Nonlinear Dyn (2011) 64: 131–145
DOI 10.1007/s11071-010-9852-x

ORIGINAL PAPER

Dynamics of three coupled limit cycle oscillators with vastly different frequencies

Hiba Sheheitli · Richard H. Rand

Received: 24 June 2010 / Accepted: 19 September 2010 / Published online: 14 October 2010

Abstract A system of three coupled limit cycle oscillators with vastly different frequencies is studied. The three oscillators, when uncoupled, have the frequencies $\omega_1 = O(1)$, $\omega_2 = O(1/\varepsilon)$ and $\omega_3 = O(1/\varepsilon^2)$, respectively, where $\varepsilon \ll 1$. The method of direct partition of motion (DPM) is extended to study the leading order dynamics of the considered autonomous system. It is shown that the limit cycles of oscillators 1 and 2, to leading order, take the form of a Jacobi elliptic function whose amplitude and frequency are modulated as the strength of coupling is varied. The dynamics of the fastest oscillator, to leading order, is unaffected by the coupling to the slower oscillator. It is also found that when the coupling strength between two of the oscillators is larger than a critical bifurcation value, the limit cycle of the slower oscillator disappears. The obtained analytical results are formal and are checked by comparison to solutions from numerical integration of the system.

Keywords Coupled oscillators · DPM · Method of direct partition of motion · Bifurcations

H. Sheheitli · R.H. Rand
Dept. Mechanical and Aerospace Engineering,
Cornell University, Ithaca, NY, USA

R.H. Rand (✉)
Dept. Mathematics and Dept. MAE, Cornell University,
Ithaca, NY, USA
e-mail: rhr2@cornell.edu

1 Introduction

The effects of high frequency excitation on nonlinear mechanical systems have been extensively studied and reviewed in recent years [1, 8, 14, 15]. These effects include apparent changes in system properties such as the number of equilibrium points, stability of equilibrium points, natural frequencies, stiffness, and bifurcation paths [15]. Such problems can be analyzed using standard perturbation methods such as the method of multiple timescales or the method of averaging [15]. However, the method of direct partition of motion (DPM) developed by Blekhman [1] serves to facilitate the study of such problems. Unlike the averaging method or the method of multiple timescales, DPM offers no systematic way to obtain higher order terms in an asymptotic expansion of the solution, and instead is limited to the leading order terms in a formal asymptotic expansion for the dynamics of the system. In return for this limitation, one gains efficiency in terms of the required mathematical manipulations. Particularly, DPM is most useful when the main interest is in the leading order slow motion of the system that is subject to the fast excitation.

More recently, Belhaq and his associates have used DPM to study the effect of high frequency excitation on systems possessing self-excited motions [2, 3, 6, 13]. It was shown that the fast excitation could lead to the disappearance of the stable limit cycle [3].

A common feature of all the aforementioned works is that the fast excitation is due to an external source, that is, all the systems considered are nonautonomous. We assert that similar non-trivial effects could occur even if the fast excitation is internal to the system, instead of coming from an external source. An example of such a case would be a nonlinear oscillator coupled to a much faster oscillator.

Systems of coupled nonlinear oscillators with widely separated frequencies have been investigated in the literature [11, 16]. Often, the method of averaging is used to study the dynamics. In this paper, the standard DPM procedure is extended to study an autonomous system of three coupled nonlinear oscillators with widely separated frequencies. When uncoupled, each of the oscillators possesses a limit cycle solution with a frequency $\omega_1 = O(1)$, $\omega_2 = O(1/\varepsilon)$ and $\omega_3 = O(1/\varepsilon^2)$, respectively, where $\varepsilon \ll 1$. We find that the coupling between such oscillators causes a change in the amplitude and frequency of the limit cycles of oscillators 1 and 2, and if the coupling between the oscillators is strong enough then the stable limit cycle of one of these two oscillators disappears. The limit cycle of the fastest oscillator, to leading order, is unchanged by the coupling.

Such a system in which several vastly different time scales interact, is ubiquitous in the nervous system, where rhythmically active subnetworks interact while oscillating at widely different frequencies [9]. It has been shown that a fast oscillatory neuron may regulate the frequency of a much slower oscillatory network [9].

Models which involve widely separated time scales also occur in astronomical applications. E.g. a study of the vibratory motion of a planet included oscillations with periods of (a) the orbital motions of the planets (tens and hundreds of years), (b) the secular orbital motions of the Solar System (tens and hundreds of thousands of years), and (c) galactic perturbations (tens and hundreds of millions of years) [10].

The system of equations, representing the coupled oscillators studied here, is presented in Sect. 2. Section 3 describes the key assumptions of the method of direct partition of motion and presents the equations which the original system is transformed into at the end of the DPM procedure. The details of the DPM implementation are given in Appendix A. Section 4 presents the approximate solution to the equations resulting from the DPM procedure, and the details of how the solution is obtained is given in Appendix B. Section 5 discusses how varying the coupling strengths affects the dynamics of the system. Finally, Sect. 6 presents a comparison between the approximate solution obtained from DPM and that from numerical integration.

2 Three coupled limit cycle oscillators

We will consider three van der Pol type limit cycle oscillators x, y, and z, which when uncoupled, are governed by the following equations:

$$\begin{aligned} &\frac{d^2x}{dt_1^2} + x + (a_1 + b_1 x^2)\frac{dx}{dt_1} = 0 \\ &\frac{d^2y}{dt_2^2} + y + (a_2 + b_2 y^2)\frac{dy}{dt_2} = 0 \\ &\frac{d^2z}{dt_3^2} + z + (a_3 + b_3 z^2)\frac{dz}{dt_3} = 0 \end{aligned} \tag{1}$$

where

$$t_1 = \omega_1 t, \quad t_2 = \frac{\omega_2}{\varepsilon} t, \quad t_3 = \frac{\omega_3}{\varepsilon^2} t, \quad \varepsilon \ll 1$$

Here, ω_1, ω_2, and ω_3 are $O(1)$ quantities. We are interested in values of a_i and b_i $(i = 1, 2, 3)$ for which each of the equations above for x, y, and z possesses a stable limit cycle solution that is an $O(\varepsilon)$ perturbation off of a simple harmonic motion. The equations posed as such indicate that oscillation along these latter limit cycles of x, y, and z occurs on the time scales t_1, t_2, and t_3, respectively, so z is a much faster oscillator than y, which is in turn a much faster oscillator than x. We will investigate the case of nearest neighbor nonlinear coupling, as shown in Fig. 1. Then the coupled system takes the following form:

$$\frac{d^2x}{dt_1^2} + x + (a_1 + b_1 x^2)\frac{dx}{dt_1}$$
$$= \gamma_1(1 + g_1 x^2)\frac{dy}{dt_1}$$

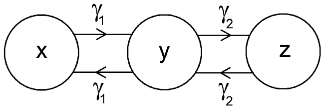

Fig. 1 Symbolic diagram for the three coupled oscillator system. γ_1 and γ_2 are coupling coefficients; see (2)

$$\begin{aligned}
&\frac{d^2y}{dt_2^2}+y+(a_2+b_2y^2)\frac{dy}{dt_2}\\
&\quad=(1+g_2y^2)\left[\gamma_1\frac{dx}{dt_2}+\gamma_2\frac{dz}{dt_2}\right]\\
&\frac{d^2z}{dt_3^2}+z+(a_3+b_3z^2)\frac{dz}{dt_3}\\
&\quad=\gamma_2(1+g_3z^2)\frac{dy}{dt_3}
\end{aligned}\tag{2}$$

