Nonlinear Dyn (2011) 64: 293–304
DOI 10.1007/s11071-010-9861-9

ORIGINAL PAPER

Study of oscillators with a non-negative real-power restoring force and quadratic damping

Ivana Kovacic · Zvonko Rakaric

Received: 11 June 2010 / Accepted: 3 October 2010 / Published online: 26 October 2010

Abstract Oscillators with a non-negative real-power restoring force and quadratic damping are considered in this paper. The equation of motion is transformed into a linear first-order differential equation for the kinetic energy. The expressions for the energy-displacement function are derived as well as the closed form exact solutions for the relationship between subsequent amplitudes. They are expressed in terms of incomplete Gamma functions. On the basis of these results, expressions for the phase trajectories and the loci of maximal velocities are obtained. It is also demonstrated that the time difference between two consecutive relative maxima and minima of the displacement can both increase and decrease with time.

Keywords Non-negative real-power restoring force · Quadratic damping · Energy-displacement function · Amplitudes · Phase trajectories · Incomplete Gamma functions

1 Introduction

One of the basic damping mechanisms is quadratic damping, which occurs, for example, when an immersed object moves through a fluid at relatively high Reynolds numbers [1, 2]. The corresponding drag force is proportional to the square of the velocity

$$R(\dot{x}) = c\,\mathrm{sgn}(\dot{x})\dot{x}^2, \tag{1}$$

where the constant c depends on the object geometry and the fluid properties.

Of interest here is to consider free oscillations of the systems having a single degree of freedom, with the damping force modelled by (1) and with the purely non-linear restoring force

$$F(x) = k\,\mathrm{sgn}(x)|x|^\alpha, \tag{2}$$

where k and α are a positive and non-negative real constants, respectively. The model (2) includes the originally multi-term restoring force tuned to have a quasi-zero stiffness characteristic [3–5], as well as those corresponding to a fractional-order restoring force [6, 7], practical examples of which can be found, for instance, in [8].

It should be noted that the Sign functions in (1) and (2) are

$$\mathrm{sgn}(\dot{x}) = \begin{cases} 1, & \dot{x} > 0 \\ 0, & \dot{x} = 0 \\ -1, & \dot{x} < 0, \end{cases}$$
$$\mathrm{sgn}(x) = \begin{cases} 1, & x > 0 \\ 0, & x = 0 \\ -1, & x < 0. \end{cases} \tag{3}$$

I. Kovacic (✉) · Z. Rakaric
Department of Mechanics, Faculty of Technical Sciences, University of Novi Sad, 21215 Novi Sad, Serbia
e-mail: ivanakov@uns.ac.rs

Thus, the equation of motion of the system considered having the mass m is

$$m\ddot{x} + c\,\mathrm{sgn}(\dot{x})\dot{x}^2 + k\,\mathrm{sgn}(x)|x|^{\alpha} = 0. \tag{4}$$

The initial conditions are assumed to be

$$x(0) = X_0, \quad \dot{x}(0) = 0, \tag{5}$$

where dots stand for the differentiation with respect to time t and X_0 is assumed to be a positive real constant.

Numerous authors have performed quantitative analyses of oscillators with restoring force non-linearities and small quadratic damping by obtaining their approximate solution. Some of the methods used have been: the method of multiple scales [1], the averaging method (Krylov–Bogoliubov method) [9] and the elliptic Krylov–Bogoliubov method [10]. In addition, some techniques have been developed for converting the given equation with a certain type of non-linearity both in the restoring force and damping force into ones that have exact closed form solutions (see, for example, [11], and references cited therein).

On the other hand, the qualitative analysis given in [12] for this type of non-linear damping includes an analytical proof that the energy of the system continuously decreases along the phase path, i.e. the energy is continuously withdrawn from the system, causing the corresponding phase path to spiral in towards the equilibrium point $x = \dot{x} = 0$. In [13], a system with quadratic damping but a linear restoring force was studied qualitatively by analyzing the phase plane. It was shown that the phase trajectories in the upper half of the phase plane are symmetric to the phase trajectories in the lower half with respect to the equilibrium point. Klotter [14] considered oscillators with purely non-linear integer-power or linear-plus-non-linear integer-power restoring forces. He transformed the equations of motion into linear first-order differential equations for the square of the velocity, which is actually twice the kinetic energy, to derive the transcendental relationship between two consecutive maximal displacement amplitudes. Cveticanin [15] used a similar approach for a system with a linear restoring force to derive the exact analytical expressions for the energy-displacement function and the amplitudes of vibration in which the Lambert W-function occurs.

The aim of this paper is to carry out a qualitative analysis of the non-linear oscillator governed by (4) for the initial conditions (5) focusing on the influence of an arbitrary non-negative real power α on the energy-displacement function, which, as far as the authors are aware, has not been examined so far. The energy-displacement function obtained is used to find the values of the amplitude of motion. These values are exact, unlike the results of the application of the methods listed above, which are approximate. They are only valid for weakly non-linear damping forces, while the procedure proposed here does not have this limitation. In addition, the expressions for phase trajectories are also derived. Examples are presented to illustrate the findings. They consist of systems with a purely non-linear restoring force of odd or even power, including a fractional-order restoring force.

2 Model

Introducing the non-dimensional variables

$$\xi = \frac{x}{X_0}, \quad \tau = \frac{t}{\sqrt{\frac{m}{k}}X_0^{\frac{1-\alpha}{2}}}. \tag{6}$$

Equation (4) can be expressed in non-dimensional form as

$$\xi'' + \mu\,\mathrm{sgn}(\xi')\xi'^2 + \mathrm{sgn}(\xi)|\xi|^{\alpha} = 0, \tag{7}$$

where the prime denotes differentiation with respect to τ and

$$\mu = \frac{ck}{m}X_0. \tag{8}$$

Thus, instead of having five parameters as in (4) and (5), the system behavior will depend on two parameters only: the coefficient μ and the power α. The initial conditions (5) are now

$$\xi(0) \equiv \xi_0^+ = 1, \quad \xi'(0) = 0, \tag{9}$$

where the superscript '+' is used to emphasize the sign of the displacement, which is of importance for the procedure developed below.

3 Energy-displacement function

As shown analytically in [12], the total mechanical energy of the system (7) decreases with time. However, instead of analyzing the change of this energy in time, it is more convenient to consider how it changes with the displacement, as this energy-displacement function can be used to carry out a qualitative analysis of the system behavior.

