Nonlinear Dyn (2011) 65:335–348
DOI 10.1007/s11071-010-9895-z

ORIGINAL PAPER

On the fundamental linear fractional order differential equation

Abdelfatah Charef · Hassene Nezzari

Received: 17 October 2009 / Accepted: 18 November 2010 / Published online: 8 December 2010

Abstract This paper deals with the rational function approximation of the irrational transfer function $G(s)=\frac{X(s)}{E(s)}=\frac{1}{[(\tau_0 s)^{2m}+2\zeta(\tau_0 s)^m+1]}$ of the fundamental linear fractional order differential equation $(\tau_0)^{2m}\frac{d^{2m}x(t)}{dt^{2m}}+2\zeta(\tau_0)^m\frac{d^m x(t)}{dt^m}+x(t)=e(t)$, for $0<m<1$ and $0<\zeta<1$. An approximation method by a rational function, in a given frequency band, is presented and the impulse and the step responses of this fractional order system are derived. Illustrative examples are also presented to show the exactitude and the usefulness of the approximation method.

Keywords Fractional order differential equations · Irrational transfer function · Rational function

1 Introduction

The theory of fractional order systems has gained some importance during the last decades due mainly to its applications in numerous fields of physics and engineering [1–4]. Because of their representation by irrational transfer functions, the fractional order systems were studied marginally in the theory and practice. Only in recent years that one can find significant progress in theoretical works which may serve as a foundation of the fractional order system theory [1–7]. Finding accurate and efficient methods for solving the linear fractional order differential equation has been an active research undertaking. Exact solutions cannot be found easily, thus approximation and numerical methods must be used. There exist two main classes of methods for solving linear fractional differential equations: the frequency-domain methods and the time-domain methods. The frequency-domain methods are based on the approximation by rational function of the irrational transfer function of the linear fractional order system using a specified error in decibels and a bandwidth to generate the poles and the zeros of the rational function [8, 9]. The time domain methods are based on the computation of analytical expression of the output or the computation of its numerical solution [10–17].

In a previous paper [9], the fundamental linear fractional order differential equation defined in [11] by the following equation, for $0<m<2$:

$$(\tau_0)^m\frac{d^m x(t)}{dt^m}+x(t)=e(t) \tag{1}$$

whose transfer function is given by the following irrational function:

$$G(s)=\frac{X(s)}{E(s)}=\frac{1}{[1+(\tau_0 s)^m]} \tag{2}$$

was analyzed and its impulse and step responses have been derived by methods based on the approximation

A. Charef (✉) · H. Nezzari
Département d'Electronique, Université Mentouri-Constantine, Route Ain El-bey, Constantine 25000, Algeria
e-mail: afcharef@yahoo.com

by a rational function of its irrational transfer function of (2) for a given frequency band.

Another fundamental linear fractional order differential equation is represented, for $0 < m < 1$, by the following equation:

$$(\tau_0)^{2m}\frac{d^{2m}x(t)}{dt^{2m}} + 2\zeta(\tau_0)^m\frac{d^m x(t)}{dt^m} + x(t) = e(t) \quad (3)$$

Its transfer function is given by the following irrational function:

$$G(s) = \frac{X(s)}{E(s)} = \frac{1}{[(\tau_0 s)^{2m} + 2\zeta(\tau_0 s)^m + 1]} \quad (4)$$

where the parameter ζ is a real number such that $0 < \zeta < 1$ and τ_0 is a positive real number. In this article, a rational function approximation method, in a given frequency band, of the irrational transfer function of (4) is presented. Then the impulse and step responses of this type of fractional system are derived. Illustrative examples are presented to show the exactitude and the usefulness of the approximation method.

2 Generalized oscillation fractional order system

2.1 Definition

The generalized oscillation fractional order system is defined in this context by the fundamental linear fractional order differential equation given in (3) as

$$(\tau_0)^{2m}\frac{d^{2m}x(t)}{dt^{2m}} + 2\zeta(\tau_0)^m\frac{d^m x(t)}{dt^m} + x(t) = e(t) \quad (5)$$

for $0 < m < 1$, $0 < \zeta < 1$ and $\tau_0 > 0$. Its transfer function is given in (4) by the following irrational function:

$$G(s) = \frac{X(s)}{E(s)} = \frac{1}{[(\tau_0 s)^{2m} + 2\zeta(\tau_0 s)^m + 1]} \quad (6)$$

2.2 Preliminaries

In dielectric studies, K.S. Cole and R.H. Cole [18] observed that dispersion/relaxation data measured from a large number of materials can be modeled by the following function:

$$F(s) = \frac{1}{[1 + (\tau_0 s)^m]} = \int_0^\infty \frac{H(\tau)}{1 + s\tau}\,d\tau \quad (7)$$

where τ_0 is a real positive number, m is a real number such that $0 < m < 1$ and $H(\tau)$ is the distribution of relaxation times function that is given as [18]:

$$H(\tau) = \frac{1}{2\pi}\left[\frac{\sin[(1-m)\pi]}{\cosh[m\log(\frac{\tau}{\tau_0})] - \cos[(1-m)\pi]}\right] \quad (8)$$

For a limited frequency band of interest $[0, \omega_H]$, we have shown that $F(s)$ can be approximated as [9]:

$$\begin{aligned} F(s) &= \frac{1}{[1 + (\tau_0 s)^m]} \\ &\cong \int_0^\infty \frac{\sum_{i=1}^{2N-1} H(\tau_i)\delta(\tau - \tau_i)}{1 + s\tau}\,d\tau \\ &= \sum_{i=1}^{2N-1} \frac{H(\tau_i)}{1 + s\tau_i} \end{aligned} \quad (9)$$

where the points τ_i are logarithmically equidistant and are given as:

$$\tau_i = \tau_0(\lambda)^{N-i} \quad \text{for } i = 1, 2, \ldots, 2N-1 \quad (10)$$

with $\tau_N = \tau_0$, $\lambda > 1$ and the number N is determined as follows:

$$N = \text{Integer}\left[\frac{\log[\tau_0\omega_{\max}]}{\log(\lambda)}\right] + 1 \quad (11)$$

where $\omega_{\max}$ is an approximation frequency which is chosen to be very large multiple of ω_H in order to have a very good approximation in the frequency band $[0, \omega_H]$.

Hence, we can write that

$$F(s) = \frac{1}{[1 + (\tau_0 s)^m]} \cong \sum_{i=1}^{2N-1} \frac{k_i}{(1 + \frac{s}{p_i})} \quad (12)$$

where the p_i's are the poles of the approximation which are given by

$$p_i = \frac{1}{\tau_i} = (\lambda)^{(i-N)} p_0 \quad \text{for } i = 1, 2, \ldots, 2N-1 \quad (13)$$

such that $p_0 = 1/\tau_0$ and $\lambda = \frac{p_{i+1}}{p_i}$ = the ratio of a pole to a previous one, the k_i's are the residues of the poles that are given as

$$k_i = H(\tau_i) = \frac{1}{2\pi}\left[\frac{\sin[(1-m)\pi]}{\cosh[m\log(\frac{\tau_i}{\tau_0})] - \cos[(1-m)\pi]}\right]$$

$$\text{for } i = 1, 2, \ldots, 2N-1 \quad (14)$$

Following the above idea, we will present a method for the approximation by a rational function of the irrational transfer function of (6) of the generalized oscillation fractional order system, for a given frequency band of interest.

