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Hopf bifurcation control for stochastic dynamical system with nonlinear random feedback method

Yong Xu · Shaojuan Ma · Huiqing Zhang

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Abstract Hopf bifurcation control in nonlinear stochastic dynamical system with nonlinear random feedback method is studied in this paper. Firstly, orthogonal polynomial approximation is applied to reduce the controlled stochastic nonlinear dynamical system with nonlinear random controller to the deterministic equivalent system, solvable by suitable numerical methods. Then, Hopf bifurcation control with nonlinear random feedback controller is discussed in detail. Numerical simulations show that the method provided in this paper is not only available to control the stochastic Hopf bifurcation in nonlinear stochastic dynamical system, but is also superior to the deterministic nonlinear feedback controller.

Keywords Stochastic dynamical system · Random parameter · Random feedback control method · Orthogonal polynomial approximation

Y. Xu (✉) · H. Zhang
Department of Applied Mathematics, Northwestern Polytechnical University, Xi'an 710072, China
e-mail: hsux3@nwpu.edu.cn

S. Ma
School of Information & Computation Science, North University for Nationalities, Yinchuan 750021, China

1 Introduction

Owning to the uncertain factors of external environment, manufacture, material and installation, some parameters in practical model are not constant and will be characterized as bound random parameters. The study of this kind of model is more accurate than the deterministic one. We maybe find that some phenomena are more prominent and can be neglected in deterministic study, especially the sensitive behavior. To control the sensitive behavior in this kind of stochastic system has become important.

The bifurcation in nonlinear dynamical system results form the variation of dynamical behaviors as the system parameters are changed: the saddle-node bifurcation can lead to a series of destructive dynamical behaviors, such as jump of the amplitude and lag; the period-doubling bifurcation can lead to the chaos state; the Hopf bifurcation can change the stability at the equilibrium point. Controlling (and anti-controlling) the bifurcation has been given much attention in recent years. The aim of bifurcation control is to design a controller to modify the bifurcation properties of a given nonlinear system, thereby achieving desirable dynamical behaviors. Typical objectives of bifurcation control include stabilizing the unstable bifurcation solution or branch [1–5]; delaying the onset of an inherent bifurcation [6, 7]; changing the parameter value of an existing bifurcation point [8, 9]; producing a new bifurcation by designing the system parameters [10–16]; changing the equilibrium; modifying the shape or type of a bifurcation [17]; monitoring the multiplicity, amplitude or frequency of the limit cycles [18, 19], and so on. The methods of bifurcation control employ linear and nonlinear state feedback controls [20, 21], time-delayed feedback [19], apply a washout-filter-aided dynamical feedback controller [16, 22, 23], use harmonic balance approximations [24–26] and utilize quadratic invariants in normal forms [1, 27–30].

The theories of bifurcation control in deterministic system have already been applied to many fields of engineering, biological and chemical systems. There are some summaries of bifurcation control [31]. However, these summaries concern only the bifurcation control in deterministic systems. The methods and theory of bifurcation control in stochastic systems are not worked out in full, especially in stochastic systems with random parameters. Orthogonal polynomial approximation [32–34] is based on the expansion theory of orthogonal polynomials without the limitation of small perturbations, which is shown to improve the study of the dynamical behavior [35] in stochastic systems with random parameters. The works [36–38] have illustrated that bifurcations and chaos in stochastic systems with random parameters are different from the deterministic system with their own features. Wu [39] has discussed in detail the chaos and its control via orthogonal polynomial approximation. So far, no study has explored the bifurcation control in this kind of a stochastic system. In this paper, the Hopf bifurcation control in nonlinear dynamical system with random parameter will be discussed.

This paper is organized as follows. In Sect. 2, transformation of stochastic controlled dynamical system into its equivalent deterministic one by orthogonal polynomial approximation is obtained. As an example, the Hopf bifurcation control with random feedback method in stochastic van der Pol system is discussed in Sect. 3. In Sect. 4 we draw some conclusions about the influence of random feedback method on Hopf bifurcation control in stochastic van der Pol system.