The non-dimensional equation of motion (7) can be written down in the form

$$\frac{d}{d\tau}\left[\frac{1}{2}(\xi')^2+\frac{1}{\alpha+1}|\xi|^{\alpha+1}\right]=-\mu\,\mathrm{sgn}(\xi')(\xi')^3. \quad (10)$$

Introducing the total mechanical energy

$$E=T+V,\quad T=\frac{1}{2}(\xi')^2,\quad V=\frac{|\xi|^{\alpha+1}}{\alpha+1}, \quad (11)$$

as well as changing the variable of differentiation from τ to ξ, (10) becomes

$$\frac{dE}{d\xi}=-\mu\,\mathrm{sgn}(\xi')\left[2\left(E-\frac{1}{\alpha+1}|\xi|^{\alpha+1}\right)\right]. \quad (12)$$

Based on (11), the following substitution is introduced:

$$T=E-\frac{1}{\alpha+1}|\xi|^{\alpha+1}, \quad (13)$$

so that (12) transforms to

$$\frac{dT}{d\xi}+2\mu\,\mathrm{sgn}(\xi')T=-\,\mathrm{sgn}(\xi)|\xi|^{\alpha}. \quad (14)$$

In this way, the problem is transformed from a second-order differential equation to a first-order differential equation. In order to solve (14), two cases are to be considered separately, depending on whether the power α is odd or even.

3.1 Case I: odd-power restoring force

When the motion starts from a positive initial displacement with respect to the equilibrium position and with zero velocity as described by (5), being directed to the left ($\leftarrow$), $\mathrm{sgn}(\xi')<0$. For odd-parity restoring force, (14) can be expressed as

$$\frac{dT_{\leftarrow}}{d\xi}-2\mu T_{\leftarrow}=-\xi^{\alpha}. \quad (15)$$

It should be noted that (15) holds both for $\xi>0$ and $\xi<0$, due to the fact that the restoring force is odd.

The general solution of (15) is

$$T_{\leftarrow}(\xi)=C_{\mathrm{I}_1}e^{2\mu\xi}-e^{2\mu\xi}\int e^{-2\mu\xi}\xi^{\alpha}d\xi, \quad (16)$$

where C_{I_1} is a constant of integration. The subscript I indicates the case considered and the subscript 1 denotes the first interval of motion.

In order to find the solution for $T_{\leftarrow}(\xi)$, the integral

$$J=\int e^{-2\mu\xi}\xi^{\alpha}d\xi, \quad (17)$$

needs to be solved. Therefore, the following substitution is evaluated. Introducing

$$2\mu\xi=u, \quad (18)$$

leads to

$$J=\left(\frac{1}{2\mu}\right)^{\alpha+1}\int e^{-u}u^{\alpha}du. \quad (19)$$

This integral can be related to the definition to the upper incomplete Gamma function [16, 17]

$$\Gamma[s,y]=\int_y^{\infty}e^{-u}u^{s-1}du. \quad (20)$$

In addition, taking $\alpha=s-1$, the integral (19) becomes

$$J=-\left(\frac{1}{2\mu}\right)^{\alpha+1}\Gamma[\alpha+1,2\mu\xi], \quad (21)$$

and the solution for $T_{\leftarrow}$ in (16) is found

$$T_{\leftarrow}=C_{\mathrm{I}_1}e^{2\mu\xi}+e^{2\mu\xi}\left(\frac{1}{2\mu}\right)^{\alpha+1}\Gamma[\alpha+1,2\mu\xi]. \quad (22)$$

On the basis of (13), the energy-displacement function is

$$E_{\mathrm{I}_1\leftarrow}=\frac{1}{\alpha+1}|\xi|^{\alpha+1}+C_{\mathrm{I}_1}e^{2\mu\xi}+\left(\frac{1}{2\mu}\right)^{\alpha+1}e^{2\mu\xi}\Gamma[\alpha+1,2\mu\xi]. \quad (23)$$

The constant C_{I_1} can be obtained by using the initial conditions (9)

$$C_{\mathrm{I}_1}=-\left(\frac{1}{2\mu}\right)^{\alpha+1}\Gamma\left[\alpha+1,2\mu\xi_0^{+}\right], \quad (24)$$

so that the energy during the first interval changes in accordance with

$$E_{I_1\leftarrow} = \frac{|\xi|^{\alpha+1}}{\alpha+1} + \left(\frac{1}{2\mu}\right)^{\alpha+1} e^{2\mu\xi}\big(\Gamma[\alpha+1, 2\mu\xi] - \Gamma\big[\alpha+1, 2\mu\xi_0^+\big]\big). \tag{25}$$

The motion changes direction when the velocity and the kinetic energy are zero. Based on (25) this occurs at $\xi = \xi_1^-$, when

$$\Gamma\big[\alpha+1, 2\mu\xi_1^-\big] - \Gamma\big[\alpha+1, 2\mu\xi_0^+\big] = 0. \tag{26}$$

When the system moves from the left to the right (labelled by '→'), $\mathrm{sgn}(\xi') > 0$, and (14) becomes

$$\frac{dT_\rightarrow}{d\xi} + 2\mu T_\rightarrow = -\xi^\alpha, \tag{27}$$

whose general solution is

$$T_\rightarrow = C_{I_2} e^{-2\mu\xi} + e^{-2\mu\xi}\left(\frac{1}{-2\mu}\right)^{\alpha+1}\Gamma[\alpha+1, -2\mu\xi]. \tag{28}$$

The corresponding energy-displacement function can be obtained analogously as before and is given by

$$E_{I_2\rightarrow} = \frac{|\xi|^{\alpha+1}}{\alpha+1} + C_{I_2} e^{-2\mu\xi} + \left(\frac{1}{2\mu}\right)^{\alpha+1} e^{-2\mu\xi}\Gamma[\alpha+1, -2\mu\xi]. \tag{29}$$

The constant of integration C_{I_2} can be obtained from the fact that at ξ_1^- the energy is equal to the potential one. Thus, one has

$$C_{I_2} = -\left(\frac{1}{2\mu}\right)^{\alpha+1}\Gamma\big[\alpha+1, -2\mu\xi_1^-\big]. \tag{30}$$

The expression for the energy-displacement function is

$$E_{I_2\rightarrow} = \frac{|\xi|^{\alpha+1}}{\alpha+1} + \left(\frac{1}{2\mu}\right)^{\alpha+1} e^{-2\mu\xi}\big(\Gamma[\alpha+1, -2\mu\xi] - \Gamma\big[\alpha+1, -2\mu\xi_1^-\big]\big). \tag{31}$$

Proceeding in the same way, the following expressions can be found for the energy-displacement functions for the motion to the left and right, respectively:

$$E_{I_i\leftarrow} = \frac{|\xi|^{\alpha+1}}{\alpha+1} + \left(\frac{1}{2\mu}\right)^{\alpha+1} e^{2\mu\xi}\big(\Gamma[\alpha+1, 2\mu\xi] - \Gamma\big[\alpha+1, 2\mu\xi_{i-1}^+\big]\big), \quad \xi_i^- \le \xi \le \xi_{i-1}^+, \tag{32}$$

$$E_{I_{i+1}\rightarrow} = \frac{|\xi|^{\alpha+1}}{\alpha+1} + \left(\frac{1}{2\mu}\right)^{\alpha+1} e^{-2\mu\xi}\big(\Gamma[\alpha+1, -2\mu\xi] - \Gamma\big[\alpha+1, -2\mu\xi_i^-\big]\big), \quad \xi_i^- \le \xi \le \xi_{i+1}^+,$$
$$i = 2j-1, \; j \in \mathbb{N}. \tag{33}$$

Extremal displacements, i.e. the minimum and maximum ξ_i^- and ξ_{i+1}^+, correspond to zero kinetic energy. They can be found by equating the second term in (32) and (33) to zero. These equations give the subsequent amplitude as a function of the previous one

$$\Gamma\big[\alpha+1, 2\mu\xi_i^-\big] - \Gamma\big[\alpha+1, 2\mu\xi_{i-1}^+\big] = 0 \Rightarrow \quad \xi_i^-, \tag{34}$$

$$\Gamma\big[\alpha+1, -2\mu\xi_{i+1}^+\big] - \Gamma\big[\alpha+1, -2\mu\xi_i^-\big] = 0 \Rightarrow \quad \xi_{i+1}^+. \tag{35}$$

The procedure developed will be illustrated next by considering a pure cubic oscillator.