2.3 Rational function approximation

Equation (6) can be rewritten in the following form:

$$G(s)=\frac{1}{[1+(\tau_1 s)^m][1+(\tau_2 s)^m]} \tag{15}$$

where τ_1 and τ_2 are such that

$$\tau_1^m=\tau_0^m\left(\zeta+j\sqrt{1-\zeta^2}\right) \quad \text{and} \qquad \tau_2^m=\tau_0^m\left(\zeta-j\sqrt{1-\zeta^2}\right) \tag{16}$$

The complex number $(\zeta+j\sqrt{1-\zeta^2})$ can be written as

$$\left(\zeta+j\sqrt{1-\zeta^2}\right)=\exp(j\varphi) \tag{17}$$

where

$$\varphi=\arctan\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right) \tag{18}$$

so, τ_1 and τ_2 will be

$$\tau_1=\tau_0\exp\left(j\frac{\varphi}{m}\right) \quad \text{and} \qquad \tau_2=\tau_0\exp\left(-j\frac{\varphi}{m}\right) \tag{19}$$

By partial fraction decomposition of (15), we will get

$$G(s)=\left[\frac{(\frac{1}{2}-j\frac{1}{2}\frac{\zeta}{\sqrt{1-\zeta^2}})}{[1+(\tau_1 s)^m]}+\frac{(\frac{1}{2}+j\frac{1}{2}\frac{\zeta}{\sqrt{1-\zeta^2}})}{[1+(\tau_2 s)^m]}\right] \tag{20}$$

The generalized oscillation fractional order system of (5) must be a stable system in order to approximate its irrational transfer function of (20) by a rational one. Therefore, to guarantee its stability, we have to set some conditions on the parameters m and ζ.

2.3.1 System stability analysis

The stability of fractional order systems has been considered in [20] where the sufficient condition of the bounded input-bounded output stability has been found to be based on the phase angle of the roots of the denominator of the transfer function of the fractional order system. In our case, the characteristic equation of the generalized oscillation fractional order system is given as

$$D(s)=\left[1+(\tau_1 s)^m\right]\left[1+(\tau_2 s)^m\right]=0 \tag{21}$$

with τ_1 and τ_2 are complex conjugates and are given in (19). We can write that

$$\left[1+(\tau_1 s)^m\right]=0 \quad \Rightarrow \quad (\tau_1 s)^m=-1$$
$$\Rightarrow \quad s^m=-\frac{1}{\tau_1^m}$$

hence,

$$\begin{aligned} s^m&=-\frac{1}{\tau_0^m(\zeta+j\sqrt{1-\zeta^2})}=-\frac{(\zeta-j\sqrt{1-\zeta^2})}{\tau_0^m}\\ &=-\frac{1}{\tau_0^m}\exp(-j\varphi)\\ s^m&=\frac{1}{\tau_0^m}\exp(j\pi)\exp(-j\varphi)\\ &=\frac{1}{\tau_0^m}\exp\left[j(\pi-\varphi)\right] \end{aligned} \tag{22}$$

The stability sufficient condition is thus given as [19]:

$$(\pi-\varphi)>\frac{\pi}{2}m \tag{23}$$

which leads to

$$\varphi<\frac{\pi}{2}(2-m) \tag{24}$$

Hence,

$$\arctan\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right)<\frac{\pi}{2}(2-m)$$
$$\Rightarrow \quad \left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right)<\tan\left[\frac{\pi}{2}(2-m)\right]$$

Rearranging the above equation, we will get

$$\frac{(1-\zeta^2)}{\zeta^2}<\left\{\tan\left[\frac{\pi}{2}(2-m)\right]\right\}^2$$
$$\Rightarrow \quad \left(1-\zeta^2\right)<\zeta^2\left\{\tan\left[\frac{\pi}{2}(2-m)\right]\right\}^2$$
$$\Rightarrow \quad 1<\left(1+\left\{\tan\left[\frac{\pi}{2}(2-m)\right]\right\}^2\right)\zeta^2$$

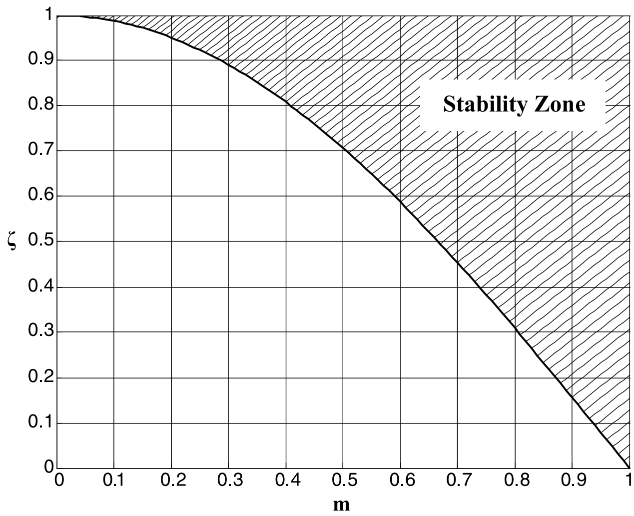

Fig. 1 Plot of the stability zone in the $[m, \zeta]$ plan of the generalized oscillation fractional order system of equation (5)

which yields, for $0 < m < 1$, to the following equation:

$$\zeta > \frac{1}{\sqrt{1 + \{\tan[\frac{\pi}{2}(2-m)]\}^2}} = -\cos\left[\frac{\pi}{2}(2-m)\right] \tag{25}$$

The above equation is the stability sufficient condition for the generalized oscillation fractional order system of (5) based on the parameters m and ζ. Figure 1 shows the plot of the stability region in the $[m, \zeta]$ plan. We note that (25) has also been derived in [21] when the stability and the resonance conditions of the fractional transfer function of (4) have been studied.

2.3.2 Approximation

Assuming that the stability sufficient condition is fulfilled, we can then use the above rational function approximation method to approximate each of the two irrational functions forming $G(s)$ of (20). So, for a limited frequency band $[0, \omega_H]$, the two irrational functions are approximated as

$$\frac{1}{[1 + (\tau_1 s)^m]} \cong \sum_{i=1}^{2N-1} \frac{H(\tau_{1i})}{1 + s\tau_{1i}} \tag{26}$$

$$\frac{1}{[1 + (\tau_2 s)^m]} \cong \sum_{i=1}^{2N-1} \frac{H(\tau_{2i})}{1 + s\tau_{2i}} \tag{27}$$

where the points τ_{1i} and τ_{2i}, for $i = 1, 2, \dots, 2N-1$, are given respectively as:

$$\tau_{1i} = \tau_1(\lambda)^{N-i} \quad \text{and} \quad \tau_{2i} = \tau_2(\lambda)^{N-i} \tag{28}$$

where the ratio $\lambda > 1$ and N is given as in (11). We also have, for $i = 1, 2, \dots, 2N-1$:

$$\frac{\tau_{1i}}{\tau_1} = \frac{\tau_{2i}}{\tau_2} = (\lambda)^{N-i} \tag{29}$$

Hence, the residues $H(\tau_{1i})$ and $H(\tau_{2i})$, for $i = 1, 2, \dots, 2N-1$, are such that:

$$k_i = H(\tau_{1i}) = H(\tau_{2i}) = \frac{1}{2\pi}\left[\frac{\sin[(1-m)\pi]}{\cosh[m\log((\lambda)^{N-i})] - \cos[(1-m)\pi]}\right] \tag{30}$$

Equations (26) and (27) are thus given as

$$\frac{1}{[1 + (\tau_1 s)^m]} \cong \sum_{i=1}^{2N-1} \frac{k_i}{1 + \frac{s}{p_{1i}}} \tag{31}$$

$$\frac{1}{[1 + (\tau_2 s)^m]} \cong \sum_{i=1}^{2N-1} \frac{k_i}{1 + \frac{s}{p_{2i}}} \tag{32}$$

where p_{1i} and p_{2i}, for $i = 1, 2, \dots, 2N-1$, are the poles of the two approximations. They are given by

$$p_{1i} = \frac{1}{\tau_{1i}} = \frac{1}{\tau_1(\lambda)^{N-i}} = (\lambda)^{(i-N)} p_1 \tag{33}$$

$$p_{2i} = \frac{1}{\tau_{2i}} = \frac{1}{\tau_2 (\lambda)^{N-i}} = (\lambda)^{(i-N)} p_2 \quad (34)$$

such that

$$p_1 = \frac{1}{\tau_1} = \frac{1}{\tau_0} \exp\left(-j\frac{\varphi}{m}\right) \quad (35)$$

$$p_2 = \frac{1}{\tau_2} = \frac{1}{\tau_0} \exp\left(j\frac{\varphi}{m}\right) \quad (36)$$