2 Orthogonal polynomial approximation for stochastic controlled dynamical system

Taking stochastic van der Pol system as an example,

$$\begin{cases} \dot{x} = -y, \\ \dot{y} = \mu y - ax^2 y + bx, \end{cases} \tag{1}$$

where μ is a random parameter, a and b are deterministic parameters. Suppose that μ can be expressed as

$$\mu = \bar{\mu} + \delta u, \tag{2}$$

where $\bar{\mu}$ and δ are the statistic parameters of μ while δ is regarded as intensity of the random parameter μ, and U is the random variable defined on $(-\infty, +\infty)$ with the probability density function $\rho_U(u)$.

Suppose that the nonlinear random feedback controller $C(x, y)$ usually can be written as $C(x, y) = (A + A\delta_2 u)x^2 y$, where A is feedback intensity and δ_2 is random intensity of random controller. Random variable U is independent and identically distributed with the random variable in (2). Taking this controller into the right side of the second equation in (1), the controlled van der Pol system can be written as

$$\begin{cases} \dot{x} = -y, \\ \dot{y} = (\bar{\mu} + \delta u)y - ax^2 y + bx + A(1 + \delta_2 u)x^2 y, \end{cases} \tag{3}$$

It follows from the orthogonal polynomial approximation that the responses of controlled system (3) can be expressed approximately by the following series under the condition of the convergence in mean square:

$$\begin{cases} x(t, u) = \sum_{i=0}^{M} x_i(t) P_i(u), \\ y(t, u) = \sum_{i=0}^{M} y_i(t) P_i(u), \end{cases} \tag{4}$$

where $x_i(t) = \int_{-\infty}^{\infty} \rho_U(u) x(t, u) P_i(u)\, du$, $y_i(t) = \int_{-\infty}^{\infty} \rho_U(u) y(t, u) P_i(u)\, du$, $P_i(u)$ is the ith orthogonal polynomial, and M is the largest order of the polynomials we have taken.

The orthogonality of polynomial can be expressed as

$$\int_{-\infty}^{\infty} \rho_U(u) P_i(u) P_j(u)\, du = \begin{cases} \delta_i, & i = j, \\ 0, & i \neq j, \end{cases} \tag{5}$$

where δ_i is the Kronecker delta function. The recurrent formula for orthogonal polynomials is

$$u P_i(u) = \alpha_i P_{i+1}(u) + \beta_i P_i(u) + \gamma_i P_{i-1}(u),$$
$$\gamma_i \neq 0,\ P_{-1}(u) = 0,\ P_0(u) = 1, \tag{6}$$

where

$$\alpha_i = \frac{a_{i-1}\sqrt{\delta_i}}{a_i\sqrt{\delta_{i-1}}}, \qquad \beta_i = \frac{1}{\delta_i}\int_{-\infty}^{\infty} u\rho_U(u)P_i^2(u)\,du,$$
$$\gamma_i = \frac{a_i\sqrt{\delta_{i+1}}}{a_{i+1}\sqrt{\delta_i}}.$$

Substituting (4) into (3), we have

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_{i=0}^{M} x_i(t)P_i(u)\right) = -\sum_{i=0}^{M} y_i(t)P_i(u), \\ \frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_{i=0}^{M} y_i(t)P_i(u)\right) = \bar{\mu}\sum_{i=0}^{M} y_i(t)P_i(u) \\ \quad + \delta u\sum_{i=0}^{M} y_i(t)P_i(u) \\ -(a - A - A\delta_2 u)\left(\sum_{i=0}^{M} x_i(t)P_i(u)\right)^2 \sum_{i=0}^{M} y_i(t)P_i(u) \\ \quad + b\sum_{i=0}^{M} x_i(t)P_i(u), \end{cases} \tag{7}$$

By the recurrent formulas, the triple product polynomial of (7) can be further reduced into a linear combination of related single polynomials. By denoting the coefficient of $P_i(u)$ in the linear combination as $S_i(t)$, we have

$$\left(\sum_{i=0}^{M} x_i(t)P_i(u)\right)^2 \sum_{i=0}^{M} y_i(t)P_i(u) = \sum_{i=0}^{3M} S_i(t)P_i(u). \tag{8}$$

where $S_i(t)$ $(i = 0, 1, 2, \dots, 3M)$ can be derived through computer algebraic system, such as MAPLE [40]. In this paper we choose a kind of bound random variable defined on $[-1, 1]$ with arch-like probability density function $p_U(u)$:

$$p_U(u) = \begin{cases} (2/\pi)\sqrt{1-u^2} & \text{as } |u| \le 1 \\ 0 & \text{as } |u| > 1. \end{cases}$$