3.1.1 *Example 1. Pure cubic oscillator*

On the basis of (34) and (35), the relative minima and maxima of the amplitude of a pure cubic oscillator, which corresponds to $\alpha = 3$, are defined by

$$\Gamma\big[4, 2\mu\xi_1^-\big] - \Gamma[4, 2\mu] = 0, \tag{36}$$

$$\Gamma\big[4, -2\mu\xi_2^+\big] - \Gamma\big[4, -2\mu\xi_1^-\big] = 0, \tag{37}$$

$$\Gamma\big[4, 2\mu\xi_3^-\big] - \Gamma\big[4, 2\mu\xi_2^+\big] = 0, \tag{38}$$

$$\Gamma\big[4, -2\mu\xi_4^+\big] - \Gamma\big[4, -2\mu\xi_3^-\big] = 0, \tag{39}$$

$$\Gamma\big[4, 2\mu\xi_5^-\big] - \Gamma\big[4, 2\mu\xi_4^+\big] = 0. \tag{40}$$

Solving these equations numerically for $\mu = 0.5$, starting from (36), the values given in Table 1 are obtained.

Then, the energy-displacement function in the first interval of motion (32), starting from the initial position ξ_0^+ and lasting until the position ξ_1^- is

$$E_{I_1\leftarrow} = \frac{|\xi|^4}{4} + \left(\frac{1}{2\mu}\right)^4 e^{2\mu\xi}(\Gamma[4, 2\mu\xi] - \Gamma[4, 2\mu]), \quad -0.71149 \le \xi \le 1. \tag{41}$$

Table 1 Values of the negative and positive amplitudes of a pure cubic oscillator for $\mu = 0.5$

ξ_0	ξ_1^-	ξ_2^+	ξ_3^-	ξ_4^+	ξ_5^-
1	−0.71149	0.55294	−0.45240	0.38288	−0.33193

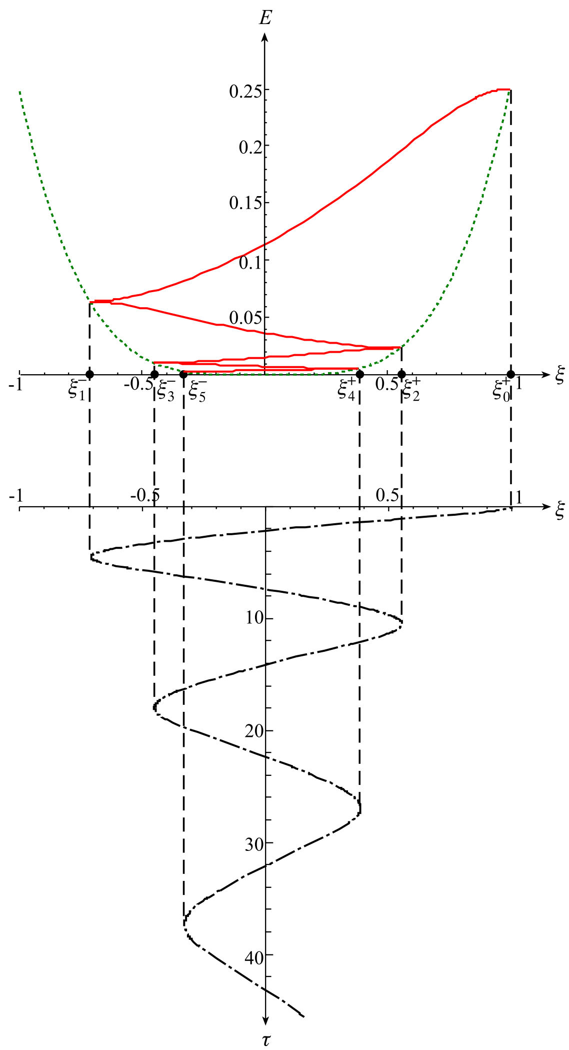

Fig. 1 Energy-displacement curves and time response for a pure cubic oscillator and $\mu = 0.5$

This function in plotted in the upper part of Fig. 1, where the potential well $V = |\xi|^4/4$ is shown, too. By using (33), the energy-displacement function in the next interval, when the system moves to the right, is obtained

$$E_{I_2\rightarrow} = \frac{|\xi|^4}{4} + \left(\frac{1}{2\mu}\right)^4 e^{-2\mu\xi}(\Gamma[4, -2\mu\xi], - \Gamma[4, 2\mu \cdot 0.71149]), \quad -0.71149 \le \xi \le 0.55294. \tag{42}$$

Further use of (32) and (33) gives the following energy-displacement functions in the subsequent intervals of motion:

$$E_{I_3\leftarrow} = \frac{|\xi|^4}{4} + \left(\frac{1}{2\mu}\right)^4 e^{2\mu\xi}(\Gamma[4, 2\mu\xi] - \Gamma[4, 2\mu \cdot 0.55294]), \quad -0.45240 \le \xi \le 0.55294, \tag{43}$$

$$E_{I_4\rightarrow} = \frac{|\xi|^4}{4} + \left(\frac{1}{2\mu}\right)^4 e^{-2\mu\xi}(\Gamma[4, -2\mu\xi] - \Gamma[4, 2\mu \cdot 0.45240]), \quad -0.45240 \le \xi \le 0.38288, \tag{44}$$

$$E_{I_5\leftarrow} = \frac{|\xi|^4}{4} + \left(\frac{1}{2\mu}\right)^4 e^{2\mu\xi}(\Gamma[4, 2\mu\xi] - \Gamma[4, 2\mu \cdot 0.38288]), \quad -0.33193 \le \xi \le 0.38288. \tag{45}$$

The energy-displacement functions (42)–(45) are also plotted in the upper part of Fig. 1, clearly indicating how the energy decreases. The positions at which the motion changes direction, i.e. the amplitudes $\xi_0^+ - \xi_5^-$, are depicted as well. The time evolution $\xi(\tau)$ obtained by integrating the equation of motion numerically is shown in the lower part of Fig. 1. The link between the amplitudes $\xi_0^+ - \xi_5^-$ in the energy-displacement function and the time evolution is also depicted to verify the results obtained. It is seen that the amplitudes obtained above coincide with the ones calculated numerically from the equation of motion.

