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ORIGINAL PAPER

Robust stability of recurrent neural networks with ISS learning algorithm

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Abstract In this paper, an input-to-state stability (ISS) approach is used to derive a new robust weight learning algorithm for dynamic neural networks with external disturbance. Based on linear matrix inequality (LMI) formulation, the ISS learning algorithm is presented to not only guarantee exponential stability but also reduce the effect of an external disturbance. It is shown that the design of the ISS learning algorithm can be achieved by solving LMI, which can be easily facilitated by using some standard numerical packages. A numerical example is presented to demonstrate the validity of the proposed learning algorithm.

Keywords Input-to-state stability (ISS) approach · Weight learning algorithm · Dynamic neural networks · Linear matrix inequality (LMI)

1 Introduction

The last few decades have witnessed the use of neural networks in many real-world applications and have offered an attractive paradigm for a broad range of adaptive complex systems. In recent years, neural networks have enjoyed a great deal of success and have proven useful in many areas such as combinatorial optimization, signal processing, and pattern recognition [1].

Stability of neural networks is a prerequisite for their successful applications as either associative memories or optimization solvers. Stability problems in neural networks can be classified into two main categories: stability of neural networks [2–7] and stability of learning algorithms [8–14]. In this paper, we focus on deriving a new learning algorithm for neural networks. Stable learning algorithms can be developed by analyzing identification or tracking errors in neural networks. In [9], the stability conditions of learning laws in a situation where neural networks were used to identify and control a nonlinear system were studied. In [15], the dynamic backpropagation was modified with NL_q stability constraints. In [11, 13], passivity-based learning laws were proposed for neural networks. Since neural networks cannot match the unknown nonlinear systems exactly, extensive modifications should be made to either the normal gradient algorithm or to the backpropagation algorithm [8, 9, 12, 14].

It is well known that real physical systems are often affected by disturbances, such as perturbations in control or errors on observation. Thus, control systems are required not only to be stable, but also to have the property of input-to-state stability (ISS). A control system is called ISS, it means that no matter what the initial state is, if the inputs are uniformly small then the state of the control system must eventually be small. ISS is an interesting concept first introduced in [16]

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to nonlinear control systems. It has been widely accepted as an important concept in control engineering and many research results have been reported in recent years [17–24]. A natural question arises: can we obtain a learning algorithm for dynamic neural networks via the ISS approach? This paper gives an answer for this question. To the best of our knowledge, however, for the ISS based learning algorithm of dynamic neural networks with disturbance, there is no result in the literature so far, which still remains open and challenging.

In this paper, we propose an ISS learning algorithm, which is a new robust learning algorithm, for dynamic neural networks with external disturbance. This learning algorithm is a new contribution to the topic of neural networks. Theoretically, a dynamic neural network adapted by the ISS learning algorithm is exponentially stable and remains bounded for any bounded disturbance. Based on linear matrix inequality (LMI) formulation, the design of the propose learning algorithm can be realized by solving the LMI, which can be facilitated readily via standard numerical algorithms [25, 26].

This paper is organized as follows. In Sect. 2, we formulate the problem. In Sect. 3, an LMI problem for the ISS leaning algorithm of dynamic neural networks is proposed. In Sect. 4, a numerical example is given, and finally, conclusions are presented in Sect. 5.

2 Problem formulation

Consider the following differential equation:

$$\dot{X}(t) = F\big(X(t), U(t)\big), \tag{1}$$

where $X(t) \in R^n$ is the state variable, $U(t) \in R^m$ is the external input. $F : R^n \times R^m \to R^n$ is continuously differentiable and satisfies $F(0,0) = 0$. Throughout this paper, we will use the following definitions:

Definition 1 A function $\gamma : R_{\geq 0} \to R_{\geq 0}$ is a $\mathcal{K}$ function if it is continuous, strictly increasing and $\gamma(0) = 0$.