Corresponding to this random variable, the orthogonal polynomial of (4) is chosen as the second Chebyshev polynomial [41]. The coefficients of the recurrent equation (6) are: $\alpha_i = \gamma_i = 1/2$, $\beta_i = 0$.

Substituting (6) and (8) into (7), we have

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_{i=0}^{M} x_i(t)P_i(u)\right) = -\sum_{i=0}^{M} y_i(t)P_i(u), \\ \frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_{i=0}^{M} y_i(t)P_i(u)\right) = -(a - A)\sum_{i=0}^{3M} S_i(t)P_i(u) \\ \quad + b\sum_{i=0}^{M} x_i(t)P_i(u) + \bar{\mu}\sum_{i=0}^{M} y_i(t)P_i(u) \\ \quad + \delta/2\sum_{i=0}^{M}\big(P_i(u)\big[y_{i-1}(t) + y_{i+1}(t)\big]\big) \\ \quad + A\delta_2/2\sum_{i=0}^{3M}\big(P_i(u)\big[S_{i-1}(t) + S_{i+1}(t)\big]\big), \end{cases} \tag{9}$$

Multiplying both sides of (9) by $P_i(u)$, $i = 0, 1, 2, \dots, M$, and taking expectation with respect to U, owing to the orthogonality of orthogonal polynomials, we can finally obtain the equivalent deterministic equation (10):

$$\begin{cases} \dot{x}_0 = -y_0, \\ \dot{y}_0 = \bar{\mu}y_0 + \delta[y_1]/2 - (a - A)S_0 \\ \quad + A\delta_2[S_1]/2 + bx_0, \\ \dot{x}_1 = -y_1, \\ \dot{y}_1 = \bar{\mu}y_1 + \delta[y_0 + y_2]/2 - (a - A)S_4 \\ \quad A\delta_2[S_0 + S_2]/2 + bx_4, \\ \dots, \\ \dot{x}_4 = -y_4, \\ \dot{y}_4 = \bar{\mu}y_4 + \delta[y_3]/2 - (a - A)S_4 \\ \quad + A\delta_2[S_3 + S_5]/2 + bx_4. \end{cases} \tag{10}$$

This is the equivalent nonlinear deterministic system reduced by the Chebyshev polynomial approximation for the controlled stochastic van der Pol system with a random variable u of an arch-like pdf. When $M \to \infty$, the responses in (4) are strictly equivalent to $x(t, u)$ of the stochastic dynamical system. Otherwise, if N is finite, (4) is just approximate value. In the following numerical analysis, we take $M = 4$ and obtain the numerical solution $x_i(t)$, $y_i(t)$ $(i = 0, 1, \dots, 4)$ of (10) by available effective numerical methods. The approximate random response of the original controlled stochastic van der Pol system can be expressed as

$$\begin{cases} x(t,u) \approx \sum_{i=0}^{4} x_i(t) P_i(u), \\ y(t,u) \approx \sum_{i=0}^{4} y_i(t) P_i(u), \end{cases}$$

and the ensemble mean response of the controlled stochastic van der Pol system can be obtained as

$$\begin{cases} E\big[x(t,u)\big] \approx \sum_{i=0}^{4} x_i(t) E\big[U_i(u)\big] = x_0(t) \\ E\big[y(t,u)\big] \approx \sum_{j=0}^{4} y_j(t) E\big[U_j(u)\big] = y_0(t) \end{cases} \tag{11}$$

The sample response of controlled mean parameter system can be obtained when $u = 0$:

$$\begin{cases} x(t,0) \approx \sum_{i=0}^{4} x_i(t) U_i(0) = x_0(t) - x_2(t) + x_4(t) \\ y(t,0) \approx \sum_{j=0}^{4} y_j(t) U_j(0) = y_0(t) - y_2(t) + y_4(t) \end{cases} \tag{12}$$