We can see that, for $i = 1, 2, \ldots, 2N-1$, p_{2i} is the complex conjugate of p_{1i}. Hence, the approximation by a rational function of $G(s)$ is given as:

$$\begin{aligned} G(s) &= \frac{1}{[(\tau_0 s)^{2m} + 2\zeta(\tau_0 s)^m + 1]} \\ &= \sum_{i=1}^{2N-1} k_i \left[\frac{(\frac{1}{2} - j\frac{1}{2}\frac{\zeta}{\sqrt{1-\zeta^2}})}{1 + \frac{s}{p_{1i}}} \right. \\ &\quad \left. + \frac{(\frac{1}{2} + j\frac{1}{2}\frac{\zeta}{\sqrt{1-\zeta^2}})}{1 + \frac{s}{p_{2i}}} \right] \end{aligned} \quad (37)$$

$$\begin{aligned} G(s) &= \frac{1}{[(\tau_0 s)^{2m} + 2\zeta(\tau_0 s)^m + 1]} \\ &= \sum_{i=1}^{2N-1} k_i \frac{1 + a_i s}{1 + 2\alpha\frac{s}{\omega_i} + \frac{s^2}{\omega_i^2}} \end{aligned} \quad (38)$$

where the coefficients α, a_i, and ω_i, for $i = 1, 2, \ldots, 2N-1$ are

$$\alpha = \cos\left(\frac{\varphi}{m}\right) \quad (39)$$

$$a_i = \tau_0 \lambda^{(N-i)} \left(\cos\left(\frac{\varphi}{m}\right) - \frac{\zeta}{\sqrt{1-\zeta^2}} \sin\left(\frac{\varphi}{m}\right) \right) \quad (40)$$

$$\omega_i = \frac{1}{\tau_0 \lambda^{(N-i)}} \quad (41)$$

2.4 Time responses

From (38), we have that

$$\begin{aligned} G(s) = \frac{X(s)}{E(s)} &= \frac{1}{[(\tau_0 s)^{2m} + 2\zeta(\tau_0 s)^m + 1]} \\ &= \sum_{i=1}^{2N-1} k_i \frac{1 + a_i s}{1 + 2\alpha\frac{s}{\omega_i} + \frac{s^2}{\omega_i^2}} \end{aligned} \quad (42)$$

so,

$$\begin{aligned} X(s) &= \frac{E(s)}{[(\tau_0 s)^{2m} + 2\zeta(\tau_0 s)^m + 1]} \\ &= \sum_{i=1}^{2N-1} k_i \frac{1 + a_i s}{1 + 2\alpha\frac{s}{\omega_i} + \frac{s^2}{\omega_i^2}} E(s) \end{aligned} \quad (43)$$

for $e(t) = \delta(t)$ the unit impulse $E(s) = 1$, we will have

$$\begin{aligned} X(s) &= \frac{1}{[(\tau_0 s)^{2m} + 2\zeta(\tau_0 s)^m + 1]} \\ &= \sum_{i=1}^{2N-1} k_i \frac{1 + a_i s}{1 + 2\alpha\frac{s}{\omega_i} + \frac{s^2}{\omega_i^2}} \end{aligned} \quad (44)$$

the impulse response of these systems is then given as

$$\begin{aligned} x(t) &= \sum_{i=1}^{2N-1} k_i C_i \exp(-\alpha\omega_i t) \\ &\quad \times \sin\left(\omega_i \left(\sqrt{1-\alpha^2}\right) t + \Phi_i\right) \end{aligned} \quad (45)$$

where the constants C_i and Φ_i, for $i = 1, 2, \ldots, 2N-1$, are given as [22]:

$$\begin{aligned} C_i &= \omega_i \sqrt{\frac{1 - 2\alpha a_i \omega_i + (a_i \omega_i)^2}{(1-\alpha^2)}} \quad \text{and} \\ \Phi_i &= \arctan\left(\frac{a_i \omega_i \sqrt{1-\alpha^2}}{1 - \alpha a_i \omega_i}\right) \end{aligned} \quad (46)$$

Now, for $e(t) = u(t)$ the unit step, $E(s) = 1/s$, (42) will be

$$\begin{aligned} X(s) &= \frac{1}{[(\tau_0 s)^{2m} + 2\zeta(\tau_0 s)^m + 1]} \frac{1}{s} \\ &= \sum_{i=1}^{2N-1} k_i \frac{1 + a_i s}{1 + 2\alpha\frac{s}{\omega_i} + \frac{s^2}{\omega_i^2}} \frac{1}{s} \end{aligned} \quad (47)$$

the step response of this system can be obtained as

$$\begin{aligned} x(t) &= \sum_{i=1}^{2N-1} k_i \left[1 - \frac{C_i}{\omega_i} \exp(-\alpha\omega_i t) \right. \\ &\quad \left. \times \sin\left(\omega_i \left(\sqrt{1-\alpha^2}\right) t + \Phi_{1i}\right) \right] \end{aligned} \quad (48)$$

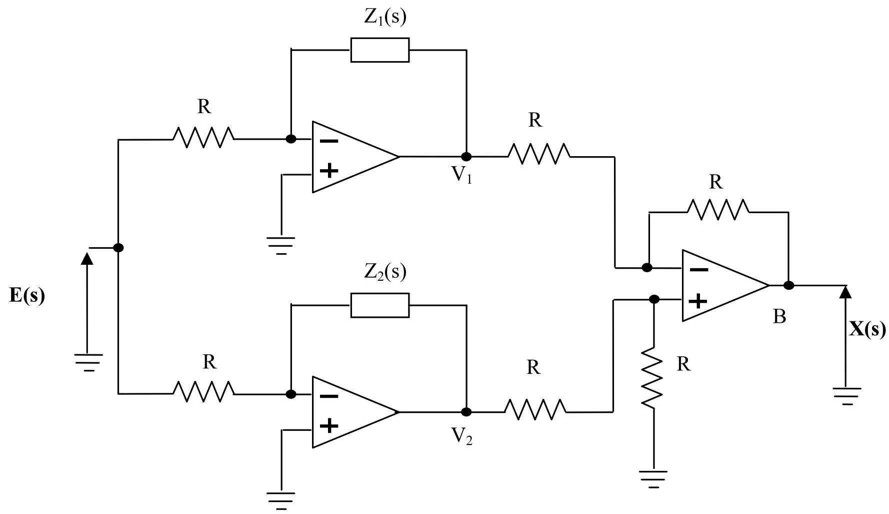

Fig. 2 Analog circuit realization of the generalized oscillation fractional order system

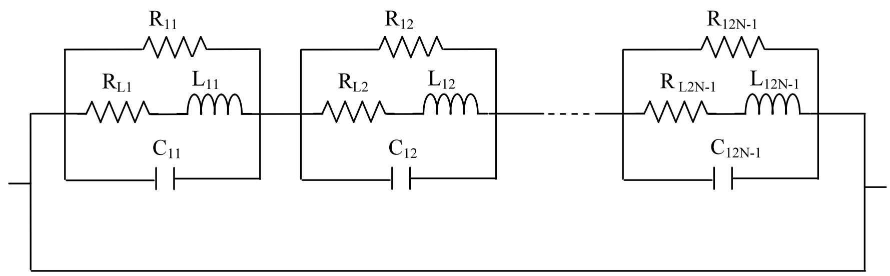

Fig. 3 Analog circuit realization of the impedance Z_1

where the constants Φ_{1i} is given as [22]:

$$\Phi_{1i} = \arctan\left(\frac{a_i\omega_i\sqrt{1-\alpha^2}}{1-\alpha a_i\omega_i}\right) - \arctan\left(\frac{\sqrt{1-\alpha^2}}{-\alpha}\right) \tag{49}$$

2.5 Analog circuit realization

The generalized oscillation fractional order system can not be directly realized by an analog circuit because its transfer function is an irrational function. However, its rational function approximation given in (38) by

$$G(s) = \sum_{i=1}^{2N-1} k_i \frac{1+a_i s}{1+2\alpha\frac{s}{\omega_i}+\frac{s^2}{\omega_i^2}} \tag{50}$$

can be realized by the analog circuit shown in Fig. 2.