Definition 2 A function $\gamma : R_{\geq 0} \to R_{\geq 0}$ is a $\mathcal{K}_\infty$ function if it is a $\mathcal{K}$ function and also $\gamma(s) \to \infty$ as $s \to \infty$.

Definition 3 A function $\beta : R_{\geq 0} \times R_{\geq 0} \to R_{\geq 0}$ is a $\mathcal{KL}$ function if, for each fixed $t \geq 0$, the function $\beta(\cdot, t)$ is a $\mathcal{K}$ function, and for each fixed $s \geq 0$, the function $\beta(s, \cdot)$ is decreasing and $\beta(s,t) \to 0$ as $t \to \infty$.

The notion of ISS can be described as follows:

Definition 4 The system (1) is said to be input-to-state stable if there exist a $\mathcal{K}$ function $\gamma(s)$ and a $\mathcal{K}L$ function $\beta(s,t)$, such that, for each input $U(t) \in L_\infty^m$ ($\sup_{t\geq 0} \|U(t)\| < \infty$) and each initial state $X(0) \in R^n$, it holds that

$$\big\|X(t)\big\| \leq \beta\big(\big\|X(0)\big\|, t\big) + \gamma\big(\big\|U(t)\big\|\big) \tag{2}$$

for each $t \geq 0$.

It is noted that, if a system is input-to-state stable, the behavior of the system should remain bounded when its inputs are bounded. Now we introduce a useful result that is employed for obtaining the ISS weight learning algorithm.

Lemma 1 [19] *A continuous function $V(\cdot) : R^n \to R_{\geq 0}$ is called an ISS-Lyapunov function for the system (1), if there exist $\mathcal{K}_\infty$ functions $\alpha_1, \alpha_2, \alpha_3$, and α_4 such that*

$$\alpha_1\big(\big\|X(t)\big\|\big) \leq V\big(X(t)\big) \leq \alpha_2\big(\big\|X(t)\big\|\big) \tag{3}$$

for any $X(t) \in R^n$ and

$$\dot{V}\big(X(t)\big) \leq -\alpha_3\big(\big\|X(t)\big\|\big) + \alpha_4\big(\big\|U(t)\big\|\big) \tag{4}$$

for any $X(t) \in R^n$ and any $U(t) \in L_\infty^m$. Then the system (1) is input-to-state stable if and only if it admits an ISS-Lyapunov function.

Consider the following dynamic neural network:

$$\begin{aligned}\dot{x}(t) &= Ax(t) + W(t)\theta\big(x(t)\big) + V(t)\phi\big(x(t)\big)u(t)\\ &\quad + Gd(t),\end{aligned} \tag{5}$$

where $x(t) = [x_1(t)\ \ldots\ x_n(t)]^T \in R^n$ is the state vector, $u(t) = [u_1(t)\ \ldots\ u_m(t)]^T \in R^m$ is the bounded control input, $d(t) = [d_1(t)\ \ldots\ d_k(t)]^T \in R^k$ is the disturbance vector, $A = \mathrm{diag}\{-a_1, \ldots, -a_n\} \in R^{n\times n}$ ($a_i > 0, i = 1, \ldots, n$) is the self-feedback matrix, $W(t) \in R^{n\times p}$ and $V(t) \in R^{n\times p}$ are the weight matrices, $\theta(x) = [\theta_1(x), \ldots, \theta_p(x)]^T : R^n \to R^p$ is the nonlinear vector field and $\phi(x) = [\phi_{ij}(x)] : R^n \to$

$R^{p\times m}$ is the nonlinear matrix function, and $G \in R^{n\times k}$ is a known constant matrix. The element functions $\theta_i(x)$ and $\phi_{ij}(x)$ $(i = 1,\dots,p, j = 1,\dots,m)$ are usually selected as sigmoid functions.