This particular form of coupling is inspired by the work of Bourkha and Belhaq [3] in which the point of suspension of a self-excited pendulum is subjected to a horizontal parametric forcing. Various derivatives of x, y, and z, with respect to the different time scales, appear in (2); these are related to the corresponding derivatives with respect to time t as follows:

$$\begin{aligned}
&\frac{dx}{dt_1}=\frac{1}{\omega_1}\frac{dx}{dt}, \qquad \frac{d^2x}{dt_1^2}=\frac{1}{\omega_1^2}\frac{d^2x}{dt^2},\\
&\frac{dx}{dt_2}=\frac{\varepsilon}{\omega_2}\frac{dx}{dt}\\
&\frac{dy}{dt_1}=\frac{1}{\omega_1}\frac{dy}{dt}, \qquad \frac{dy}{dt_2}=\frac{\varepsilon}{\omega_2}\frac{dy}{dt},\\
&\frac{dy}{dt_3}=\frac{\varepsilon^2}{\omega_3}\frac{dy}{dt}, \qquad \frac{d^2y}{dt_2^2}=\frac{\varepsilon^2}{\omega_2^2}\frac{d^2y}{dt^2}\\
&\frac{dz}{dt_2}=\frac{\varepsilon}{\omega_2}\frac{dz}{dt}, \qquad \frac{dz}{dt_3}=\frac{\varepsilon^2}{\omega_3}\frac{dz}{dt},\\
&\frac{d^2z}{dt_3^2}=\frac{\varepsilon^4}{\omega_3^2}\frac{d^2z}{dt^2}
\end{aligned}\tag{3}$$

Without loss of generality, from now on, we will assume $\omega_1=1$. Now, making use of these relations, with the dot denoting differentiation with respect to time t, the coupled system becomes:

$$\begin{aligned}
&\ddot{x}+x+(a_1+b_1x^2)\dot{x}=\gamma_1(1+g_1x^2)\dot{y}\\
&\ddot{y}+\frac{\omega_2^2}{\varepsilon^2}y+\frac{\omega_2}{\varepsilon}(a_2+b_2y^2)\dot{y}\\
&\quad=\frac{\omega_2}{\varepsilon}(1+g_2y^2)[\gamma_1\dot{x}+\gamma_2\dot{z}]\\
&\ddot{z}+\frac{\omega_3^2}{\varepsilon^4}z+\frac{\omega_3}{\varepsilon^2}(a_3+b_3z^2)\dot{z}\\
&\quad=\frac{\omega_3}{\varepsilon^2}\gamma_2(1+g_3z^2)\dot{y}
\end{aligned}\tag{4}$$

3 Direct partition of motion (DPM)

The method of direct partition of motion (DPM) has been developed to study the nontrivial effects of fast excitation on nonlinear systems [1, 8, 14]. Systems previously studied in the literature using DPM tend to be nonautonomous, like the classic problem of a pendulum with a vibrating support [15]. Typically, the fast excitation is due to an external harmonic forcer with a frequency that is much larger than that of the free response of the system. The main idea of DPM is that for such problems, the solution is partitioned into a slow motion and a fast motion. In other words, one assumes that the solution can be written as a sum of two functions: a function changing on the slow time scale only, and another function changing on the fast time scale as well as the slow time scale. Here, the slow time scale refers to the time scale of the free response of the system, while the fast time scale refers to that of the fast excitation that the system is subjected to. In such problems, one is mainly interested in the slow dynamics, that is, the slow component of the solution, and the fast component is interesting only in how it affects the dynamics of the slow component. For the system described by (4), the x oscillator can be considered to be subject to fast excitation by the y oscillator, and similarly, the y oscillator can be seen to be subject to fast excitation by the z oscillator. Accordingly, we will look for a solution partitioned in the following manner:

$$\begin{aligned}
&x=X(t_1)+\varepsilon\xi(t_1,t_2,t_3)\\
&y=Y(t_2)+\varepsilon\eta(t_1,t_2,t_3)\\
&z=Z(t_3)+\varepsilon\zeta(t_1,t_2,t_3)
\end{aligned}\tag{5}$$

We will use DPM to investigate the dynamics of the leading order motions X, Y, and Z. We start by substituting (5) into (4). Terms of like powers of ε are collected and then the key assumptions of DPM are utilized [1, 8, 14]. These key assumptions can be stated as follows:

1. ξ is periodic and has a zero average over the t_2 and t_3 time scales.
2. η is periodic and has a zero average over the t_3 timescale.

These assumptions lead to the following conditions on ξ and η:

$$\begin{aligned}\langle\xi\rangle_2 &= \left\langle\frac{\partial\xi}{\partial t_2}\right\rangle_2 = \left\langle\frac{\partial^2\xi}{\partial t_2^2}\right\rangle_2\\ &= \langle\xi\rangle_3 = \left\langle\frac{\partial\xi}{\partial t_3}\right\rangle_3 = \left\langle\frac{\partial^2\xi}{\partial t_3^2}\right\rangle_3\\ &= \langle\eta\rangle_3 = \left\langle\frac{\partial\eta}{\partial t_3}\right\rangle_3 = \left\langle\frac{\partial^2\eta}{\partial t_3^2}\right\rangle_3 = 0 \qquad (6)\end{aligned}$$

Where the two operators $\langle\bullet\rangle_2$ and $\langle\bullet\rangle_3$ are defined to be the average over one period of oscillations on the t_2 and t_3 time scales, respectively. Denoting those periods by T_2 and T_3:

$$\langle\bullet\rangle_2 = \frac{1}{T_2}\int_0^{T_2}\bullet\, dt_2, \qquad \langle\bullet\rangle_3 = \frac{1}{T_3}\int_0^{T_3}\bullet\, dt_3$$

The calculations performed are algebraically complicated and typical of DPM calculations [1, 8, 14]. We present the details in Appendix A, with the result that the coupled system in (4) is transformed to the following equations governing X, Y, Z, and ξ, η, ζ:

$$\frac{d^2X}{dt_1^2} + X + (a_1 + b_1X^2)\frac{dX}{dt_1} - 2\gamma_1 g_1\omega_2 X\left\langle\xi\frac{dY}{dt_2}\right\rangle_2 = 0 \qquad (7)$$

$$\frac{d^2Y}{dt_2^2} + Y + (a_2 + b_2Y^2)\frac{dY}{dt_2} - 2\gamma_2 g_2\frac{\omega_3}{\omega_2}Y\left\langle\eta\frac{dZ}{dt_3}\right\rangle_3 = 0 \qquad (8)$$

$$\frac{d^2Z}{dt_3^2} + Z + (a_3 + b_3Z^2)\frac{dZ}{dt_3} = 0 \qquad (9)$$

$$\frac{\partial^2\xi}{\partial t_2^2} - \frac{\gamma_1}{\omega_2}(1 + g_1X^2)\frac{dY}{dt_2} = 0 \qquad (10)$$

$$,\frac{\partial^2\eta}{\partial t_3^2} - \gamma_2\frac{\omega_2}{\omega_3}(1 + g_2Y^2)\frac{dZ}{dt_3} = 0 \qquad (11)$$

$$\frac{d^2\zeta}{dt_3^2} + \zeta + (a_3 + b_3Z^2)\frac{d\zeta}{dt_3} + 2b_3\zeta\frac{dZ}{dt_3}Z - \gamma_2\left[\left(\frac{\omega_2}{\omega_3}\frac{dY}{dt_2} + \frac{d\eta}{dt_3}\right)(1 + g_3Z^2)\right] = 0 \qquad (12)$$

Recall that our goal is to understand the motion of X, Y, and Z. Note that (9) governing Z, is independent of ζ, unlike (7) on X and (8) on Y which depend on ξ and η, respectively. Thus, we will not need to solve (12) on ζ, which we nevertheless list here for completeness.