Where, $Z_1(s)$ and $Z_2(s)$ are the impedances of the two $(2N-1)$ parallel RLC cells connected in series as shown in Figs. 3 and 4, respectively, and the resistor $R = 1\ \Omega$. We have chosen this type of analog circuit because from Sect. 2.3 we can easily show that the parameters a_i of (8), for $i = 1, \ldots, 2N-1$, are all negatives.

Thus, the impedances $Z_1(s)$ and $Z_2(s)$ of the above circuits are given as follows:

$$Z_1(s) = \sum_{i=1}^{2N-1}\left(\frac{R_{1i}R_{Li}}{R_{1i}+R_{Li}}\right) \times \left[\frac{1+(\frac{L_{1i}}{R_{Li}})s}{1+(\frac{L_{1i}+C_{1i}R_{1i}R_{Li}}{R_{1i}+R_{Li}})s+(\frac{R_{1i}C_{1i}L_{1i}}{R_{1i}+R_{Li}})s^2}\right] \tag{51}$$

$$Z_2(s) = \sum_{i=1}^{2N-1}\left[\frac{L_{2i}s}{1+\frac{L_{2i}}{R_{2i}}s+C_{2i}L_{2i}s^2}\right] \tag{52}$$

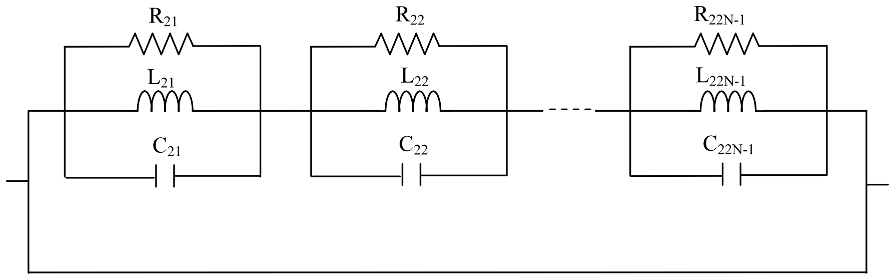

Fig. 4 Analog circuit realization of the impedance Z_2

The output voltage $X(s)$ of the circuit of Fig. 2 is given by

$$X(s)=\frac{R}{R}\big[V_2(s)-V_1(s)\big]=\big[V_2(s)-V_1(s)\big] \quad (53)$$

where the voltages $V_1(s)$ and $V_2(s)$ are, respectively, given from Fig. 1 by

$$V_1(s)=-\frac{Z_1}{R}E(s)=-Z_1E(s) \quad \text{and}$$
$$V_2(s)=-\frac{Z_2}{R}E(s)=-Z_2E(s) \quad (54)$$

The voltage $X(s)$ will then be

$$X(s)=(Z_1-Z_2)E(s) \quad (55)$$

Hence, the transfer function $G(s)$ of the circuit of Fig. 2 is given by

$$G(s)=\frac{X(s)}{E(s)}=(Z_1-Z_2) \quad (56)$$

From (51) and (52) the transfer function $G(s)$ can be written as

$$G(s)=\sum_{i=1}^{2N-1}\Bigg\{\left(\frac{R_{1i}R_{Li}}{R_{1i}+R_{Li}}\right)\times\left[\frac{1+(\frac{L_{1i}}{R_{Li}})s}{1+(\frac{L_{1i}+C_{1i}R_{1i}R_{Li}}{R_{1i}+R_{Li}})s+(\frac{R_{1i}C_{1i}L_{1i}}{R_{1i}+R_{Li}})s^2}\right]-\left[\frac{L_{2i}s}{1+\frac{L_{2i}}{R_{2i}}s+C_{2i}L_{2i}s^2}\right]\Bigg\} \quad (57)$$

Now, if we let

$$L_{2i}=L_{1i},\qquad C_{2i}=\frac{R_{1i}}{R_{1i}+R_{Li}}C_{1i}\quad\text{and}$$
$$R_{2i}=\frac{L_{1i}(R_{1i}+R_{Li})}{L_{1i}+C_{1i}R_{1i}R_{Li}} \quad (58)$$

Equation (57) becomes:

$$G(s)=\frac{X(s)}{E(s)}=\sum_{i=1}^{2N-1}\left(\frac{R_{1i}R_{Li}}{R_{1i}+R_{Li}}\right)\times\left[\frac{1-(\frac{L_{1i}}{R_{1i}})s}{1+\frac{L_{2i}}{R_{2i}}s+C_{2i}L_{2i}s^2}\right] \quad (59)$$

So, from (50) and (59), and for $i=1,\dots,2N-1$, we can write that

$$k_i=\frac{R_{1i}R_{Li}}{R_{1i}+R_{Li}},\qquad a_i=-\frac{L_{1i}}{R_{1i}},$$
$$\frac{2\alpha}{\omega_i}=\frac{L_{1i}+C_{1i}R_{1i}R_{Li}}{R_{1i}+R_{Li}}\quad\text{and}$$
$$\omega_i^2=\frac{R_{1i}+R_{Li}}{R_{1i}C_{1i}L_{1i}} \quad (60)$$

Therefore, for $i=1,\dots,2N-1$, the resistors, inductors, and the capacitors values of the analog circuit modeling the generalized oscillation fractional order system, in a given frequency band, are given by:

$$R_{Li}=k_i\frac{(1-2\alpha\omega_i a_i)}{2}+\frac{\sqrt{(1-2\alpha\omega_i a_i)+4\omega_i a_i(\alpha^2-1)}}{2} \quad (61)$$

$$R_{1i}=\frac{k_iR_{Li}}{R_{Li}-k_i},\qquad L_{1i}=\frac{-a_ik_iR_{Li}}{R_{Li}-k_i}\quad\text{and}$$
$$C_{1i}=\frac{R_{Li}-k_i}{-a_ik_i^2\omega_i^2} \quad (62)$$

$$R_{2i}=\frac{L_{1i}(R_{1i}+R_{Li})}{L_{1i}+C_{1i}R_{1i}R_{Li}},\qquad L_{2i}=L_{1i}\quad\text{and}$$
$$C_{2i}=\frac{R_{1i}}{R_{1i}+R_{Li}}C_{1i} \quad (63)$$

2.6 Illustrative examples

In this section, we will present two examples simulated on a PC using MATLAB to show the effectiveness and the exactitude of the proposed approach for the approximation of the generalized oscillation fractional order system. In the first example, the parameter ζ is such that $\frac{\sqrt{2}}{2} < \zeta < 1$, but in the second example ζ is such that $0 < \zeta < \frac{\sqrt{2}}{2}$.