The initial condition of (1) with deterministic parameters is defined as $x_0 = x(0)$, $y_0 = y(0)$. Next, the Hopf bifurcation of the controlled stochastic van der Pol system with random parameters will be explored by comparing three kinds of responses: the response of deterministic controlled van der Pol system (DR, for short), the sample response of controlled mean parameter system (SRM, for short) and the ensemble mean response (EMR, for short). In this paper, for the intensity δ small values are taken, so we take the initial conditions of the controlled deterministic equivalent system (10) the same as the initial conditions of the controlled deterministic system, namely

$$x_0 = x_0(0), \qquad y_0 = y_0(0),$$
$$x_i(0) = y_i(0) = 0 \quad (i = 1, 2, 3, 4).$$

In this text, we take

$$\mathbf{x}(0) = [3.0, 0, \ldots, 0, 0]^T$$
$$\mathbf{y}(0) = [2.0, 0, \ldots, 0, 0]^T$$
$$x_0 = 3.0,\ y_0 = 2.0.$$

3 Hopf bifurcation control with nonlinear random feedback method

It is well known that in order to control the Hopf bifurcation, the nonlinear feedback controller can be used to control the amplitude of the limit cycle. In this section we will discuss the influence of nonlinear random feedback controller on the Hopf bifurcation. We suppose that the random intensity of the controller varies from 0 to 0.25. Next we analyze the Hopf bifurcation control with nonlinear random feedback controller based on deterministic controlled system (10). The parameters considered in this section are: $\bar{\mu} = 0.1$, $b = 1.0$, $a = 1.0$, $\delta = 0.1$.

As $A = -3$, $\delta_2 = 0.0$, the controller is deterministic nonlinear feedback controller. The phase trajectories of SRM and EMR for the controlled stochastic van der Pol system are shown in Fig. 1(a). The amplitudes of two limit cycles are equal to 0.31. Increasing the random intensity of the controller, $\delta_2 = 0.1$, Fig. 1(b) shows that the amplitude of the limit cycle for EMR in controlled stochastic van der Pol system has declined, and the amplitude of the limit cycle for SRM has decreased indistinctively. As we continue to increase $\delta_2 = 0.2$, the amplitudes of two limit cycles are all changing, which is shown in Fig. 1(c), especially for EMR of controlled stochastic van der Pol system. As we increase the random intensity of the controller up to 0.25, the amplitudes of two limit cycles are becoming smaller, and the limit cycle of stochastic controlled van der Pol system is becoming dramatically smaller, which is shown in Fig. 1(d). Figure 1(e) shows the time history diagrams of phase trajectories of EMR for the stochastic controlled van der Pol system for different random intensity of the random controller. Figure 1(f) is a local portrait of Fig. 1(e).

From Fig. 1 we can find that the effect of the random feedback controller on decreasing the amplitude of the limit cycle is available. In other words, we can control the stochastic Hopf bifurcation with nonlinear random feedback method.

Two kinds of controller are:

$$\begin{aligned} &C_1(x,y) = A_1 x^2 y, \\ &C_2(x,y) = A_2(1 + \delta_2 u) x^2 y, \end{aligned} \tag{13}$$

where A_1 and A_2 are feedback intensities.

If $\delta_2 = 0$, $A_1 = A_2$, then $C_1 = C_2$. As $A_1 = -4$ and $A_2 = -3$, $\delta_2 = 0.1$, the time history diagrams of the EMR of controlled stochastic van der Pol system