Remark 1 It can be seen that the Hopfield model [27] is the special case of the neural network (5) with $A = \operatorname{diag}\{a_i\}$, $a_i = -\frac{1}{R_iC_i}$, $R_i > 0$, and $C_i > 0$. R_i and C_i are the resistance and capacitance at the ith node of the network, respectively.

The purpose of this paper is to derive the ISS learning algorithm guaranteeing the ISS of dynamic neural networks if there exists the disturbance $d(t)$. In addition, this ISS learning algorithm will be shown to guarantee the exponential stability when the disturbance $d(t)$ disappears.

3 ISS weight learning algorithm

This section is dedicated to designing the ISS weight learning algorithm of dynamic neural networks. The following theorem gives an LMI condition for the existence of the desired learning algorithm.

Theorem 1 *For a given $Q = Q^T > 0$, assume that there exist $X = X^T > 0$ and Y such that*

$$\begin{bmatrix} AX + XA^T + Y + Y^T & G & X \\ G^T & -I & 0 \\ X & 0 & -Q^{-1} \end{bmatrix} < 0. \tag{6}$$

If the weight matrices $W(t)$ and $V(t)$ are updated as

$$\begin{bmatrix} W(t) & V(t) \end{bmatrix} = \begin{cases} \frac{YX^{-1}x(t)}{\Phi(t)} \begin{bmatrix} \theta(x(t)) \\ \phi(x(t))u(t) \end{bmatrix}^T, & \Phi(t) \neq 0, \\ 0, & \Phi(t) = 0, \end{cases} \tag{7}$$

where $\Phi(t) = \|\theta(x(t))\|^2 + \|\phi(x(t))u(t)\|^2$, then the ISS of the neural network (5) is achieved.

Proof The neural network (5) can be represented by

$$\dot{x}(t) = Ax(t) + \begin{bmatrix} W(t) & V(t) \end{bmatrix} \begin{bmatrix} \theta(x(t)) \\ \phi(x(t))u(t) \end{bmatrix} + Gd(t). \tag{8}$$

In order to guarantee the ISS, the following relation must be satisfied:

$$Kx(t) = \begin{bmatrix} W(t) & V(t) \end{bmatrix} \begin{bmatrix} \theta(x(t)) \\ \phi(x(t))u(t) \end{bmatrix}, \tag{9}$$

where $K \in R^{n\times n}$ is the gain matrix of the ISS learning algorithm. Then we obtain the following equation:

$$\dot{x}(t) = (A + K)x(t) + Gd(t). \tag{10}$$

Selecting the weight matrices $[W(t)\ V(t)]$ such that (9) is fulfilled, we can obtain (10). One of the possible weight selections $[W(t)\ V(t)]$ to fulfill (9) (perhaps excepting a subspace of a smaller dimension) is given by

$$\begin{bmatrix} W(t) & V(t) \end{bmatrix} = Kx(t) \begin{bmatrix} \theta(x(t)) \\ \phi(x(t))u(t) \end{bmatrix}^+, \tag{11}$$

where $[\cdot]^+$ stands for the pseudoinverse matrix in the Moore–Penrose sense [28]. This learning law is just an algebraic relation depending on $x(t)$ and $u(t)$, which can be evaluated directly. Taking into account that [28]

$$x^+ = \begin{cases} \frac{x^T}{\|x\|^2}, & x \neq 0, \\ 0, & x = 0, \end{cases} \tag{12}$$

the formula (11) can be rewritten as follows:

$$\begin{bmatrix} W(t) & V(t) \end{bmatrix} = \begin{cases} \frac{Kx(t)}{\Phi(t)} \begin{bmatrix} \theta(x(t)) \\ \phi(x(t))u(t) \end{bmatrix}^T, & \Phi(t) \neq 0, \\ 0, & \Phi(t) = 0. \end{cases} \tag{13}$$