2.6.1 Example 1

The first example of the generalized oscillation fractional order system is represented by the following fundamental linear fractional order differential equation of (5) with $\zeta = 0.87$, $m = 0.46$ and $\tau_0 = 1$ as

$$\frac{d^{0.92}x(t)}{dt^{0.92}} + 1.74\frac{d^{0.46}x(t)}{dt^{0.46}} + x(t) = e(t)$$

its transfer function is given by

$$G(s) = \frac{X(s)}{E(s)} = \frac{1}{[s^{0.92} + 1.74s^{0.46} + 1]}$$

For a given frequency band of interest $[0, \omega_H] = [0, 1000 \text{ rad/s}]$ and an approximation ratio $\lambda = 1.2$, the approximation frequency $\omega_{\max}$, the number N, the parameters φ and α can be easily calculated from Sect. 2 as

$$\omega_{\max} = 10^8\omega_H = 10^{11} \text{ rad/s}$$

$$N = \text{Integer}\left[\frac{\log[10^{11}]}{\log(1.2)}\right] + 1 = 139$$

$$\varphi = \arctan\left(\frac{\sqrt{1-(0.87)^2}}{0.87}\right) = 0.516$$

$$\alpha = \cos\left(\frac{0.516}{0.46}\right) = 0.434$$

then the relaxation times τ_i, the frequencies ω_i, the residues k_i and the parameters a_i of the approximation are also given from Sect. 2, for $i = 1, 2, \ldots, 277$, as

$$\tau_i = (1.2)^{(139-i)} \quad \text{and} \quad \omega_i = (1.2)^{(i-139)}$$

$$k_i = \frac{0.158}{\cosh\{0.084(139-i)\} + 0.125}$$

$$a_i = -1.154(1.2)^{(139-i)}$$

Hence, the transfer function approximation of the generalized oscillation fractional order system is given by

$$\begin{aligned} G(s) &= \frac{1}{[s^{0.92} + 1.74s^{0.46} + 1]} \\ &\cong \sum_{i=1}^{277}\left[\frac{0.158}{\cosh\{0.084(139-i)\} + 0.125}\right] \\ &\quad \times \left[\frac{1 - [1.154(1.2)^{(139-i)}]s}{1 + 0.868[(1.2)^{(139-i)}]s + [(1.44)^{(139-i)}]s^2}\right] \end{aligned}$$

To verify the validity of the proposed approach, we have used Oustaloup's method [2] to obtain the rational function approximation of the fractional order differentiator $s^{0.46}$ as

$$\begin{aligned} s^{0.46} &= \frac{N(s)}{D(s)} \\ &= (2.5119)10^{-5}\prod_{i=-131}^{131}\frac{(1 + \frac{s}{9.9368\cdot 10^9(1.2037)^i})}{(1 + \frac{s}{9.1247\cdot 10^9(1.2037)^i})} \end{aligned}$$

Then we have derived the rational function approximation of the transfer function of the generalized oscillation fractional order system, with almost the same order as the proposed rational approximation method in the frequency band $[0, \omega_H] = [0, 1000 \text{ rad/s}]$, as

$$\begin{aligned} G(s) &= \frac{1}{[s^{0.92} + 1.74s^{0.46} + 1]} \\ &= \frac{1}{[(s^{0.46})^2 + 1.74s^{0.46} + 1]} \\ &= \frac{1}{[\frac{N(s)}{D(s)}]^2 + 1.74[\frac{N(s)}{D(s)}] + 1} \end{aligned}$$

Figures 5 and 6 show the bode plots of the fractional order system transfer function and its proposed rational function approximation along with the rational function approximation by Oustaloup's method. We can easily see that they are all quite overlapping over the frequency band of interest.

Figure 7 shows the impulse responses of this generalized oscillation fractional order system obtained

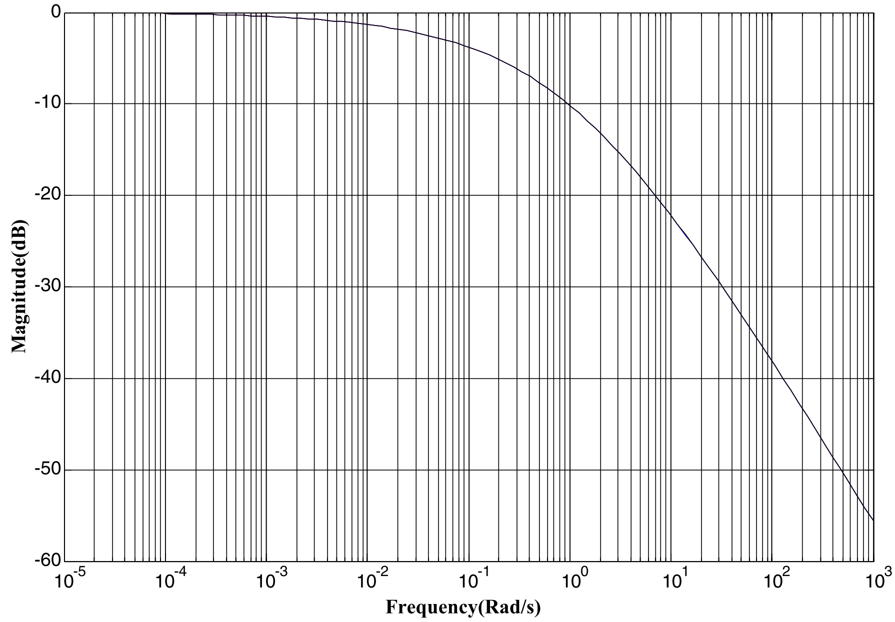

Fig. 5 Magnitude Bode plots of $G(s) = \frac{1}{s^{0.92}+1.74s^{0.46}+1}$, its proposed rational function approximation and the rational function approximation by Oustaloup's method

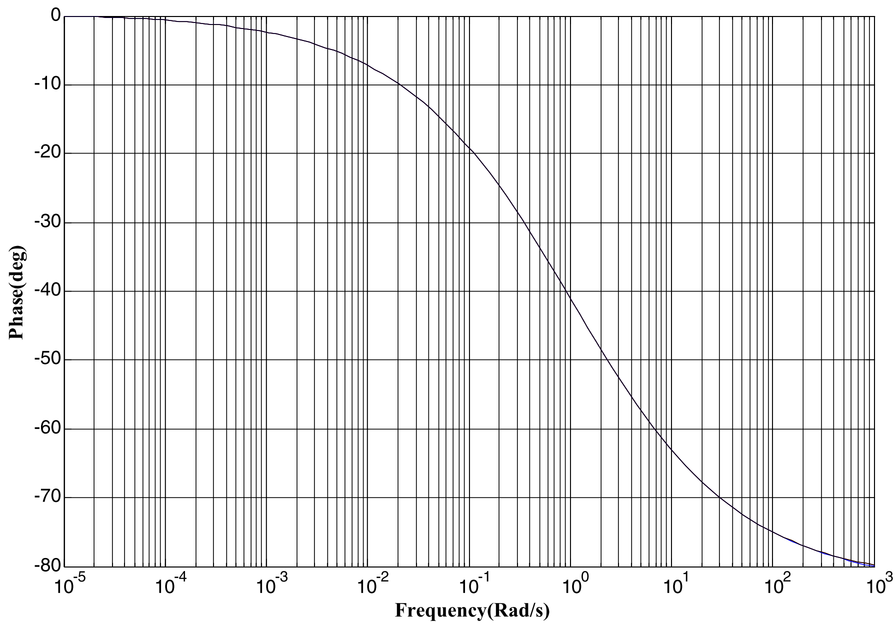

Fig. 6 Phase Bode plots of $G(s) = \frac{1}{s^{0.92}+1.74s^{0.46}+1}$, its proposed rational function approximation and the rational function approximation by Oustaloup's method

from its proposed rational function approximation as

$$x(t) = \sum_{i=1}^{277}\left(\frac{0.158}{\cosh\{0.084(139-i)\}+0.125}\right) \times (1.2)^{(i-139)} 2.028 \exp\left(-0.435(1.2)^{(i-139)}t\right) \times \sin\left(0.9\left((1.2)^{(i-139)}\right)t - 0.6053\right)$$

and from the generalized impulse response function given in [11] as

$$\begin{aligned} x(t) &= (-1.141j)F_{0.46}\left[(-0.87+0.493j), t\right] \\ &\quad + (1.141j)F_{0.46}\left[(-0.87-0.493j), t\right] \\ &= (-1.141j)t^{-0.54}\sum_{n=0}^{100}\frac{(-0.87+0.493j)^n t^{0.46n}}{\Gamma(0.46n+0.46)} \end{aligned}$$