3.2 Case II: even-power restoring force

When the restoring force is an even-power function, the initial motion to the left needs to be divided into two parts, depending on the sign of the displacement. When the displacement is positive, the analysis given above for the motion to the left holds and, consequently, the energy changes in accordance with

$$E_{\mathrm{II}_{1\leftarrow+}} = \frac{|\xi|^{\alpha+1}}{\alpha+1} + \left(\frac{1}{2\mu}\right)^{\alpha+1} e^{2\mu\xi}\big(\Gamma[\alpha+1, 2\mu\xi] - \Gamma\big[\alpha+1, 2\mu\xi_0^+\big]\big), \tag{46}$$

where the subscript II indicates Case II and '+' stands for the sign of the displacement.

However, when the displacement is negative, which will be labelled by '−', (14) can be written as

$$\frac{dT_{\leftarrow-}}{d\xi} - 2\mu T_{\leftarrow-} = \xi^{\alpha}. \tag{47}$$

By carrying out a procedure similar to the one described above, the following expression for the energy-displacement function can be found:

$$E_{\mathrm{II}_{1\leftarrow-}} = \frac{|\xi|^{\alpha+1}}{\alpha+1} + C_{\mathrm{II}_1} e^{2\mu\xi} - e^{2\mu\xi}\left(\frac{1}{2\mu}\right)^{\alpha+1}\Gamma[\alpha+1, 2\mu\xi], \tag{48}$$

where the constant C_{II_1} can be obtained from the condition

$$E_{\mathrm{II}_{1\leftarrow+}}(\xi=0) = E_{\mathrm{II}_{1\leftarrow-}}(\xi=0). \tag{49}$$

Knowing that $\Gamma[\alpha+1, 0] \equiv \Gamma[\alpha+1]$, i.e. that the incomplete gamma function turns into the Euler Gamma function when the second argument is equal to zero, one obtains

$$C_{\mathrm{II}_1} = \left(\frac{1}{2\mu}\right)^{\alpha+1}\big(2\Gamma[\alpha+1] - \Gamma\big[\alpha+1, 2\mu\xi_0^+\big]\big). \tag{50}$$

Thus, the energy-displacement function is

$$E_{\mathrm{II}_{1\leftarrow-}} = \frac{|\xi|^{\alpha+1}}{\alpha+1} + \left(\frac{1}{2\mu}\right)^{\alpha+1} e^{2\mu\xi}\big(2\Gamma[\alpha+1] - \Gamma\big[\alpha+1, 2\mu\xi_0^+\big] - \Gamma[\alpha+1, 2\mu\xi]\big). \tag{51}$$

The motion changes direction at the position ξ_1^- when

$$2\Gamma[\alpha+1] - \Gamma\big[\alpha+1, 2\mu\xi_0^+\big] - \Gamma\big[\alpha+1, 2\mu\xi_1^-\big] = 0. \tag{52}$$

When the motion is directed towards the right and the displacement is negative, (14) becomes

$$\frac{dT_{\rightarrow-}}{d\xi} + 2\mu T_{\rightarrow-} = \xi^{\alpha}, \tag{53}$$

which results in the following solution for the energy-displacement function:

$$E_{\mathrm{II}_{2\rightarrow-}} = \frac{1}{\alpha+1}|\xi|^{\alpha+1} + C_{\mathrm{II}_2} e^{-2\mu\xi} - \left(\frac{1}{-2\mu}\right)^{\alpha+1} e^{-2\mu\xi}\Gamma[\alpha+1, -2\mu\xi], \tag{54}$$

with the constant C_{II_2} being defined by the condition

$$E_{\mathrm{II}_{1\leftarrow-}}\big(\xi_1^-\big) = E_{\mathrm{II}_{2\rightarrow-}}\big(\xi_1^-\big). \tag{55}$$

Its value is then

$$C_{\mathrm{II}_2} = \left(\frac{1}{-2\mu}\right)^{\alpha+1}\Gamma\big[\alpha+1, -2\mu\xi_1^-\big], \tag{56}$$

and the energy-displacement function is

$$E_{\mathrm{II}_{2\rightarrow-}} = \frac{1}{\alpha+1}|\xi|^{\alpha+1} + \left(\frac{1}{2\mu}\right)^{\alpha+1} e^{-2\mu\xi}\big(\Gamma[\alpha+1, -2\mu\xi] - \Gamma\big[\alpha+1, -2\mu\xi_1^-\big]\big). \tag{57}$$

Finally, when the displacement is positive, (14) corresponds to (27). The corresponding solution can be used then together with the condition $E_{\mathrm{II}_{2\rightarrow-}}(0) = E_{\mathrm{II}_{2\rightarrow+}}(0)$ to derive the constant C_{II_3}

$$C_{\mathrm{II}_3} = \left(\frac{1}{-2\mu}\right)^{\alpha+1}\big(\Gamma\big[\alpha+1, -2\mu\xi_1^-\big] - 2\Gamma[\alpha+1]\big), \tag{58}$$

which completes the expression for the energy-displacement function

$$E_{\mathrm{II}_{2\rightarrow+}} = \frac{|\xi|^{\alpha+1}}{\alpha+1} + \left(\frac{1}{2\mu}\right)^{\alpha+1} e^{-2\mu\xi}\big(2\Gamma[\alpha+1] - \Gamma\big[\alpha+1, -2\mu\xi_1^-\big] - \Gamma[\alpha+1, -2\mu\xi]\big). \tag{59}$$

In the subsequent intervals of motion, the energy-displacement functions are defined by

$$E_{\mathrm{II}_{i\leftarrow+}} = V + \left(\frac{1}{2\mu}\right)^{\alpha+1} e^{2\mu\xi}\big(\Gamma[\alpha+1,2\mu\xi] - \Gamma\big[\alpha+1,2\mu\xi_{i-1}^{+}\big]\big), \quad 0\le\xi\le\xi_{i-1}^{+}, \tag{60}$$

$$E_{\mathrm{II}_{i\leftarrow-}} = V + \left(\frac{1}{2\mu}\right)^{\alpha+1} e^{2\mu\xi}\big(2\Gamma[\alpha+1] - \Gamma[\alpha+1,2\mu\xi] - \Gamma\big[\alpha+1,2\mu\xi_{i-1}^{+}\big]\big), \quad \xi_i^{-}\le\xi\le 0, \tag{61}$$

$$E_{\mathrm{II}_{i+1\to-}} = V + \left(\frac{1}{2\mu}\right)^{\alpha+1} e^{-2\mu\xi}\big(\Gamma[\alpha+1,-2\mu\xi] - \Gamma\big[\alpha+1,-2\mu\xi_i^{-}\big]\big), \quad \xi_i^{-}\le\xi\le 0, \tag{62}$$