To obtain the gain matrix K of the ISS learning algorithm, consider the following Lyapunov function:

$$V(t) = x^T(t)Px(t), \tag{14}$$

where $P = P^T > 0$. Note that $V(t)$ satisfies the following Rayleigh inequality [29]:

$$\lambda_{\min}(P)\|x(t)\|^2 \le V(t) \le \lambda_{\max}(P)\|x(t)\|^2 \tag{15}$$

where $\lambda_{\min}(\cdot)$ and $\lambda_{\max}(\cdot)$ are the maximum and minimum eigenvalues of the matrix. The time derivative of $V(t)$ along the trajectory of (10) is

$$\begin{aligned} \dot{V}(t) &= \dot{x}(t)^T Px(t) + x^T(t)P\dot{x}(t) \\ &= x^T(t)\left[A^TP + K^TP + PA + PK\right]x(t) \end{aligned}$$

$$+ x^T(t)PGd(t) + d^T(t)G^TPx(t). \quad (16)$$

If we use the inequality $X^TY + Y^TX \le X^T\Lambda X + Y^T\Lambda^{-1}Y$, which is valid for any matrices $X \in R^{n\times m}$, $Y \in R^{n\times m}$, $\Lambda = \Lambda^T > 0$, $\Lambda \in R^{n\times n}$, we have

$$x(t)^TPGd(t) + d^T(t)G^TPx(t) \le d^T(t)d(t) + x(t)^TPGG^TPx(t). \quad (17)$$

Using (17), we obtain

$$\dot{V}(t) \le x^T(t)\big[A^TP + PA + PK + K^TP + PGG^TP\big]x(t) + d^T(t)d(t).$$

If the following matrix inequality is satisfied,

$$A^TP + PA + PK + K^TP + Q + PGG^TP < 0, \quad (18)$$

we have

$$\dot{V}(t) < -x^T(t)Qx(t) + d^T(t)d(t) \quad (19)$$

$$\le -\lambda_{\min}(Q)\|x(t)\|^2 + \|d(t)\|^2. \quad (20)$$

Define functions $\alpha_1(r)$, $\alpha_2(r)$, $\alpha_3(r)$, and $\alpha_4(r)$ as

$$\alpha_1(r) \triangleq \lambda_{\min}(P)r^2, \quad (21)$$

$$\alpha_2(r) \triangleq \lambda_{\max}(P)r^2, \quad (22)$$

$$\alpha_3(r) \triangleq \lambda_{\min}(Q)r^2, \quad (23)$$

$$\alpha_4(r) \triangleq r^2. \quad (24)$$

Note that $\alpha_1(r)$, $\alpha_2(r)$, $\alpha_3(r)$, and $\alpha_4(r)$ are $\mathcal{K}_\infty$ functions. From (15) and (20), we can obtain

$$\alpha_1(\|x(t)\|) \le V(t) \le \alpha_2(\|x(t)\|), \quad (25)$$

$$\dot{V}(t) \le -\alpha_3(\|x(t)\|) + \alpha_4(\|d(t)\|). \quad (26)$$

According to Lemma 1, we can conclude that $V(t)$ is an ISS-Lyapunov function and the ISS weight learning is achieved. From Schur complement, the matrix inequality (18) is equivalent to

$$\begin{bmatrix} A^TP + PA + PK + K^TP & PG & I \\ G^TP & -I & 0 \\ I & 0 & -Q^{-1} \end{bmatrix} < 0. \quad (27)$$

Pre- and post-multiplying (27) by $\mathrm{diag}(P^{-1}, I, I)$ and introducing change of variables such as $X = P^{-1}$ and $Y = KP^{-1}$, (27) is equivalently changed into the LMI (6). Then the gain matrix of the ISS learning algorithm is given by $K = YX^{-1}$. The learning algorithm (13) is changed to (7). This completes the proof. □

Corollary 1 *Without the external disturbance, if we use the learning algorithm* (7), *the exponential stability is obtained.*