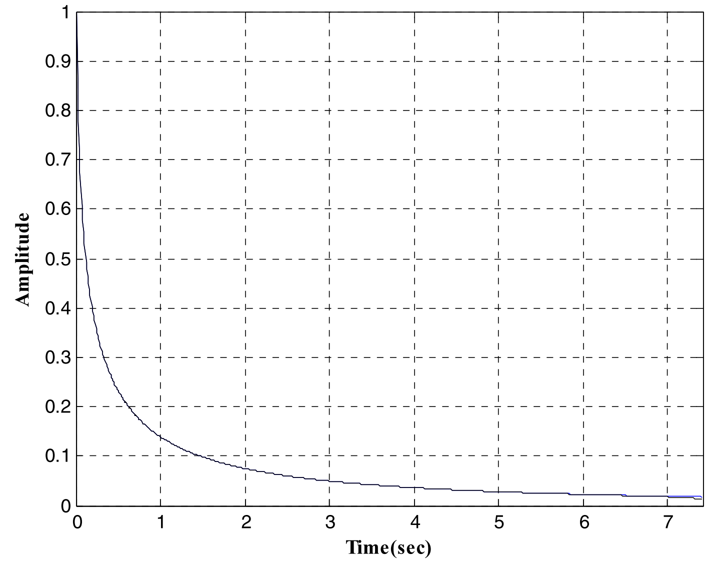

Fig. 7 Impulse responses of the fractional system $\frac{d^{0.92}x(t)}{dt^{0.92}} + 2(0.87)\frac{d^{0.46}x(t)}{dt^{0.46}} + x(t) = e(t)$ from its proposed rational function approximation method and from the generalized impulse response function

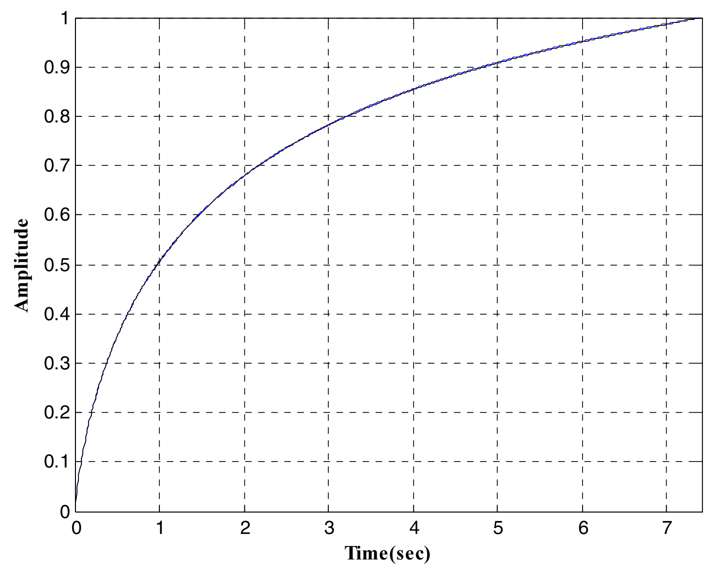

Fig. 8 Step responses of the fractional system $\frac{d^{0.92}x(t)}{dt^{0.92}} + 2(0.87)\frac{d^{0.46}x(t)}{dt^{0.46}} + x(t) = e(t)$ from its proposed rational function approximation method and from the generalized impulse response function

$$+ (1.141j)t^{-0.54} \sum_{n=0}^{100} \frac{(-0.87 - 0.493j)^n t^{0.46n}}{\Gamma(0.46n + 0.46)}$$

Figure 8 shows also the step responses of this generalized oscillation fractional order system obtained from its proposed rational function approximation as

$$x(t) = \sum_{i=1}^{2N-1} k_i \left[1 - \frac{C_i}{\omega_i} \exp(-\alpha\omega_i t) \times \sin\left(\omega_i\left(\sqrt{1-\alpha^2}\right)t + \Phi_{1i}\right)\right]$$

$$x(t) = \sum_{i=1}^{277} \left(\frac{0.158}{\cosh\{0.084(139 - i)\} + 0.125}\right) \times \left[1 - 2.028\exp\left(-0.435(3)^{(i-139)}t\right) \times \sin\left(0.9\left((1.2)^{(i-139)}\right)t + 0.5156\right)\right]$$

and from the Mittag-Leffler function given in [11] as

$$x(t) = (0.5 - 0.882j) \times \left\{1 - E_{0.46}\left[(-0.87 + 0.493j)t^{0.46}\right]\right\}$$

$$+(0.5+0.882j) \times \left\{1-E_{0.46}\left[(-0.87-0.493j)t^{0.46}\right]\right\}$$
$$=(0.5-0.882j) \times \left\{1-\sum_{n=0}^{100}\frac{(-0.87+0.493j)^n t^{0.46n}}{\Gamma(0.46n+1)}\right\}$$
$$+(0.5+0.882j) \times \left\{1-\sum_{n=0}^{100}\frac{(-0.87-0.493j)^n t^{0.46n}}{\Gamma(0.46n+1)}\right\}$$

2.6.2 Example 2

We will take a second numerical example for a generalized oscillation fractional order system represented by the following fundamental linear fractional order differential equation of (5) with $\zeta = 0.35, m = 0,9$ and $\tau_0 = 1$ as

$$\frac{d^{1.8}x(t)}{dt^{1.8}} + 0.7\frac{d^{0.9}x(t)}{dt^{0.9}} + x(t) = e(t)$$

its transfer function is given by

$$G(s) = \frac{X(s)}{E(s)} = \frac{1}{[s^{1.8}+0.7s^{0.9}+1]}$$

For a given frequency band of interest $[0, \omega_H] = [0, 1000 \text{ rad/s}]$ and an approximation ratio $\lambda = 1.15$, the approximation frequency ω_{max}, the number N, the parameters φ and α can be easily calculated from Sect. 2 as:

$$\omega_{max} = 10^6\omega_H = 10^9 \text{ rad/s}$$

$$N = \text{Integer}\left[\frac{\log[10^9]}{\log(1.15)}\right] + 1 = 149$$

$$\varphi = \arctan\left(\frac{\sqrt{1-(0.35)^2}}{0.35}\right) = 1.213$$

$$\alpha = \cos\left(\frac{1.213}{0.9}\right) = 0.221$$

then the relaxation times τ_i, the frequencies ω_i, the residues k_i and the parameters a_i of the approximation are also given from Sect. 2, for $i = 1, 2, \dots, 297$, as:

$$\tau_i = (1.15)^{(149-i)} \quad \text{and} \quad \omega_i = (1.15)^{(i-149)}$$

$$k_i = \frac{0.0492}{\cosh\{0.126(149-i)\}+0.951}$$

$$a_i = -0.1435(1.15)^{(149-i)}$$

Hence, the transfer function approximation of the generalized oscillation fractional order system is given by

$$G(s) = \frac{1}{[s^{1.8}+0.7s^{0.46}+1]} \cong \sum_{i=1}^{297}\left[\frac{0.049}{\cosh\{0.126(149-i)\}+0.951}\right] \times \left[\frac{1-[0.144(1.15)^{(149-i)}]s}{1+1.8[(1.15)^{(149-i)}]s+[(1.323)^{(149-i)}]s^2}\right]$$

To verify the validity of the proposed approach, we have used Oustaloup's method [2] to obtain the rational function approximation of the fractional order differentiator $s^{0.9}$ as

$$s^{0.9} = \frac{N(s)}{D(s)} = (1.26)10^{-10}\prod_{i=-150}^{150}\frac{(1+\frac{s}{2.5287\cdot 10^9(1.1659)^i})}{(1+\frac{s}{2.322\cdot 10^9(1.1659)^i})}$$

Then we have derived the rational function approximation of the transfer function of the generalized oscillation fractional order system, with the same order as the proposed rational approximation method in the frequency band $[0, \omega_H] = [0, 1000 \text{ rad/s}]$, as

$$G(s) = \frac{1}{[s^{1.8}+0.7s^{0.9}+1]} = \frac{1}{[(s^{0.9})^2+0.7s^{0.9}+1]} = \frac{1}{[\frac{N(s)}{D(s)}]^2+0.7[\frac{N(s)}{D(s)}]+1}$$

Figures 9 and 10 show the bode plots of the fractional order system transfer function and its proposed rational function approximation along with the rational function approximation by Oustaloup's method. We can easily see that they are all quite overlapping over the frequency band of interest.