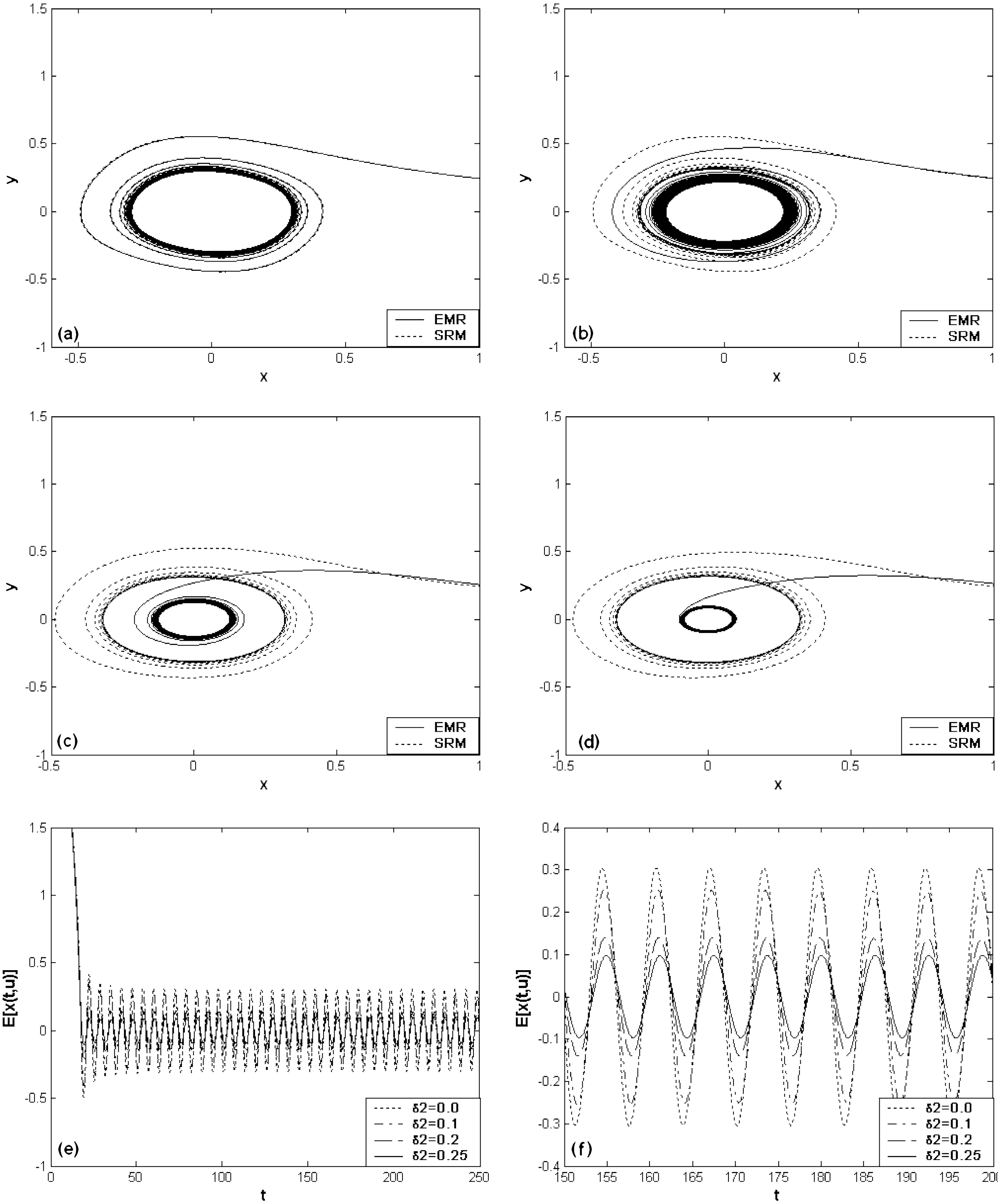

Fig. 1 Phase portraits (**a**, **b**, **c**, **d**) and time history diagrams (**e**, **f**) of the controlled stochastic van der Pol system: (**a**) $\delta_2 = 0.0$, (**b**) $\delta_2 = 0.1$, (**c**) $\delta_2 = 0.2$, (**d**) $\delta_2 = 0.25$