$$E_{\mathrm{II}_{i+1\to+}} = V + \left(\frac{1}{2\mu}\right)^{\alpha+1} e^{-2\mu\xi}\big(2\Gamma[\alpha+1] - \Gamma[\alpha+1,-2\mu\xi] - \Gamma\big[\alpha+1,-2\mu\xi_i^{-}\big]\big),$$

$$0\le\xi\le\xi_{i+1}^{+}, \tag{63}$$

$$i = 2j-1,\ \ j\in\mathbb{N}, \tag{64}$$

where V is defined by the third expression in (11), while the minima and maxima ξ_i^- and ξ_{i+1}^+ can be calculated, respectively, from

$$2\Gamma[\alpha+1] - \Gamma\big[\alpha+1,2\mu\xi_i^{-}\big] - \Gamma\big[\alpha+1,2\mu\xi_{i-1}^{+}\big] = 0 \Rightarrow \xi_i^{-}, \tag{65}$$

$$2\Gamma[\alpha+1] - \Gamma\big[\alpha+1,-2\mu\xi_{i+1}^{+}\big] - \Gamma\big[\alpha+1,-2\mu\xi_i^{-}\big] = 0 \Rightarrow \xi_{i+1}^{+}. \tag{66}$$

Thus, it is seen that, unlike in Case I with an even restoring case, in the case with an odd restoring force twice more expressions for the energy-displacement functions have been derived due to the fact that the motion in one direction needed to be split into two parts: one for positive and the other one for negative displacements.

The use of the expressions (60)–(66) will be illustrated on the example of a restoring force the power of which is a ratio involving an even integer.

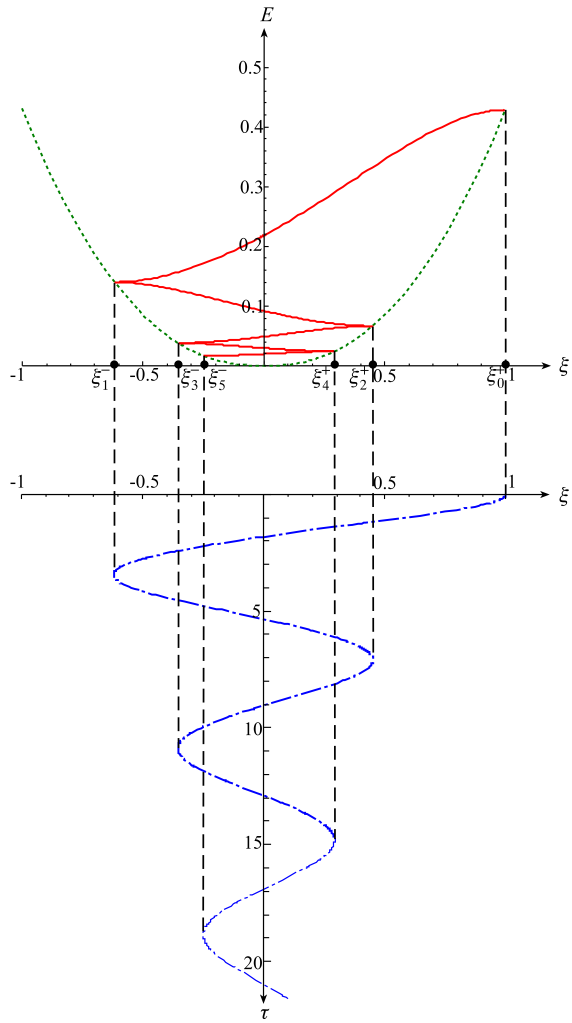

Fig. 2 Energy-displacement curves and time response for a fractional-order oscillator with $\alpha = 4/3$ and $\mu = 0.5$

Table 2 Values of the negative and positive amplitudes of a fractional-order oscillator for $\alpha = 4/3$, $\mu = 0.5$

ξ_0	ξ_1^-	ξ_2^+	ξ_3^-	ξ_4^+	ξ_5^-
1	−0.619508	0.450347	−0.354093	0.291847	−0.248257

3.2.1 *Example 2. Fractional-order oscillator with the power involving an even integer*

A fractional-order restoring force with $\alpha = 4/3$ is considered here. The amplitudes of motion are defined by (65) and (66). Starting from $i = 1$ in (60) and knowing the value of ξ_0^+, the energy-displacement function (60) can be plotted (Fig. 2). Then, using (65), the value of

ξ_1^- can be obtained numerically and it is given in Table 2. Further, for the known ξ_1^-, (61) can be used to plot the energy-displacement diagram from the zero position until $\xi_1^- \leq \xi \leq 0$. Then, for the known ξ_1^- and $i = 1$, (66) yields the value of ξ_2^+ and (63) for the energy-displacement function is completed. By proceeding in an analogous way, the energy-displacement function shown in Fig. 2 can be obtained. The accuracy of the obtained amplitudes is confirmed by comparing them with the numerically obtained amplitudes on the time response plotted in the lower part of Fig. 2.

4 Phase trajectories and some characteristics of motion

In this section, the phase trajectories of system (7) will be obtained analytically. To that end, (7) is written as

$$\frac{dT}{d\xi} + \mathrm{sgn}(\xi)|\xi|^\alpha = -\mu\,\mathrm{sgn}(\xi')(\xi')^2, \tag{67}$$

so that the velocity can be expressed as

$$\xi' = \pm\sqrt{-\frac{1}{\mu}\frac{dT}{d\xi} - \frac{1}{\mu}\mathrm{sgn}(\xi)|\xi|^\alpha}. \tag{68}$$

The expressions for the energy-displacement function for the two cases distinguished above can be used to define the velocity (68) completely as the function of ξ, i.e. to obtain the expressions for the phase trajectories. The kinetic energy T is described by the second term in (32) and (33) for Case I and by the second term in (60)–(63) for Case II.