Proof When $d(t) = 0$, from (20), we obtain

$$\dot{V}(t) < -\lambda_{\min}(Q)\|x(t)\|^2 \le -\frac{\lambda_{\min}(Q)}{\lambda_{\max}(P)}V(t). \quad (28)$$

From (28), we have

$$V(t) < V(0)e^{-\kappa t} \quad (29)$$

where

$$\kappa = \frac{\lambda_{\min}(Q)}{\lambda_{\max}(P)} > 0.$$

Using (15), we have

$$\|x(t)\| < \sqrt{\frac{V(0)}{\lambda_{\min}(P)}e^{-\kappa t}} = \sqrt{\frac{x^T(0)Px(0)}{\lambda_{\min}(P)}}e^{-\frac{\kappa}{2}t}. \quad (30)$$

Since

$$\sqrt{\frac{x^T(0)Px(0)}{\lambda_{\min}(P)}} > 0, \qquad \frac{\kappa}{2} > 0, \quad (31)$$

the relation (30) guarantees the exponential stability. This completes the proof. □

With zero initial condition and a given level $\gamma > 0$, the dynamic neural network (5) is $\mathcal{H}_\infty$ stable if the state vector $x(t)$ of the neural network (5) satisfies

$$\int_0^\infty x^T(t)Rx(t)\,dt < \gamma^2\int_0^\infty d^T(t)d(t)\,dt, \quad (32)$$

where R is a positive symmetric matrix. The parameter γ is called the $\mathcal{H}_\infty$ norm bound or the disturbance attenuation level. If the LMI (6) is slightly modified, the learning algorithm (7) can guarantee the $\mathcal{H}_\infty$ stability.

Corollary 2 *For given level $\gamma > 0$ and $R = R^T > 0$, assume that there exist $X = X^T > 0$ and Y such that*

$$\begin{bmatrix} AX + XA^T + Y + Y^T & G & X \\ G^T & -\gamma^2 I & 0 \\ X & 0 & -R^{-1} \end{bmatrix} < 0. \tag{33}$$

If the weight matrices $W(t)$ and $V(t)$ are updated as (7), then the $\mathcal{H}_\infty$ stability of the neural network (5) is achieved.

Proof Consider the following Lyapunov function:

$$V(t) = x^T(t)Px(t), \tag{34}$$

where $P = P^T > 0$. By using the proof of Theorem 1, the time derivative of $V(t)$ along the trajectory of (10) is given by (16). If we use the inequality $X^TY + Y^TX \le X^T\Lambda X + Y^T\Lambda^{-1}Y$, which is valid for any matrices $X \in R^{n\times m}$, $Y \in R^{n\times m}$, $\Lambda = \Lambda^T > 0$, $\Lambda \in R^{n\times n}$, we have

$$x(t)^T PGd(t) + d^T(t)G^T Px(t) \le \gamma^2 d^T(t)d(t) + \frac{1}{\gamma^2}x(t)^T PGG^T Px(t). \tag{35}$$

Using (35), we obtain

$$\dot V(t) \le x^T(t)\Big[A^TP + PA + PK + K^TP + \frac{1}{\gamma^2}PGG^TP\Big]x(t) + \gamma^2 d^T(t)d(t).$$

If the following matrix inequality is satisfied,

$$A^TP + PA + PK + K^TP + R + \frac{1}{\gamma^2}PGG^TP < 0, \tag{36}$$

we have

$$\dot V(t) < -x^T(t)Rx(t) + \gamma^2 d^T(t)d(t). \tag{37}$$

Integrating both sides of (37) from 0 to ∞ gives

$$V(\infty) - V(0) < -\int_0^\infty e^T(t)Re(t)\,dt + \gamma^2\int_0^\infty d^T(t)d(t)\,dt. \tag{38}$$