4 Solving for X, Y and Z

The equations listed at the end of the previous section can be tackled successively. First, we will seek an approximate solution for the Z equation, since it is uncoupled from X and Y. This allows us to solve (11) for an expression of η in terms of Y and t_3. Plugging the expression for η and Z into (8) allows us to evaluate the definite integral and solve for an approximate expression for Y. Again, plugging the obtained expression for Y in (10) provides an expression for ξ in terms of X and t_2. Then plugging the expression for ξ and Y into (7) allows us to evaluate the corresponding definite integral and solve for an approximate solution for X. For the convenience of the reader, we present the final results here, and give the details of the described process in Appendix B.

- The limit cycle solution for the Z equation is approximated as:

$$Z(t_3) \approx C_3\cos(t_3), \qquad C_3 = 2\sqrt{-\frac{a_3}{b_3}} \qquad (13)$$

- The equation governing Y takes the following form:

$$\frac{d^2Y}{dt_2^2} + \alpha_2 Y + \beta_2 Y^3 + (a_2 + b_2Y^2)\frac{dY}{dt_2} = 0 \qquad (14)$$

where

$$\alpha_2 = 1 + \gamma_2^2C_3^2g_2 = 1 - 4\gamma_2^2\frac{a_3}{b_3}g_2,$$

$$\beta_2 = \gamma_2^2C_3^2g_2^2 = -4\gamma_2^2\frac{a_3}{b_3}g_2^2$$

The latter equation admits an approximate steady state solution that can be written as a Jacobi elliptic function:

$$Y(t_2) \approx C_2\mathrm{cn}(A_2t_2, k_2) \qquad (15)$$

where

$$A_2^2 = \alpha_2 + \beta_2C_2^2, \qquad k_2^2 = \frac{\beta_2C_2^2}{2A_2^2},$$

$$\alpha_2 = 1 - 4\gamma_2^2 \frac{a_3}{b_3} g_2, \qquad \beta_2 = -4\gamma_2^2 \frac{a_3}{b_3} g_2^2$$

Here, C_2 is the amplitude of the solution, the coefficient A_2 affects the frequency of the solution, and the modulus k_2 affects both the amplitude and frequency of the solution. The above expressions for A_2 and k_2 in terms of C_2 represent the frequency-amplitude relation for the solution.

The amplitude C_2 is a root of a Melnikov integral and takes the value of the solution to the following equation:

$$H_2 = a_2 \mathrm{I}_1(k_2) + b_2 C_2^2 \mathrm{I}_2(k_2) = 0 \tag{16}$$

where

$$\mathrm{I}_1(k_2) = \frac{1}{3k_2^2}\left[\left(2k_2^2 - 1\right)\mathrm{E}(k_2) + \left(1 - k_2^2\right)\mathrm{K}(k_2)\right]$$

$$\mathrm{I}_2(k_2) = \frac{1}{15k_2^4}\left[2\left(k_2^4 - k_2^2 + 1\right)\mathrm{E}(k_2) - \left(k_2^4 - 3k_2^2 + 2\right)\mathrm{K}(k_2)\right]$$

k_2 denotes the value of the modulus corresponding to C_2, $\mathrm{K}(k)$ is the complete elliptic integral of the first kind and $\mathrm{E}(k)$ is the complete elliptic integral of the second kind.

Such a limit cycle solution has a period T_2 expressed as

$$T_2 = \frac{4\mathrm{K}(k_2)}{A_2}$$

From (14) and (15), we can see that the steady state behavior of Y depends only on a_2, b_2, a_3, b_3, g_2, and γ_2.

It is worth noting that while the uncoupled y oscillator in (1) has only one equilibrium point at the origin, (14) shows that Y can posses two additional equilibrium points given by

$$Y = Y^* = \pm\sqrt{\frac{-\alpha_2}{\beta_2}}, \qquad \frac{dY}{dt_2} = 0 \tag{17}$$

Such a change of the number of equilibrium points of a system is a well-known nontrivial effect of fast excitation [15]. We can see that these equilibrium points exist only if α_2 and β_2 have opposite signs.

- The equation governing X takes the following form:

$$\frac{d^2X}{dt_1^2} + \alpha_1 X + \beta_1 X^3 + \left(a_1 + b_1 X^2\right)\frac{dX}{dt_1} = 0 \tag{18}$$

where

$$\alpha_1 = 1 - 2\gamma_1^2 g_1 F, \qquad \beta_1 = -2\gamma_1^2 g_1^2 F$$

$$F = \frac{4C_2^2}{A_2^2 k_2^2 T_2}\left[\left(1 - k_2^2\right)\mathrm{K}(k_2) - \mathrm{E}(k_2)\right]$$

Note that the equation governing X equation is analogous in form to the Y equation and is treated similarly. Then an approximate steady state solution for X can be written as

$$X(t_1) \approx C_1 \mathrm{cn}(A_1 t_1, k_1) \tag{19}$$

where

$$A_1^2 = \alpha_1 + \beta_1 C_1^2, \qquad k_1^2 = \frac{\beta_1 C_1^2}{2A_1^2},$$

$$\alpha_1 = 1 - 2\gamma_1^2 g_1 F, \qquad \beta_1 = -2\gamma_1^2 g_1^2 F$$

where C_1 is a solution to the following equation:

$$H_1 = a_1 \mathrm{I}_1(k_1) + b_1 C_1^2 \mathrm{I}_2(k_1) = 0 \tag{20}$$

with

$$\mathrm{I}_1(k_1) = \frac{1}{3k_1^2}\left[\left(2k_1^2 - 1\right)\mathrm{E}(k_1) + \left(1 - k_1^2\right)\mathrm{K}(k_1)\right]$$

$$\mathrm{I}_2(k_1) = \frac{1}{15k_1^4}\left[2\left(k_1^4 - k_1^2 + 1\right)\mathrm{E}(k_1) - \left(k_1^4 - 3k_1^2 + 2\right)\mathrm{K}(k_1)\right]$$

The corresponding period of X in t_1 is

$$T_1 = \frac{4\mathrm{K}(k_1)}{A_1}$$

Again, while the uncoupled x oscillator in (1) has only one equilibrium point at the origin, (18) shows that X can posses two additional equilibrium points given by

$$X = X^* = \pm\sqrt{\frac{-\alpha_1}{\beta_1}}, \qquad \frac{dX}{dt_1} = 0 \tag{21}$$

These equilibrium points exist only if α_1 and β_1 have opposite signs.

From (18) and (19), we can see that the steady state behavior of X depends on the following parameters: a_i and b_i for $i = 1, 2, 3$, as well as g_i and γ_i for $i = 1, 2$.

5 Bifurcation of limit cycles

Notice that there are many parameters in the obtained results. In this section, we discuss the effect of varying the coupling strengths γ_1 and γ_2 while holding the other parameters fixed.

From (14), we can see that the behavior of Y does not depend on γ_1. However, as γ_2 is varied while the other parameters are fixed, α_2 and β_2 vary. Detailed analysis of the codimension two bifurcation that occurs in a system of the form of (14) can be found in the literature [7]. In particular, for certain parameter values, there exists an unstable limit cycle solution, in addition to the stable one [7]. This unstable limit cycle is born through a Hopf bifurcation followed by a homoclinic bifurcation [7]. When that unstable limit cycle is first born, it encloses the origin and has a smaller amplitude than the stable limit cycle. As parameters are varied, the two limit cycles approach each other until they suddenly coalesce and disappear [7]. Once the cycles disappear, the only stable attractors for the system are the two equilibrium points given in (17).