4.1 Case I: odd-power restoring force

For the system with an odd-power restoring force and during the motion to the left, the use of (32) and (68) yields

$$\xi'_{\mathrm{I}_{i\leftarrow}} = -\frac{1}{\sqrt{\mu}}\left(\frac{1}{2\mu}\right)^{\frac{\alpha}{2}} \times e^{\mu\xi}\sqrt{\Gamma[\alpha+1, 2\mu\xi] - \Gamma\big[\alpha+1, 2\mu\xi_{i-1}^+\big]}, \quad \xi_i^- \leq \xi \leq \xi_{i-1}^+. \tag{69}$$

During the motion to the right, based on (33) and (68), the part of the phase trajectory is defined by

$$\xi'_{\mathrm{I}_{i+1\rightarrow}} = +\frac{1}{\sqrt{\mu}}\left(\frac{1}{2\mu}\right)^{\frac{\alpha}{2}} \times e^{-\mu\xi}\sqrt{\Gamma[\alpha+1, -2\mu\xi] - \Gamma\big[\alpha+1, -2\mu\xi_i^-\big]}, \quad \xi_i^- \leq \xi \leq \xi_{i+1}^+, \tag{70}$$

$$i = 2j-1, \; j \in \mathbb{N}. \tag{71}$$

4.2 Case II: even-power restoring force

If the restoring force is even, one can combine (60)–(63) with (68) to derive the following expressions for the parts of the phase trajectories, from one relative maximum to the next one:

$$\xi'_{\mathrm{II}_{i\leftarrow+}} = -\frac{1}{\sqrt{\mu}}\left(\frac{1}{2\mu}\right)^{\frac{\alpha}{2}} e^{\mu\xi}\sqrt{\Gamma[\alpha+1, 2\mu\xi] - \big[\Gamma\alpha+1, 2\mu\xi_{i-1}^+\big]}, \quad 0 \leq \xi \leq \xi_{i-1}^+, \tag{72}$$

$$\xi'_{\mathrm{II}_{i\leftarrow-}} = -\frac{1}{\sqrt{\mu}}\left(\frac{1}{2\mu}\right)^{\frac{\alpha}{2}} e^{\mu\xi}\sqrt{2\Gamma[\alpha+1] - \Gamma[\alpha+1, 2\mu\xi] - \Gamma\big[\alpha+1, 2\mu\xi_{i-1}^+\big]}, \quad \xi_i^- \leq \xi \leq 0, \tag{73}$$

$$\xi'_{\mathrm{II}_{i+1\rightarrow-}} = \frac{1}{\sqrt{\mu}}\left(\frac{1}{2\mu}\right)^{\frac{\alpha}{2}} e^{-\mu\xi}\sqrt{\Gamma[\alpha+1, -2\mu\xi] - \Gamma\big[\alpha+1, -2\mu\xi_i^-\big]}, \quad \xi_i^- \leq \xi \leq 0, \tag{74}$$

$$\xi'_{\mathrm{II}_{i+1\rightarrow+}} = \frac{1}{\sqrt{\mu}}\left(\frac{1}{2\mu}\right)^{\frac{\alpha}{2}} e^{-\mu\xi}\sqrt{2\Gamma[\alpha+1] - \Gamma[\alpha+1, -2\mu\xi] - \Gamma\big[\alpha+1, -2\mu\xi_i^-\big]}, \quad 0 \leq \xi \leq \xi_{i+1}^+. \tag{75}$$

4.3 Examples

The expressions for the phase trajectories derived above are plotted for Example 1 and Example 2 considered in Sect. 3. They are shown in Fig. 3a and 3b, for the characteristic values of ξ_{i-1}^{+} and ξ_i^{-} given in Tables 1 and 2 (Fig. 3c contains one more example, which will be discussed in the next section). Different types of lines are used to emphasize the use of different expressions depending on the interval. It should be pointed out that these analytically obtained results were compared with the phase trajectories obtained by integrating the equation of motion numerically and the exact match was found. However, they are not shown here for clarity. The phase trajectories spiral in from the initial position and have velocity maxima in the second and fourth quadrant.

4.4 Maximal velocities

The maximal velocity can be obtained by differentiating (69) and (72) (or (70) and (74)) with respect to ξ. As a result of some of that, it follows that

$$\frac{d\xi'}{d\xi} = -\mu\xi' + \frac{\xi^{\alpha}}{\xi'}. \tag{76}$$

In case of a maximum, the left side of (76) is zero, yielding

$$\left(\xi_*'\right)^2 = \frac{\xi_*^{\alpha}}{\mu}, \tag{77}$$

where the subscript '$*$' stands for the maximal velocity and the corresponding displacement. If plotted in the $\xi - \xi'$ plane, (77) represents the locus of maximal velocities. The loci corresponding to Example 1 and Example 2 are shown in Fig. 3a and 3b as thick lines. It is interesting to note that in the case of a quadratic oscillator, the locus is a straight line, i.e. there is proportionality between the square of the maximal velocities and the corresponding displacements. This case is plotted in Fig. 3c.

Equation (77) implies that the displacement corresponding to the maximum velocity cannot be zero, unlike the case of a conservative system.

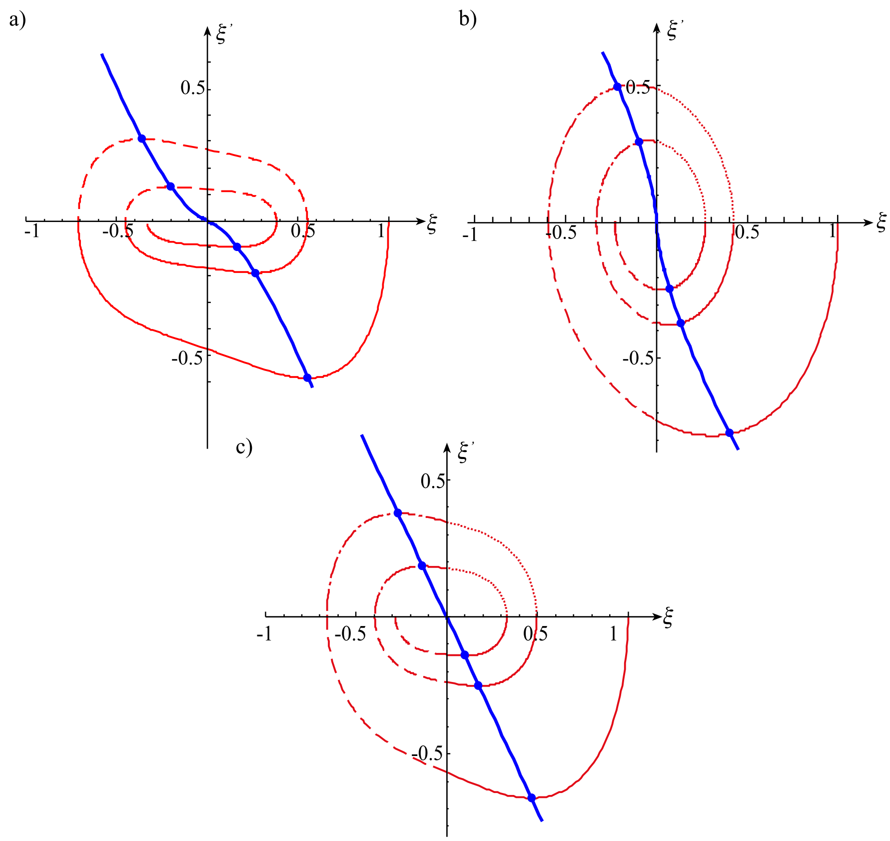

Fig. 3 Phase trajectories and loci of maximal velocities (*thick solid line*) for: (**a**) a pure cubic oscillator; (**b**) a fractional-order oscillator with $\alpha = 4/3$; (**c**) a pure quadratic oscillator. In all cases $\mu = 0.5$

In addition, (77) indicates that the ratio of the square of the maximal velocity and the corresponding displacement raised to the power of the restoring force, remains constant during the motion and it is equal to the reciprocal of the damping coefficient. Thus, this ratio represents an invariant of the system with a non-negative real-power restoring force and quadratic damping. This conclusion is in agreement with the physical mechanism of the system described by the equation of motion (7). Namely, when the velocity is maximal, the acceleration is zero because the motion considered is rectilinear. This fact yields the conclusion about the equilibrium between the damping force and the restoring force at the position corresponding to maximal velocities.