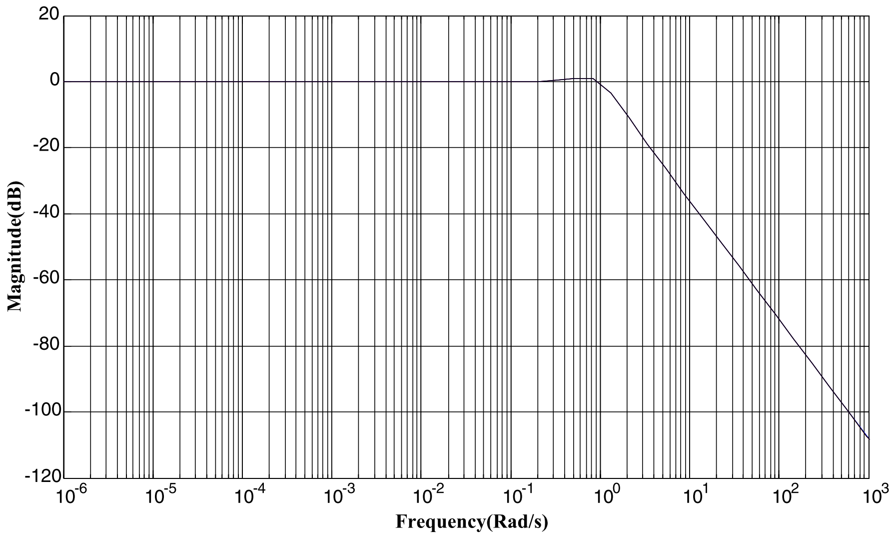

Fig. 9 Magnitude Bode plots of $G(s) = \frac{1}{s^{1.8}+0.7s^{0.9}+1}$, its proposed rational function approximation and the rational function approximation by Oustaloup's method

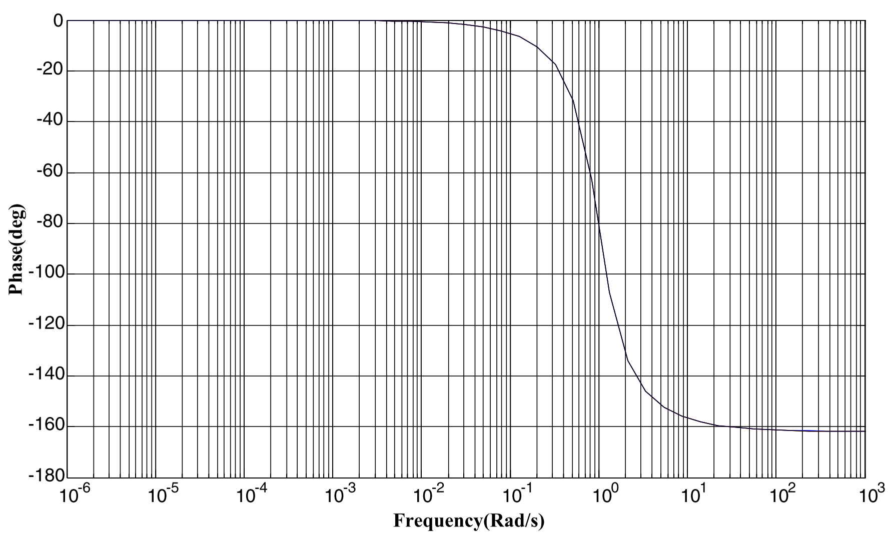

Fig. 10 Phase Bode plots of $G(s) = \frac{1}{s^{1.8}+0.7s^{0.9}+1}$, its proposed rational function approximation and the rational function approximation by Oustaloup's method

Figure 11 shows the impulse responses of this generalized oscillation fractional order system obtained from its proposed rational function approximation as

$$x(t) = \sum_{i=1}^{297}\left(\frac{0.049(1.15)^{(i-149)}}{\cosh\{0.126(149-i)\}+0.951}\right) \times \left(1.068\exp\left(-0.221(1.15)^{(i-149)}t\right) \times \sin\left(0.975\left((1.15)^{(i-149)}\right)t - 0.135\right)\right)$$

and from the generalized impulse response function given in [11] as

$$\begin{aligned} x(t) &= (-0.534j)F_{0.9}\left[(-0.35+0.938j), t\right] + (0.534j)F_{0.9}\left[(-0.35-0.938j), t\right] \\ &= (-0.534j)t^{-0.1}\sum_{n=0}^{100}\frac{(-0.35-0.938j)^n t^{0.9n}}{\Gamma(0.9n+0.9)} + (0.534j)t^{-0.1}\sum_{n=0}^{100}\frac{(-0.35-0.938j)^n t^{0.9n}}{\Gamma(0.9n+0.9)} \end{aligned}$$

Figure 12 shows also the step responses of this generalized oscillation fractional order system obtained from its proposed rational function approximation as

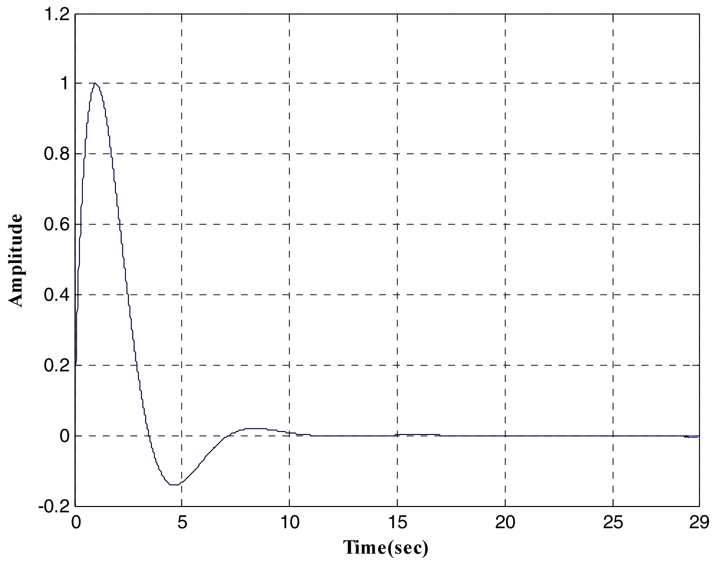

Fig. 11 Impulse responses of the fractional system $\frac{d^{1.8}x(t)}{dt^{0.9}} + 2(0.35)\frac{d^{0.9}x(t)}{dt^{0.9}} + x(t) = e(t)$ from its proposed rational function approximation method and from the generalized impulse response function