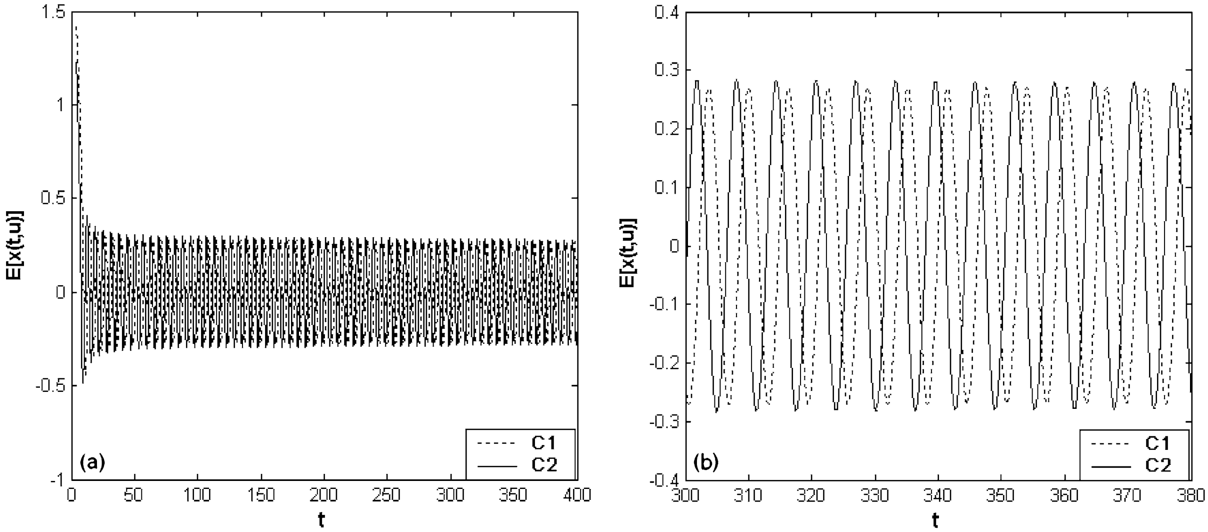

Fig. 2 Time history diagrams of the controlled stochastic van der Pol system: (**a**) $A_1 = -4$, (**b**) $A_2 = -3$, $\delta_2 = 0.1$

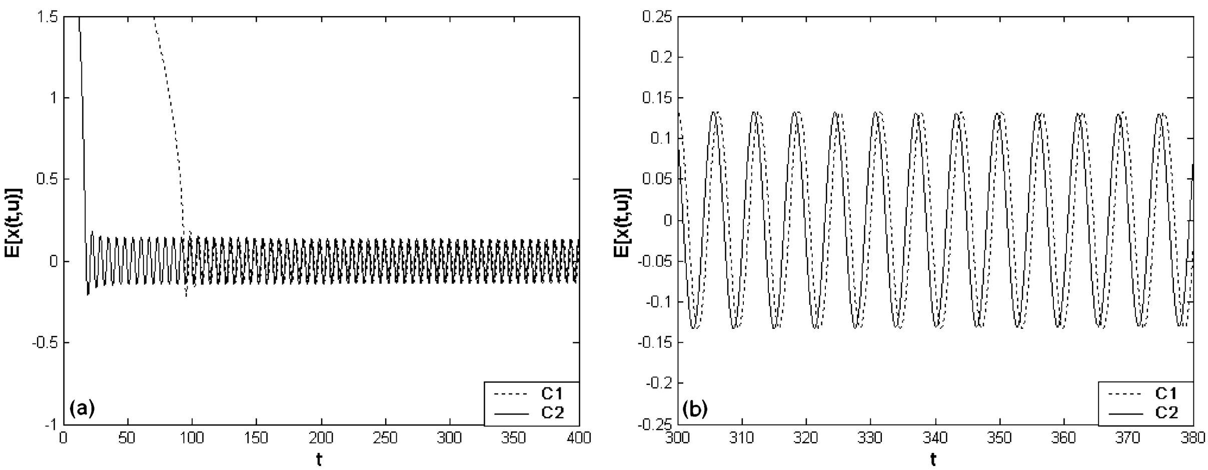

Fig. 3 Time history diagrams of the controlled stochastic van der Pol system: (**a**) $A_1 = -20$, (**b**) $A_2 = -3$, $\delta_2 = 0.2$

with controllers C_1 and C_2 are shown in Fig. 2(a). Figure 2(b) is a local portrait of Fig. 2(a). From Fig. 2 we see that the amplitudes of limit cycles of the controlled stochastic van der Pol system with the different controllers are decreased in the same degree.