4.5 On the time difference between two consecutive relative extrema

By using the previously derived results for the velocity, it is possible to obtain the expression for the time difference between two consecutive relative maxima and minima. For example, on the basis of (69), this time difference $\tilde{T}$ for the case of odd-power restoring forces is defined by

$$\tilde{T} = 2^{\frac{\alpha}{2}}\mu^{\frac{\alpha+1}{2}} \times \int_{\xi_{i-1}^{+}}^{\xi_i^{-}} \left| \frac{d\xi}{e^{\mu\xi}\sqrt{\Gamma[\alpha+1, 2\mu\xi] - \Gamma[\alpha+1, 2\mu\xi_{i-1}^{+}]}} \right|. \tag{78}$$

The form of the integrand in (78) does not enable one to obtain a closed form solution for $\tilde{T}$ analytically. However, this quantity can be obtained numerically for the boundaries of the integral defined by (34) and (35).

This approach is used to calculate the time difference between two consecutive relative maxima and minima for the oscillators with different values of the power and for two different values of the parameter μ, as shown in Tables 3 and 4. It is seen that for powers smaller than unity the time difference becomes shorter as time passes. This is opposite to the behavior of the oscillator with the powers higher than unity. It can be seen from Tables 3 and 4, as well as from the time responses shown in Figs. 1 and 2, that this time difference increases with time for these oscillators.

It is interesting to note that these findings can partly be recovered from the first-order averaging method recently developed by Cveticanin [18] for fractional-order non-conservative oscillators ('partly' was used here since the method presented in [18] is applicable to the case when the damping force is weakly non-linear; however, in the procedure developed above no requirement of this type is imposed). In order to use Cveticanin's method, the equation of motion (7) is written down as

$$\xi'' + \mathrm{sgn}(\xi)|\xi|^{\alpha} = \mu f(\xi, \xi'), \tag{79}$$

$$f(\xi, \xi') = -\mathrm{sgn}(\xi')\xi'^{2}. \tag{80}$$

According to the results presented in [18] for $\mu \ll 1$, the solution for motion can be assumed in the form

$$\xi = a\cos\psi, \tag{81}$$

where the amplitude a and the complete phase ψ are defined by the following first-order differential equations:

$$\frac{da}{d\tau} = -\frac{2\mu}{\pi(\alpha+3)\omega(a)} \times \int_0^{2\pi} f(a\cos\psi, -a\omega\sin\psi)\sin d\psi, \tag{82}$$

Table 3 Duration of the time difference between two consecutive relative maxima and minima for $\mu = 0.1$ and different values of the power of the restoring force

α	$\tilde{T}_1$	$\tilde{T}_2$	$\tilde{T}_3$	$\tilde{T}_4$	$\tilde{T}_5$
1/3	2.8647	2.72561	2.61031	2.51248	2.428
3/5	2.98105	2.90077	2.83211	2.77227	2.7194
9/11	3.0724	3.03658	3.00532	2.9776	2.9527
1	3.14622	3.14525	3.14455	3.14398	3.1437
11/9	3.23388	3.27311	3.30898	3.34203	3.3726

Table 4 Duration of the time difference between two consecutive relative maxima and minima for $\mu = 0.5$ and different values of the power of the restoring force

α	$\tilde{T}_1$	$\tilde{T}_2$	$\tilde{T}_3$	$\tilde{T}_4$	$\tilde{T}_5$
1/3	2.73687	2.25834	2.01376	1.85239	1.73384
3/5	2.93448	2.6206	2.44894	2.33098	2.2414
9/11	3.09284	2.92173	2.82865	2.76418	2.7147
1	3.2224	3.17509	3.16008	3.15333	3.1497
11/9	3.37772	3.48692	3.58236	3.663	3.7325

$$\frac{d\psi}{d\tau} = \omega(a) - \frac{\mu}{2\pi a \omega(a)} \times \int_0^{2\pi} f(a\cos\psi, -a\omega\sin\psi)\cos d\psi, \quad (83)$$

and

$$\omega(a) = q\,|a|^{\frac{\alpha-1}{2}}, \quad q = \sqrt{\frac{\alpha+1}{2}}\frac{\sqrt{\pi}\,\Gamma(\frac{\alpha+3}{2(\alpha+1)})}{\Gamma(\frac{1}{\alpha+1})}. \quad (84)$$

For the non-conservative function (80), (82) and (83) yield

$$a = (1+\mu B\tau)^{-\frac{2}{\alpha+1}}, \quad (85)$$

$$\psi = -\frac{3(\alpha+3)\pi}{16\mu} + \frac{3(\alpha+3)\pi(1+\mu B\tau)^{\frac{2}{\alpha+1}}}{16\mu}, \quad (86)$$

where

$$B = \frac{4\sqrt{\frac{2}{\pi}}(\alpha+1)^{3/2}\Gamma(\frac{\alpha+3}{2(\alpha+1)})}{3(\alpha+3)\Gamma(\frac{1}{\alpha+1})}. \quad (87)$$

It should be noted that for a linear oscillator, the amplitude decay (85) and the complete phase are in complete agreement with the solution found in [1]. For the confirmation of the accuracy of the results derived for $\alpha \neq 1$, the time response obtained, i.e. (81), (85)–(87) is compared with the numerical results and shown in Fig. 4 for two different cases: under-linear (sublinear) case $\alpha = 9/11$ and over-linear (superlinear) case $\alpha = 11/9$. In both cases, the coefficient μ is kept constant and equal to 0.1. It is seen that the approximate solutions agree well with the numerical results. The expression for the complete phase ψ given by (86) can be utilized to examine how the ratio ψ/τ changes with time for different values of the power α. This is plotted in Fig. 5. It is seen that for the under-linear oscillator, this ratio increases with time, which implies that the time difference between two consecutive relative maxima and minima decreases with time, and this is in agreement with the findings discussed above. In the case of the over-linear oscillator, the opposite is true. It should also be noticed that this procedure leads to the conclusion that the frequency, to which this ratio corresponds now, is constant and equal to unity. This implies that quadratic damping does not influence it, which is a consequence of having use a first-order approximation. Hence, higher-order averaging is needed to define this influence. However, the results of the exact analysis presented above and in Tables 3 and 4 clearly show that the time difference between two consecutive relative maxima and minima in the linear oscillator is affected by quadratic damping in a way that it decreases with time.