It is found that as γ_2 is increased, the period of oscillation of the y oscillator increases, and for γ_2 equal to a critical value $\gamma_{2_{cr}}$, the limit cycle suddenly disappears. That is, for $\gamma_2 \geq \gamma_{2_{cr}}$, (16) has no real solution.

Similarly, holding all other parameters fixed, as γ_1 is increased, α_1 and β_1 in (18) vary. Then the equation governing X undergoes the same bifurcations as the Y equation. It is found that as γ_1 is increased the period of the x oscillations increases and then the limit cycle of x suddenly disappears for γ_1 equal to a critical value $\gamma_{1_{cr}}$. That is, for $\gamma_1 \geq \gamma_{1_{cr}}$, (20) has no real solution.

While the Y equation is independent of γ_1, the X equation depends on both γ_2 and γ_1. This is because the X equation depends implicitly on the amplitude and period of Y through the factor F, as expressed in (18). Consequently, the value of $\gamma_{1_{cr}}$ varies as γ_2 is varied.

The bifurcation diagram in Fig. 2 summarizes the dependence of the existence of stable limit cycle solutions for X and Y on the value of γ_1 and γ_2. The rest of the parameters were fixed to the following typical values:

$$a_1 = a_2 = a_3 = -0.1$$

$$b_1 = b_2 = b_3 = 0.5$$

$$g_1 = g_2 = g_3 = -0.5$$

$$\omega_1 = \omega_2 = \omega_3 = 1$$

$$\varepsilon = 0.04$$

6 Numerical validation

In order to check the approximate formal solution to (4) that we obtained, we start by fixing $\gamma_2 = 1$ and varying γ_1. Throughout this section, the other parameters are fixed to the set of values given in the previous section. Figures 3, 4, 5, and 6 show the approximate formal solution compared to that obtained from numerical integration. In all these plots, dotted lines correspond to the approximate formal solution of the X, Y, and Z equations, and solid lines correspond to numerical solutions of the full system in (4).

For a fixed $\gamma_2 = 1$, as γ_1 is increased, we can see that the limit cycle of x disappears for $\gamma_1 = 1.73$ (Fig. 4). Now, if instead, we fix $\gamma_1 = 1.73$ but increase γ_2, x regains its limit cycle solution as shown in Fig. 5. As γ_2 is increased further, as in Fig. 6, the limit cycle of the y oscillator disappears.

Thus, the appearance of limit cycles in the numerical solution agrees well with the bifurcation predictions obtained from the approximate formal solution. The observed difference in oscillation period between numerical and perturbation solutions (see, e.g., x in Fig. 3 and y in Fig. 5) is due to the approximate nature of the perturbation method, which involves a formal expansion of frequency in a power series in ϵ, and which produces inaccuracies due to the assumption that $\epsilon \ll 1$. Note that since the system is autonomous, the phase of the steady state periodic solution is arbitrary, which accounts for the difference in phase between numerical and perturbation solutions when the oscillation periods agree, e.g., y and z in Fig. 3.

7 Conclusion

The standard DPM procedure was extended to study the dynamics of three coupled nonlinear oscillators.

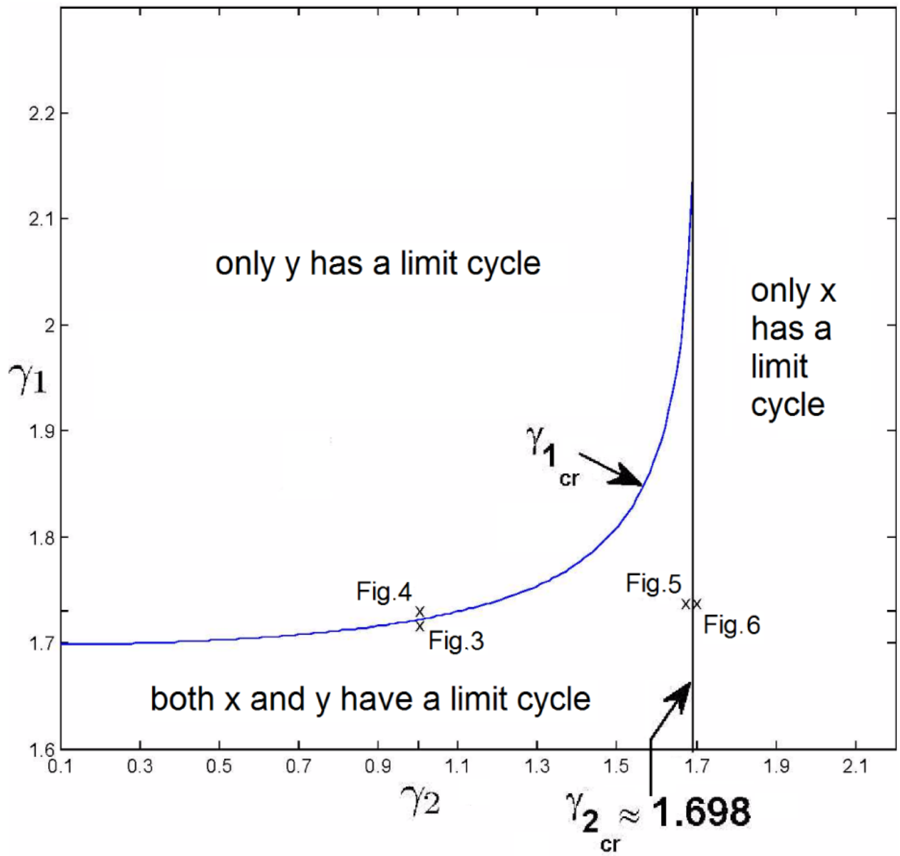

Fig. 2 Regions are displayed in the γ_1 ,γ_2 plane for which different steady state solutions exist. The $\gamma_{1_{\mathrm{cr}}}$ and $\gamma_{1_{\mathrm{cr}}}$ curves are the boundaries on which bifurcations occur. $\varepsilon = 0.04$

The oscillators, when uncoupled, have a steady state limit cycle solution due to a van der Pol type nonlinearity. The frequencies of these limit cycle solutions are widely separated such that $\omega_1 = O(1)$, $\omega_2 = O(1/\varepsilon)$ and $\omega_3 = O(1/\varepsilon^2)$, where $\varepsilon \ll 1$. To leading order, the approximate motion of each of the oscillators is found to be only affected by the coupling to a faster oscillator. So, the fastest oscillator with frequency ω_3 is unaffected by the coupling to the slower oscillator, while the amplitude and the frequency of the limit cycles of the other two oscillators with frequencies ω_1 and ω_2 are found to vary as the strength of the nearest neighbor coupling is varied. We note that in a system of two such coupled oscillators, the faster oscillator acts like a forcing function and the system behaves like a forced single degree of freedom system comparable to that studied by Bourkha and Belhaq [3].

The steady state limit cycle motions of oscillators 1 and 2 take the form of a Jacobi elliptic function. It was shown that for coupling strength greater than certain critical values, such a stable limit cycle solution of one of the oscillators disappears. It is worth noting that since the fastest oscillator is unaffected by the form of coupling used, replacing the fastest oscillator by an external harmonic forcer of the same frequency would lead to a nonautonomous system of two coupled oscillators with a similar behavior to the system considered here.