Since $V(\infty) \ge 0$ and $V(0) = 0$, we have the relation (32). From Schur complement, the matrix inequality (36) is equivalent to

$$\begin{bmatrix} A^TP + PA + PK + K^TP & PG & I \\ G^TP & -\gamma^2 I & 0 \\ I & 0 & -R^{-1} \end{bmatrix} < 0. \tag{39}$$

Pre- and post-multiplying (39) by $\mathrm{diag}(P^{-1}, I, I)$ and introducing change of variables such as $X = P^{-1}$ and $Y = KP^{-1}$, (39) is equivalently changed into the LMI (33). This completes the proof. □

Remark 2 The LMI problem given in Theorem 1 is to determine whether the solution exists or not. It is called the feasibility problem. The LMI problem can be solved efficiently by using recently developed convex optimization algorithms [25]. In this paper, in order to solve the LMI problem, we utilize MATLAB LMI Control Toolbox [26], which implements state-of-the-art interior-point algorithms.

4 Numerical example

Consider the following dynamic neural network:

$$\dot x(t) = Ax(t) + W(t)\theta\big(x(t)\big) + V(t)\phi\big(x(t)\big)u(t) + Gd(t), \tag{40}$$

where

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \qquad d(t) = \begin{bmatrix} d_1(t) \\ d_2(t) \end{bmatrix},$$

$$A = \begin{bmatrix} -4 & 0 \\ 0 & -3 \end{bmatrix}, \qquad G = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix},$$

$$\theta(x(t)) = \begin{bmatrix} \frac{1}{1+e^{-x_1(t)}} \\ \frac{1}{1+e^{-x_2(t)}} \end{bmatrix},$$

$$\phi(x(t)) = \begin{bmatrix} \frac{2}{1+e^{-2x_1(t)}} - 0.5 & \frac{2}{1+e^{-2x_1(t)}} - 0.5 \\ \frac{2}{1+e^{-2x_2(t)}} - 0.5 & \frac{2}{1+e^{-2x_2(t)}} - 0.5 \end{bmatrix},$$

$$u(t) = 0.1 \begin{bmatrix} \sin(t) \\ \cos(t) \end{bmatrix}.$$

Applying Theorem 1 to the neural network (40) with $Q = I$, where $I \in R^{2\times 2}$ is an identity matrix, yields

$$X = \begin{bmatrix} 0.6624 & 0.0284 \\ 0.0284 & 0.6339 \end{bmatrix},$$

$$Y = \begin{bmatrix} 1.0386 & -0.4338 \\ -0.1210 & 0.6678 \end{bmatrix}.$$

Figure 1 shows state trajectories when the initial conditions are given by

$$x(0) = \begin{bmatrix} 1.5 \\ -1 \end{bmatrix}, \qquad W(0) = \begin{bmatrix} 0.08 & -0.15 \\ 0.15 & -0.08 \end{bmatrix}, \qquad (41)$$

$$V(0) = \begin{bmatrix} -0.21 & 0.15 \\ -0.15 & -0.13 \end{bmatrix}, \qquad (42)$$

and the external disturbance $d_i(t)$ $(i = 1, 2)$ is given by $w(t)$, where $w(t)$ means a Gaussian noise with mean 0 and variance 1. Figure 1 shows that the proposed learning algorithm reduces the effect of the external disturbance $d(t)$ on the state vector $x(t)$. The evolutions of the weights $W(t)$ and $V(t)$ are shown in Figs. 2 and 3, respectively. Although the weights are not convergent, they are bounded. This result appears to be of little use for the storage of patterns, but it is very important for identification and control using neural networks [30]. Figure 4 shows state trajectories for $d_1(t) = d_2(t) = 0$. This figure shows the exponential stability of the neural network (40), which was presented in Corollary 1.