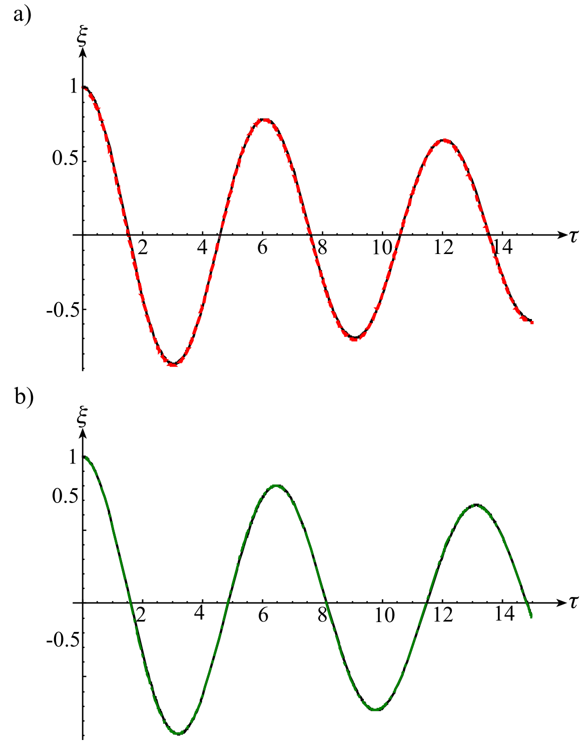

Fig. 4 Comparison of the time response obtained numerically (*solid line*) and (**a**) approximate solution (*dashed line*) for $\alpha = 9/11$; (**b**) approximate solution (*dashed-dotted line*) for $\alpha = 11/9$. In both cases $\mu = 0.1$

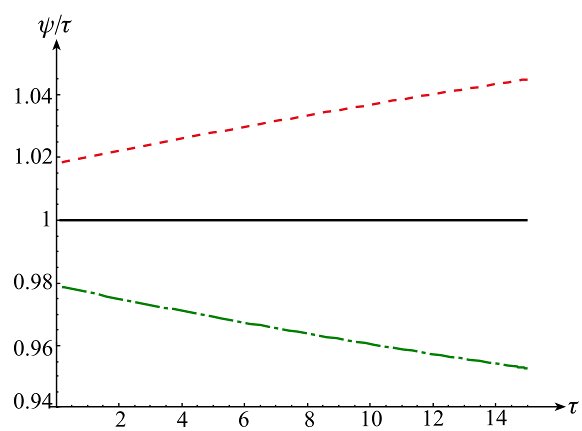

Fig. 5 Ratio ψ/τ versus τ for $\mu = 0.1$ and $\alpha = 9/11$ (*dashed line*), $\alpha = 1$ (*solid line*) and $\alpha = 11/9$ (*dashed-dotted line*)

5 Conclusions

In this paper, oscillators with a non-negative real-power restoring force and quadratic damping have been considered. A second-order differential equation of motion has been transformed into a first-order differential equation with respect to the kinetic energy. In order to find its solution, two cases have been considered depending on the type of the restoring force: the case of an odd-power restoring case and the case of an even-power restoring case. In both cases the exact expressions for the energy-displacement function have been derived in closed form in terms of incomplete Gamma functions. The exact implicit expressions for the positions at which the motion changes direction, i.e. at relative displacement maxima and minima have also been derived. In addition, the exact expressions for the phase trajectories have been obtained as well as the loci of maximal velocities. In the case of a quadratic oscillator, the locus is a straight line, i.e. there is proportionality between the square of the maximal velocities and the square of the corresponding displacements. It has been demonstrated that the time difference between two consecutive relative maxima and minima can decrease in time, and this occurs for the under-linear (sublinear) restoring force, the powers of which are smaller than unity.

Acknowledgement This investigation has financially been supported by the Ministry of Science and Technological Development, Republic of Serbia (Project No. 144008).

References

1. Nayfeh, A.H., Mook, D.T.: Nonlinear Oscillations. Wiley, New York (1979)
2. Ruzicka, J.E., Derby, T.E.: Influence of Damping in Vibration Isolation. T.S.a.V.I. Centre, US Department of Defense, Washington (1971)
3. Kovacic, I., Brennan, M.J., Waters, T.P.: A study of a nonlinear vibration isolator with quasi-zero stiffness characteristic. J. Sound Vib. **315**, 700–711 (2008)
4. Gatti, G., Kovacic, I., Brennan, M.J.: On the response of a harmonically excited two degree-of-freedom system consisting of linear and non-linear quasi-zero stiffness oscillators. J. Sound Vib. **329**, 823–1835 (2010)
5. Alabudzev, P., Gritchin, A., Kim, L., Migirenko, G., Chon, V., Stepanov, P.: Vibration Protecting and Measuring Systems with Quasi-Zero Stiffness. Hemisphere, New York (1989)
6. Mickens, R.E.: Truly Nonlinear Oscillations: Harmonic Balance, Parametric Expansions, Iteration, and Averaging Methods. World Scientific, Singapore (2010)
7. Kovacic, I., Rakaric, Z.: Oscillators with a fractional-order restoring force: higher-order approximations for motion via a modified Ritz method. Commun. Non-Linear Sci. Numer. Simul. **15**, 2651–2658 (2010)
8. Cveticanin, L., Zukovic, M.: Melnikov's criteria and chaos in systems with fractional order deflection. J. Sound Vib. **326**, 768–779 (2009)
9. Bogoliubov, N.N., Mitropolski, Y.A.: Asymptotic Methods in the Theory of Non-Linear Vibrations. Nauka, Moscow (1974) (in Russian)
10. Yuste, S.B.: Quasi-pure-cubic oscillators studied using a Krylov-Bogoliubov method. J. Sound Vib. **158**, 267–275 (1992)
11. Dasarathy, B.V.: Analysis of a class of non-linear systems. J. Sound Vib. **11**, 139–144 (1970)
12. Jordan, D.W., Smith, P.: Nonlinear Ordinary Differential Equations: An Introduction to Dynamical Systems. Oxford University Press, Oxford (1999)
13. Andronov, A.A., Vitt, A.A., Hajkin, S.E.: Theory of Vibrations. Nauka, Moscow (1981) (in Russian)
14. Klotter, K.: Free oscillations of systems having quadratic damping and arbitrary restoring forces. J. Appl. Mech. **22**, 493–499 (1955)
15. Cvetićanin, L.: Oscillator with strong quadratic damping force. Publ. Inst. Math. Nouvelle Sér. **85**, 119–130 (2009)
16. Aslam Chaudhry, M., Zubair, M.S.: A Class of Incomplete Gamma Functions with Applications. Chapman & Hall/CRC, Boca Raton (2002)
17. http://functions.wolfram.com/GammaBetaErf/Gamma2/03/01/01/0002/. Accessed 9 June 2010
18. Cveticanin, L.: Oscillator with fraction order restoring force. J. Sound Vib. **320**, 1064–1077 (2009)