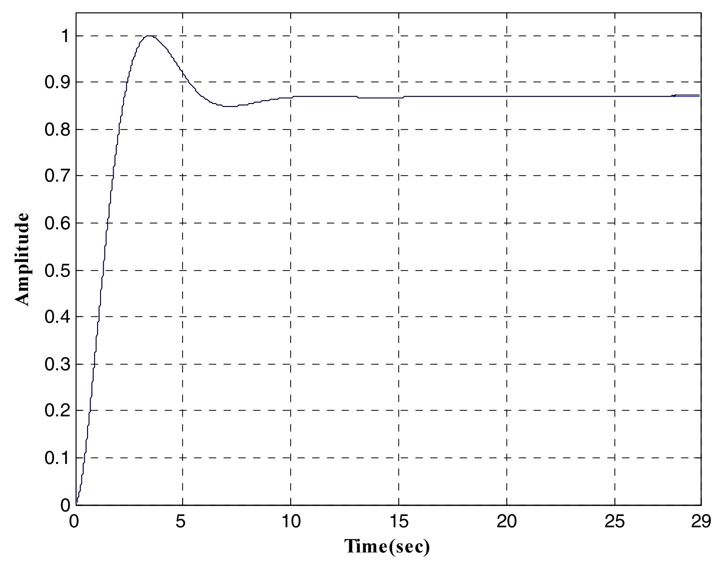

Fig. 12 Step responses of the fractional system $\frac{d^{1.8}x(t)}{dt^{1.8}} + 2(0.35)\frac{d^{0.9}x(t)}{dt^{0.9}} + x(t) = e(t)$ from its proposed rational function approximation method and from the generalized impulse response function

$$x(t) = \sum_{i=1}^{2N-1} k_i \left(1 - \frac{C_i}{\omega_i}\exp(-\alpha\omega_i t) \times \sin\big(\omega_i\big(\sqrt{1-\alpha^2}\big)t + \Phi_{1i}\big)\right)$$

$$x(t) = \sum_{i=1}^{297}\left(\frac{0.049}{\cosh\{0.126(149-i)\}+0.951}\right) \times \big(1 - 1.068\exp\big(-0.221(1.15)^{(i-149)}t\big) \times \sin\big(0.975\big((1.15)^{(i-149)}\big)t + 1.213\big)\big)$$

and from the Mittag-Leffler function given in [11] as

$$\begin{aligned} x(t) &= (0.5 - 0187j) \\ &\quad \times \left\{1 - E_{0.9}\big[(-0.35 + 0.937j)t^{0.46}\big]\right\} \\ &\quad + (0.5 + 0.187j) \\ &\quad \times \left\{1 - E_{0.9}\big[(-0.35 - 0.937j)t^{0.46}\big]\right\} \\ &= (0.5 - 0.187j) \\ &\quad \times \left\{1 - \sum_{n=0}^{100}\frac{(-0.35+0.937j)^n t^{0.9n}}{\Gamma(0.9n+1)}\right\} \end{aligned}$$

$$+(0.5+0.187j)$$
$$\times\left\{1-\sum_{n=0}^{100}\frac{(-0.350.937j)^n t^{0.9n}}{\Gamma(0.9n+1)}\right\}$$

3 Conclusion

In this paper, we have presented an effective method for rational function approximation of the irrational function given by $G(s)=\frac{X(s)}{E(s)}=\frac{1}{[(\tau_0 s)^{2m}+2\zeta(\tau_0 s)^m+1]}$ for $0<m<1$ representing the transfer function of the fundamental linear fractional order differential equation $(\tau_0)^{2m}\frac{d^{2m}x(t)}{dt^{2m}}+2\zeta(\tau_0)^m\frac{d^m x(t)}{dt^m}+x(t)=e(t)$. From this rational function, closed form of the impulse and step responses of this type of systems are derived and simple analog circuit realization are also obtained. Illustrative examples have been treated to demonstrate the usefulness and efficiency of the approximation method. The frequency responses obtained have been compared to the ones where the approximation of the fractional order differentiator has been used instead of the approximation of the whole transfer function as proposed in this work. The time responses have also been compared to the responses obtained by the Mittag-Leffler method. These comparisons have been made to show the accuracy of the proposed approximation technique.

We point out that this work is the continuation of a previous work by the first author in [9]. All the approximations obtained of the transfer functions of the different fundamental linear fractional order differential equations are part of the resolution of the fractional order differential equations framework where these functions will be used as fundamental functions very similar to that used in the study of ordinary linear differential equations.

References

1. Oustaloup, A.: Systèmes Asservis Linéaires d'Ordre Fractionnaire: Théorie et Pratique. Masson, Paris (1983)
2. Petras, I., Podlubny, I., O'Leary, P., Dorcak, L., Vinagre, B.M.: Analogue realization of fractional order controllers. Fakulta Berg, TU Kosice (2002)
3. Hilfer, R. (ed.): Applications of Calculus in Physics. World Scientific, Singapore (2000)
4. Sabatier, J., et al. (eds.): Advances in Fractional Calculus: Theoretical Development and Applications in Physics and Engineering. Springer, Dordrecht (2007)
5. Miller, K.S., Ross, B.: An Introduction to the Fractional Calculus and Fractional Differential Equations. Wiley, New York (1993)
6. Podlubny, I.: Fractional Differential Equations. Academic Press, San Diego (1999)
7. Kilbas, A.A., Srivastava, H.M., Trujillo, J.J.: Theory and Applications of Fractional Differential Equations. Elsevier, Amsterdam (2006)
8. Aoun, M., Malti, R., Levron, F., Oustaloup, A.: Numerical simulations of fractional systems: an overview of existing methods and improvements. Nonlinear Dyn. **38**, 117–131 (2004)
9. Charef, A.: Modeling and analog realization of the fundamental linear fractional order differential equation. Nonlinear Dyn. **46**, 195–210 (2006)
10. Diethelm, K., Walz, G.: Numerical solution of fractional order differential equations by extrapolation. Numer. Algorithms **16**, 231–253 (1997)
11. Hartley, T.T., Lorenzo, C.F.: A solution of the fundamental linear fractional order differential equation. NASA TP-1998-208693, December 1998
12. Diethelm, K., Ford, N.J., Freed, A.D., Luchko, Y.: Algorithms for the fractional calculus: a selection of numerical methods. Comput. Methods Appl. Mech. Eng. **194**, 743–773 (2005)
13. Momani, S., Al-Khaled, K.: Numerical solutions for systems of fractional differential equations by the decomposition method. Appl. Math. Comput. **162**, 1351–1365 (2005)
14. Kumar, P., Agrawal, O.P.: An approximate method for numerical solution of fractional differential equations. Signal Process. **86**, 2602–2610 (2006)
15. Momani, S., Odibat, Z.: Numerical comparison of methods for solving linear differential equations of fractional order. Chaos Solitons Fractals **31**, 1248–1255 (2007)
16. Arikoglu, A., Ozkol, I.: Solution of fractional differential equations by using differential transform method. Chaos Solitons Fractals **34**, 1473–1481 (2007)
17. Bonilla, B., Rivero, M., Trujillo, J.J.: On systems of linear fractional differential equations with constant coefficients. Appl. Math. Comput. **187**, 68–78 (2007)
18. Cole, K.S., Cole, R.H.: Dispersion and absorption in dielectrics, alternation current characterization. J. Chem. Phys. **9** (1941)
19. MacDonald, J.R.: Impedance Spectroscopy. Wiley, New York (1987)
20. Matignon, D.: Stability results for fractional differential equation with application to control processing. In: Proceedings of the Symposium on Control, Optimization and Supervision, CESA'96, Lille, France, pp. 963–968 (1996)
21. Malti, R., Moreau, X., Khemane, F.: Resonance of fractional transfer functions of the second kind. In: Proceedings of IFAC Workshop on Fractional Differentiation and its Applications, Ankara, Turkey, 5–7 November 2008 (2008)
22. Kuo, B.C.: Automatic Control Systems. Prentice-Hall, Englewood Cliffs (1981)