \psi_1(\lambda_j)}, \quad (1 \leq j \leq 2N), \tag{38}$$

where $\varphi = (\varphi_1, \varphi_2)^T$ and $\psi = (\psi_1, \psi_2)^T$ are two basic solutions of Spectral Problem (5), and λ_j and r_j ($\lambda_k \neq \lambda_j$, $r_k \neq r_j$ as $k \neq j$) are some parameters suitably chosen such that the determinant of the coefficients for (37) is nonzero.

If the transformations between the old potentials (q and r) and the new ones ($\overline{q}$ and $\overline{r}$) are given by [44]

$$\overline{q} = q - 2B_{N-1}, \tag{39}$$

$$\overline{r} = r + 2C_{N-1}, \tag{40}$$

then the matrices $\overline{U}$ and $\overline{V}$ have the same forms as U and V, respectively.

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