Finally, we note that the particular bifurcations that occur in such a system are highly dependent on the form of the coupling used. The coupling used here had the form of $(1 + gx^2)$ that is reminiscent of the linearization of the cosine function that often appears multiplying a fast forcer function in problems of mechanics and leads to the birth of new equilibrium points [15]. When the equation of a slow oscillator is averaged over a faster timescale, any fast variable present in that equation will average out to zero or a constant unless it multiplies the slower variable. This is because according to the DPM assumptions, any fast component is to have zero average over that fast

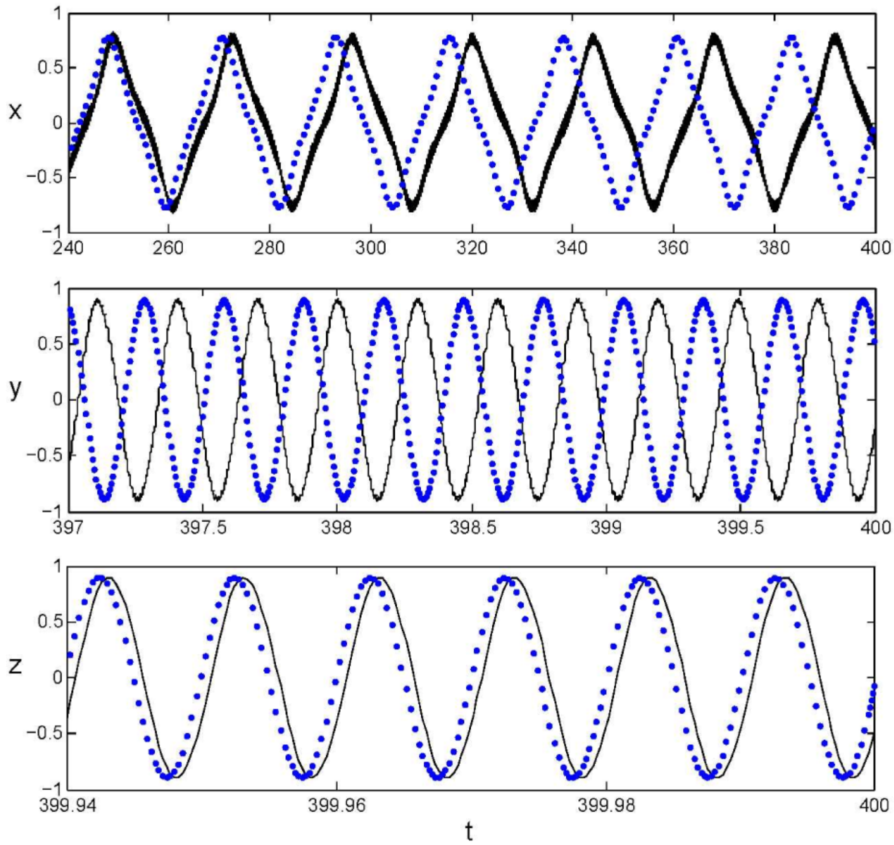

Fig. 3 Approximate formal solution (*dotted lines*) compared to that obtained from numerical integration of (4) (*solid lines*) for $\gamma_2 = 1$, γ_1=1.72, $\varepsilon = 0.04$, cf. Fig. 2

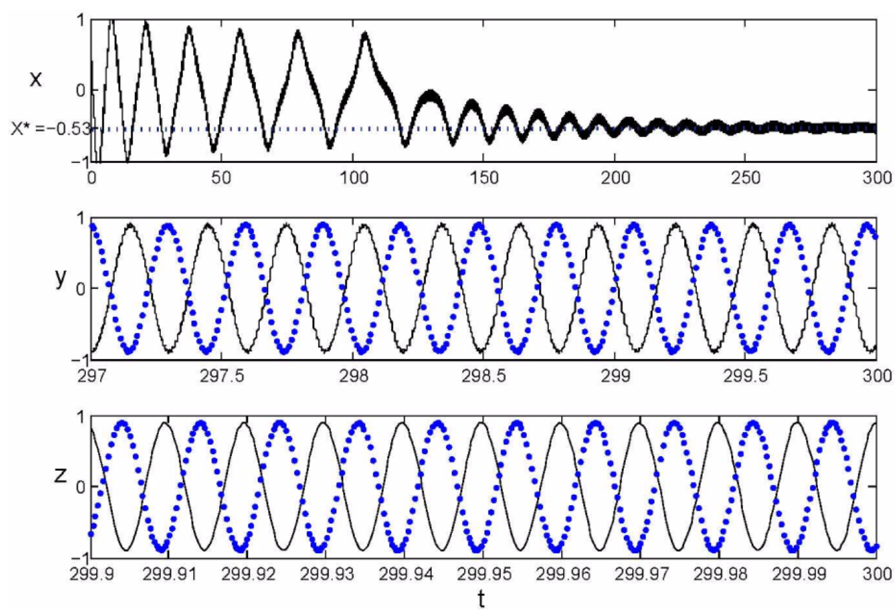

Fig. 4 Approximate formal solution (*dotted lines*) compared to that obtained from numerical integration of (4) (*solid lines*) for $\gamma_2 = 1$, γ_1=1.73, $\varepsilon = 0.04$, cf. Fig. 2

timescale. This leads to the need for a mixed nonlinear coupling term in order for nontrivial effects to occur. We expect different bifurcations to occur if different forms of nonlinear coupling are used.

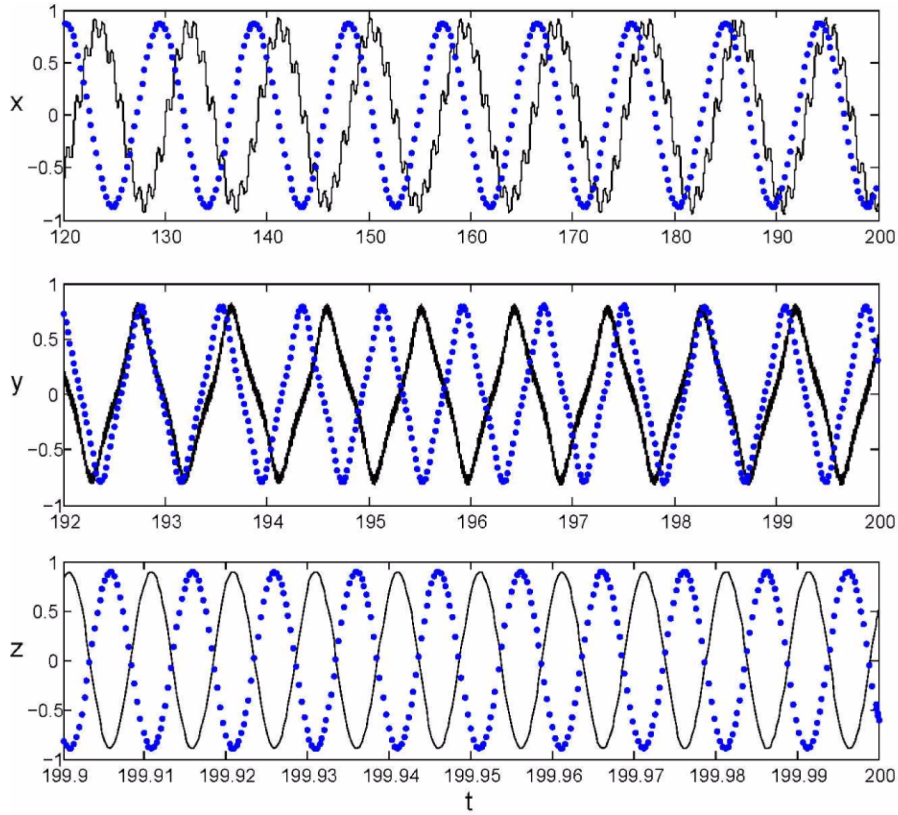

Fig. 5 Approximate formal solution (*dotted lines*) compared to that obtained from numerical integration of (4) (*solid lines*) for $\gamma_2 = 1.69$, $\gamma_1 = 1.73$, $\varepsilon = 0.04$, cf. Fig. 2