By continuing to increase the feedback intensity of controller C_1, $A_1 = -20$, and random intensity of the controller C_2, $\delta_2 = 0.2$, we find from Fig. 3 that the control effect with the controller C_1 is the same as of the controller C_2. If we increase the feedback intensity of the controller C_1 considerably, $A_1 = -40$, and increase the random intensity of the controller C_2 a little, $\delta_2 = 0.25$, the time history diagrams of the EMR of controlled stochastic van der Pol system with controllers C_1 and C_2 are shown in Fig. 4(a). Figure 4(b) is a local portrait of Fig. 4(a). From Fig. 4 we see that the amplitudes of the limit cycles of the controlled stochastic van der Pol system with the different controllers are decreased in the same degree.

As stated previously, the Hopf bifurcation can be controlled both by the deterministic nonlinear feedback controller and the random nonlinear feedback controller. To note, we must adjust the feedback intensity A_1 in great degree to achieve the control purpose,

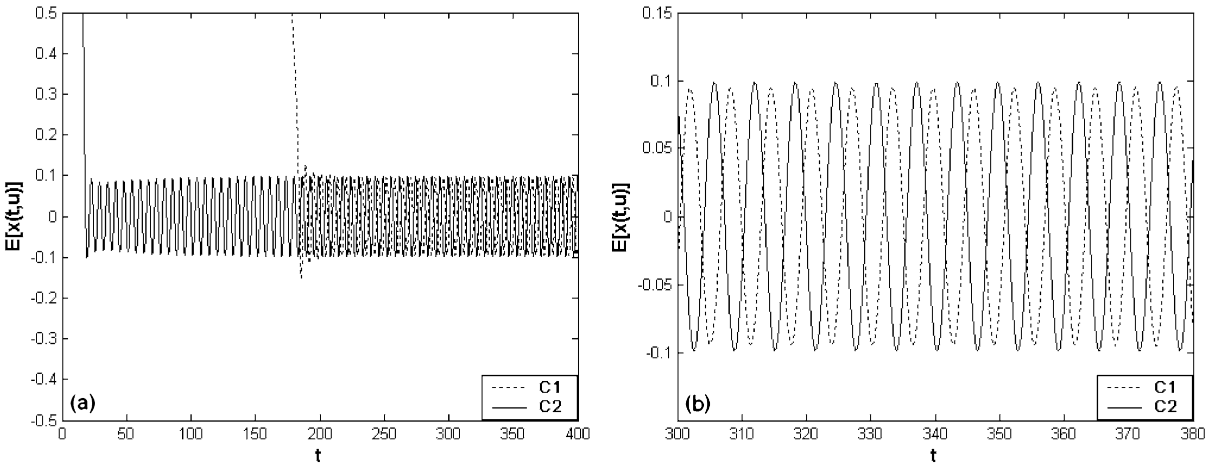

Fig. 4 Time history diagrams of the controlled stochastic van der Pol system: (**a**) $A_1 = -40$, (**b**) $A_2 = -3$, $\delta_2 = 0.25$

but to control the stochastic Hopf bifurcation we will change the random intensity δ_2 a little as the feedback intensity A_2 is small. Therefore, to control the stochastic Hopf bifurcation, the nonlinear random controller is superior to the nonlinear deterministic controller.

4 Summary

We have applied nonlinear random feedback method to control the stochastic Hopf bifurcation in van der Pol system with random parameters. Firstly, the controlled stochastic van der Pol system is reduced to its equivalent deterministic system by orthogonal polynomial approximation. Then the responses of the equivalent deterministic controlled system can be readily obtained by conventional Runge–Kutta method. By means of numerical simulations, we find that nonlinear random feedback method to control the stochastic Hopf bifurcation in stochastic van der Pol system is available. Besides, comparing with nonlinear deterministic feedback controller, we must adjust the feedback intensity in nonlinear deterministic feedback controller in great degree to control the stochastic Hopf bifurcation, but to achieve the same control effect we need to change the random intensity of nonlinear random feedback controller only a little as the feedback intensity is very small. Therefore, combining with orthogonal polynomial approximation, the random feedback method may be also helpful for further controlling other nonlinear phenomena in nonlinear stochastic dynamical system.

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