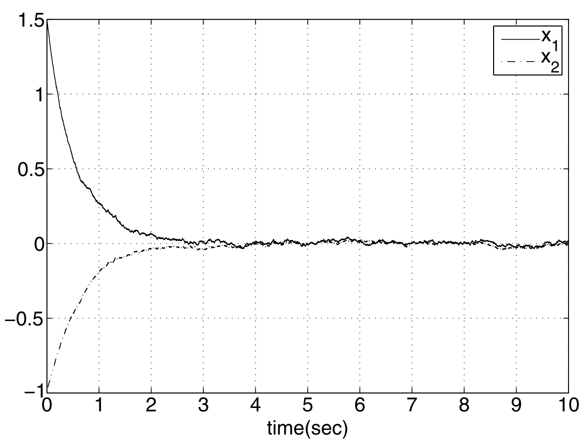

Fig. 1 State trajectories

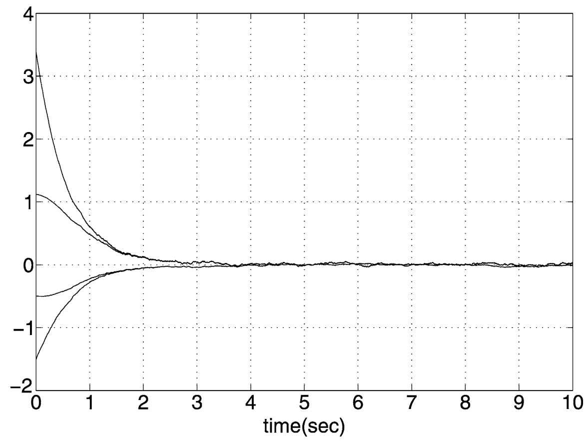

Fig. 2 Elements of the weight matrix $W(t)$

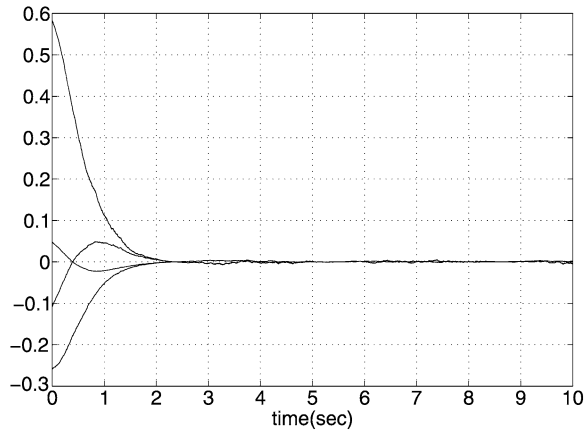

Fig. 3 Elements of the weight matrix $V(t)$

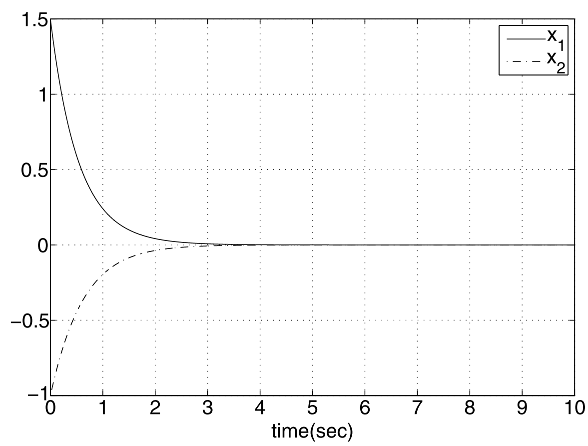

Fig. 4 State trajectories for $d_1(t) = d_2(t) = 0$

5 Conclusion

For the first time, this paper proposes an ISS learning algorithm for weight adjustment of dynamic neural networks with external disturbance. This learning algorithm guarantees exponential stability and reduces the effect of the external disturbance on the state vector. The proposed learning algorithm can be obtained by solving the LMI. A numerical simulation was performed to demonstrate the effectiveness of the proposed learning algorithm. It is expected that the results obtained in this study can be extended to discrete-time dynamic neural networks.

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