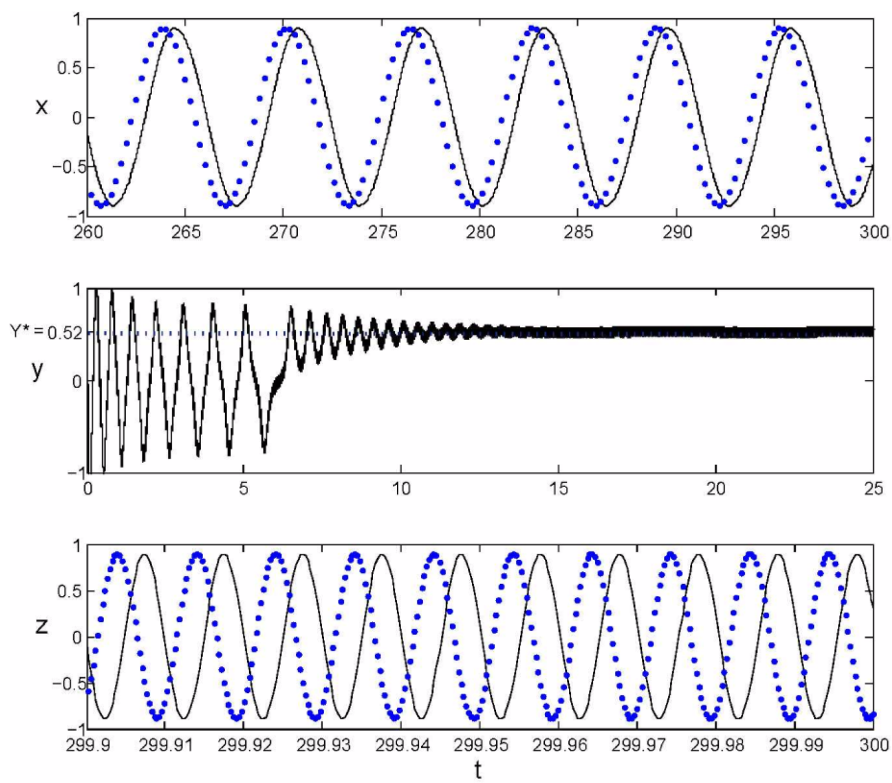

Fig. 6 Approximate formal solution (*dotted lines*) compared to that obtained from numerical integration of (4) (*solid lines*) for $\gamma_2 = 1.7$, γ_1=1.73, $\varepsilon = 0.04$, cf. Fig. 2

Acknowledgement We thank Dr. Si Mohamed Sah for sharing with us references on the method of direct partition of motion and for reviewing the MS.

Appendix A: Details of direct partition of motion

The method of direct partition of motion (DPM) is used to study the dynamics of the following system:

$$\begin{aligned}
&\ddot{x} + x + (a_1 + b_1 x^2)\dot{x} = \gamma_1(1 + g_1 x^2)\dot{y} \\
&\ddot{y} + \frac{\omega_2^2}{\varepsilon^2} y + \frac{\omega_2}{\varepsilon}(a_2 + b_2 y^2)\dot{y} \\
&\quad = \frac{\omega_2}{\varepsilon}(1 + g_2 y^2)[\gamma_1 \dot{x} + \gamma_2 \dot{z}] \\
&\ddot{z} + \frac{\omega_3^2}{\varepsilon^4} z + \frac{\omega_3}{\varepsilon^2}(a_3 + b_3 z^2)\dot{z} = \frac{\omega_3}{\varepsilon^2}\gamma_2(1 + g_3 z^2)\dot{y}
\end{aligned} \tag{22}$$

where we seek an approximate solution of partitioned as follows:

$$\begin{aligned}
x &= X(t_1) + \varepsilon\xi(t_1, t_2, t_3) \\
y &= Y(t_2) + \varepsilon\eta(t_1, t_2, t_3) \\
z &= Z(t_3) + \varepsilon\zeta(t_1, t_2, t_3)
\end{aligned} \tag{23}$$

The key assumption of DPM is that a fast component of motion is periodic and has a zero average over that fast time scale [1]. Such a condition on ξ and η and their derivatives can be written as:

$$\begin{aligned}
\langle\xi\rangle_2 &= \left\langle\frac{\partial\xi}{\partial t_2}\right\rangle_2 = \left\langle\frac{\partial^2\xi}{\partial t_2^2}\right\rangle_2 \\
&= \langle\xi\rangle_3 = \left\langle\frac{\partial\xi}{\partial t_3}\right\rangle_3 = \left\langle\frac{\partial^2\xi}{\partial t_3^2}\right\rangle_3 \\
&= \langle\eta\rangle_3 = \left\langle\frac{\partial\eta}{\partial t_3}\right\rangle_3 = \left\langle\frac{\partial^2\eta}{\partial t_3^2}\right\rangle_3 = 0
\end{aligned} \tag{24}$$

In addition, a slow function, that is, a function of one time scale only, is assumed not to change significantly over a period of a faster time scale, such that

$$\begin{aligned}
&\langle X(t_1)\rangle_2 = X(t_1), \qquad \langle X(t_1)\rangle_3 = X(t_1), \\
&\langle Y(t_2)\rangle_3 = Y(t_2)
\end{aligned} \tag{25}$$

$$\left\langle\frac{dX}{dt_1}\right\rangle_2 = \frac{dX}{dt_1}, \qquad \left\langle\frac{d^2X}{dt_1^2}\right\rangle_2 = \frac{d^2X}{dt_1^2},$$

$$\left\langle\frac{dX}{dt_1}\right\rangle_3 = \frac{dX}{dt_1}, \qquad \left\langle\frac{d^2X}{dt_1^2}\right\rangle_3 = \frac{d^2X}{dt_1^2}$$

$$\left\langle\frac{dY}{dt_2}\right\rangle_3 = \frac{dY}{dt_2}, \qquad \left\langle\frac{d^2Y}{dt_2^2}\right\rangle_3 = \frac{d^2Y}{dt_2^2}$$

The strategy is to make use of the assumptions in (24) and (25), in order to derive equations that govern X, Y, and Z. To that end, we perform the following steps:

- substitute (23) into the system in (22); the derivatives are given by the following expressions:

$$\begin{aligned}
\dot{x} &= \frac{dX}{dt_1} + \varepsilon\frac{\partial\xi}{\partial t_1} + \omega_2\frac{\partial\xi}{\partial t_2} + \frac{\omega_3}{\varepsilon}\frac{\partial\xi}{\partial t_3} \\
\ddot{x} &= \frac{d^2X}{dt_1^2} + \varepsilon\frac{\partial^2\xi}{\partial t_1^2} + \frac{\omega_2^2}{\varepsilon}\frac{\partial^2\xi}{\partial t_2^2} + \frac{\omega_3^2}{\varepsilon^3}\frac{\partial^2\xi}{\partial t_3^2} \\
&\quad + 2\omega_2\frac{\partial^2\xi}{\partial t_1\partial t_2} + 2\frac{\omega_3}{\varepsilon}\frac{\partial^2\xi}{\partial t_1\partial t_3} + 2\frac{\omega_2\omega_3}{\varepsilon^2}\frac{\partial^2\xi}{\partial t_2\partial t_3} \\
\dot{y} &= \frac{\omega_2}{\varepsilon}\frac{dY}{dt_2} + \varepsilon\frac{\partial\eta}{\partial t_1} \\
&\quad + \omega_2\frac{\partial\eta}{\partial t_2} + \frac{\omega_3}{\varepsilon}\frac{\partial\eta}{\partial t_3} \\
\ddot{y} &= \frac{\omega_2^2}{\varepsilon^2}\frac{d^2Y}{dt_2^2} + \varepsilon\frac{\partial^2\eta}{\partial t_1^2} + \frac{\omega_2^2}{\varepsilon}\frac{\partial^2\eta}{\partial t_2^2} + \frac{\omega_3^2}{\varepsilon^3}\frac{\partial^2\eta}{\partial t_3^2} \\
&\quad + 2\omega_2\frac{\partial^2\eta}{\partial t_1\partial t_2} + 2\frac{\omega_3}{\varepsilon}\frac{\partial^2\eta}{\partial t_1\partial t_3} + 2\frac{\omega_2\omega_3}{\varepsilon^2}\frac{\partial^2\eta}{\partial t_2\partial t_3} \\
\dot{z} &= \frac{\omega_3}{\varepsilon^2}\frac{dZ}{dt_3} + \varepsilon\frac{\partial\zeta}{\partial t_1} + \omega_2\frac{\partial\zeta}{\partial t_2} + \frac{\omega_3}{\varepsilon}\frac{\partial\zeta}{\partial t_3} \\
\ddot{z} &= \frac{\omega_3^2}{\varepsilon^4}\frac{d^2Z}{dt_3^2} + \varepsilon\frac{\partial^2\zeta}{\partial t_1^2} + \frac{\omega_2^2}{\varepsilon}\frac{\partial^2\zeta}{\partial t_2^2} + \frac{\omega_3^2}{\varepsilon^3}\frac{\partial^2\zeta}{\partial t_3^2} \\
&\quad + 2\omega_2\frac{\partial^2\zeta}{\partial t_1\partial t_2} + 2\frac{\omega_3}{\varepsilon}\frac{\partial^2\zeta}{\partial t_1\partial t_3} + 2\frac{\omega_2\omega_3}{\varepsilon^2}\frac{\partial^2\zeta}{\partial t_2\partial t_3}
\end{aligned}$$

- collect terms of $O(1/\varepsilon^4)$ in the z equation, this results in an equation governing $Z(t_3)$:

$$\frac{d^2Z}{dt_3^2} + Z + (a_3 + b_3 Z^2)\frac{dZ}{dt_3} = 0 \tag{26}$$

- collect terms of $O(1/\varepsilon^3)$ in the y equation, this provides an expression for η in terms of Y and Z:

$$\frac{\partial^2\eta}{\partial t_3^2} - \gamma_2\frac{\omega_2}{\omega_3}(1 + g_2 Y^2)\frac{dZ}{dt_3} = 0 \tag{27}$$

- collect terms of $O(1/\varepsilon^2)$ in the y equation:

$$\frac{d^2Y}{dt_2^2} + Y + (a_2 + b_2 Y^2)\frac{dY}{dt_2} - 2\gamma_2 g_2\frac{\omega_3}{\omega_2}Y\left(\eta\frac{dZ}{dt_3}\right)$$

$$+\frac{\omega_3}{\omega_2}Y^2\left[b_2\frac{\partial\eta}{\partial t_3}-\gamma_2 g_2\frac{\partial\zeta}{\partial t_3}+\gamma_1 g_2\frac{\partial\xi}{\partial t_3}\right]$$
$$+\frac{\omega_3}{\omega_2}\left[a_2\frac{\partial\eta}{\partial t_3}-\gamma_2\frac{\partial\zeta}{\partial t_3}-\gamma_1\frac{\partial\xi}{\partial t_3}+2\frac{\partial^2\eta}{\partial t_2\partial t_3}\right]=0 \tag{28}$$

- Average (28) over the fastest time scale t_3, making use of (24) and (25), the result is an equation governing $Y(t_2)$:

$$\frac{d^2Y}{dt_2^2}+Y+\left(a_2+b_2Y^2\right)\frac{dY}{dt_2}-2\gamma_2 g_2\frac{\omega_3}{\omega_2}Y\left\langle\eta\frac{dZ}{dt_3}\right\rangle_3=0 \tag{29}$$

- collect terms of $O(1/\varepsilon^3)$ from the x equation:

$$\frac{\partial^2\xi}{\partial t_3^2}=0 \quad\Rightarrow\quad \xi=\tilde{\xi}(t_1,t_2)+t_3\hat{\xi}(t_1,t_2)$$
$$\Rightarrow\quad \xi=\tilde{\xi}(t_1,t_2)=\xi(t_1,t_2 \text{ only}) \tag{30}$$

We have set $\hat{\xi}$ to zero, to abide by the assumption that a fast component of motion is to be periodic on the fast time scale. Note, however, that we have not enforced the condition given in (24) which requires that ξ is to have a zero average over a period in t_3, and that is because the periodicity restriction we have just enforced has established that ξ is not a function of t_3, so ξ instead would be unchanged under the operation of averaging over a period of t_3.
- looking at $O(1/\varepsilon^2)$ terms in the x equation

$$\frac{\partial^2\xi}{\partial t_2\partial t_3}=0$$

we see that this condition is readily satisfied by the result in (30).
- collect terms of $O(1/\varepsilon)$ in the x equation:

$$\frac{\partial^2\xi}{\partial t_2^2}-\frac{\gamma_1}{\omega_2}\left(1+g_1X^2\right)\frac{dY}{dt_2}$$
$$+2\frac{\omega_3}{\omega_2^2}X^2\left[b_1\frac{\partial\xi}{\partial t_3}-\gamma_1 g_1\frac{\partial\eta}{\partial t_3}\right]$$
$$+\frac{\omega_3}{\omega_2^2}\left[2\frac{\partial^2\xi}{\partial t_1\partial t_3}+a_1\frac{\partial\xi}{\partial t_3}-\gamma_1\frac{\partial\eta}{\partial t_3}\right]=0 \tag{31}$$

- substitute the expression for ξ from (30) in (31), then average (31) over the fastest time scale t_3, always using (24) and (25). We obtain an equation governing ξ in terms of X and Y:

$$\frac{\partial^2\xi}{\partial t_2^2}-\frac{\gamma_1}{\omega_2}\left(1+g_1X^2\right)\frac{dY}{dt_2}=0 \tag{32}$$

- collect terms of $O(1)$ in the x equation:

$$\frac{d^2X}{dt_1^2}+X+\left(a_1+b_1X^2\right)\frac{dX}{dt_1}$$
$$+X\left[2b_1\omega_3\xi\frac{\partial\xi}{\partial t_3}-2\gamma_1 g_1\omega_2\xi\frac{dY}{dt_2}\right.$$
$$\left.-2\gamma_1 g_1\omega_3\xi\frac{\partial\eta}{\partial t_3}\right]$$
$$+\omega_2X^2\left[b_1\frac{\partial\xi}{\partial t_2}-\gamma_1 g_1\frac{\partial\eta}{\partial t_2}\right]$$
$$+\left[a_1\omega_2\frac{\partial\xi}{\partial t_2}-\gamma_1\omega_2\frac{\partial\eta}{\partial t_2}+2\omega_2\frac{\partial^2\xi}{\partial t_1\partial t_2}\right]=0 \tag{33}$$

- substitute (32) into (33) and then average (33) over the fastest time scale t_3, finally average the resulting equation over the t_2 time scale. The result is an equation governing X:

$$\frac{d^2X}{dt_1^2}+X+\left(a_1+b_1X^2\right)\frac{dX}{dt_1}-2\gamma_1 g_1\omega_2X\left\langle\xi\frac{dY}{dt_2}\right\rangle_2=0 \tag{34}$$

Appendix B: Details of solving for X, Y, and Z

We start by finding an approximate solution for Z. We rescale a_3 and b_3 so that the Z equation looks as follows:

$$\frac{d^2Z}{dt_3^2}+Z+\varepsilon\left(a_3+b_3Z^2\right)\frac{dZ}{dt_3}=0 \tag{35}$$

We expand Z in an asymptotic series:

$$Z(t_3)=Z_0(t_3)+\varepsilon Z_1(t_3)+\cdots$$

First, we collect the leading order terms to get an equation for Z_0:

$$\frac{d^2Z_0}{dt_3^2}+Z_0=0 \quad\Rightarrow\quad Z_0=C_3\cos(t_3)$$

Then we substitute this expression for Z_0 into the equation we obtain from collecting the $O(\varepsilon)$ terms in (35). The result is an equation governing Z_1:

$$\frac{d^2Z_1}{dt_3^2} + Z_1 - \left(a_3 + b_3Z_0^2\right)\frac{dZ_0}{dt_3} = 0$$

$$\Rightarrow \quad \frac{d^2Z_1}{dt_3^2} + Z_1 = a_3C_3\sin(t_3) + b_3C_3^3\cos^2(t_3)\sin(t_3)$$

$$= a_3C_3\sin(t_3) + \frac{b_3C_3^3}{4}\left[\sin(3t_3) + \sin(t_3)\right]$$

Removing secular terms provides the following values for C_3, the amplitude of Z_0:

$$C_3 = 0 \quad \text{or} \quad C_3 = 2\sqrt{-\frac{a_3}{b_3}}$$

Hence, the limit cycle solution for the Z equation is expressed as

$$Z(t_3) \approx C_3\cos(t_3), \quad C_3 = 2\sqrt{-\frac{a_3}{b_3}} \tag{36}$$

Substituting this in the equation governing η, we get

$$\frac{\partial^2\eta}{\partial^2 t_3} + \gamma_2\frac{\omega_2}{\omega_3}\left(1 + g_2Y^2\right)C_3\sin(t_3) = 0$$

We solve for η by integrating twice over t_3:

$$\eta = \gamma_2C_3\frac{\omega_2}{\omega_3}\left(1 + g_2Y^2\right)\sin(t_3) + c_1(t_1, t_2)t_3 + c_2(t_1, t_2)$$

Now, to satisfy the assumption that η is periodic and has a zero average over t_3, the functions c_1 and c_2 need to be identically zero. So, the expression for η reduces to

$$\eta = \eta(t_2, t_3) = \gamma_2C_3\frac{\omega_2}{\omega_3}\left(1 + g_2Y^2\right)\sin(t_3) \tag{37}$$

The two expressions for Z and η in (36) and (37) are substituted in the equation for Y which we restate here:

$$\frac{d^2Y}{dt_2^2} + Y + \left(a_2 + b_2Y^2\right)\frac{dY}{dt_2} - 2\gamma_2 g_2\frac{\omega_3}{\omega_2}Y\left\langle\eta\frac{dZ}{dt_3}\right\rangle_3 = 0$$

The definite integral that Z and η appear in can now be evaluated as follows:

$$\left\langle\eta\frac{dZ}{dt_3}\right\rangle_3 = \frac{1}{2\pi}\int_0^{2\pi}\gamma_2C_3\frac{\omega_2}{\omega_3}\left(1 + g_2Y^2\right)\sin(t_3) \times \left(-C_3\sin(t_3)\right)dt_3$$

$$= -\frac{1}{2\pi}\gamma_2C_3^2\frac{\omega_2}{\omega_3}\left(1 + g_2Y^2\right)\int_0^{2\pi}\sin^2(t_3)\,dt_3$$

$$= -\frac{\gamma_2C_3^2}{2}\frac{\omega_2}{\omega_3}\left(1 + g_2Y^2\right)$$

Then the equation governing Y becomes:

$$\frac{d^2Y}{dt_2^2} + Y + \left(a_2 + b_2Y^2\right)\frac{dY}{dt_2} + \gamma_2^2C_3^2g_2Y\left(1 + g_2Y^2\right) = 0 \tag{38}$$

This equation can be rewritten as

$$\frac{d^2Y}{dt_2^2} + \alpha_2Y + \beta_2Y^3 + \left(a_2 + b_2Y^2\right)\frac{dY}{dt_2} = 0 \tag{39}$$

where

$$\alpha_2 = 1 + \gamma_2^2C_3^2g_2 = 1 - 4\gamma_2^2\frac{a_3}{b_3}g_2,$$

$$\beta_2 = \gamma_2^2C_3^2g_2^2 = -4\gamma_2^2\frac{a_3}{b_3}g_2^2$$

We rescale a_2 and b_2 such that the equation becomes

$$\frac{d^2Y}{dt_2^2} + \alpha_2Y + \beta_2Y^3 + \varepsilon\left(a_2 + b_2Y^2\right)\frac{dY}{dt_2} = 0$$

The global bifurcations that occur in such a system as α_2 and β_2 are varied, are presented in the literature [7]. It is known that for a range of parameter values, the system has a stable limit cycle, but as the parameters are varied, this stable limit cycle could disappear after colliding with an unstable limit cycle. After that, the system will have two stable equilibrium points, other than the origin which is a saddle for such parameter values. Here, we will seek an expression for that stable limit cycle solution, which we assume to be an $O(\varepsilon)$ perturbation off of a closed orbit of the conservative system corresponding to $\varepsilon = 0$. These closed orbits are known to take the form of Jacobi elliptic functions [5]. First, we rewrite the second order equation as a system of two first order equations:

$$\frac{dY}{dt_2} = W,$$

$$\frac{dW}{dt_2} = -\alpha_2Y - \beta_2Y^3 - \varepsilon\left(a_2 + b_2Y^2\right)W$$

As we mentioned, the solutions of the $\varepsilon = 0$ system can be written as

$$Y = C\text{cn}(At_2, k)$$
$$\text{where } A^2 = \alpha_2 + \beta_2 C^2,\ k^2 = \frac{\beta_2 C^2}{2A^2}$$

Here, cn is one of the periodic Jacobi elliptic functions which has a period in time t_2 given by

$$T_2 = \frac{4\text{K}(k)}{A}$$

where $\text{K}(k)$ is the complete elliptic integral of the first kind. We can see that the period depends on the coefficient A and the modulus k, which in turn depend on the amplitude C and the parameters α_2 and β_2 of the system. This relation captures the frequency–amplitude dependence brought about by the cubic nonlinearity that was introduced into the equation on Y due to the coupling with Z.

When ε is nonzero, depending on the parameters of the system, some of these closed orbits might persist and turn into limit cycles. The amplitude of such a limit cycle, when it exists, takes the values that make the Melnikov integral around that orbit equal to zero [12]. The Melnikov integral around these orbits can be written as

$$M = \int_0^{T_2} -(a_2 + b_2 Y^2) W^2 \, dt_2$$

with

$$Y = C\text{cn}(At_2, k),$$
$$W = \frac{dY}{dt_2} = -CA\text{sn}(At_2, k)\text{dn}(At_2, k)$$
$$\Rightarrow \quad M = -\int_0^{T_2} a_2 C^2 A^2 \text{sn}^2(At_2, k)\text{dn}^2(At_2, k) dt_2$$
$$- \int_0^{T_2} b_2 C^4 A^2 \text{cn}^2(At_2, k)\text{sn}^2(At_2, k) \times \text{dn}^2(At_2, k)\, dt_2$$

where sn and dn are periodic Jacobi elliptic functions. For a fixed set of parameter values: a_3, b_3, a_2, b_2, γ_2, and g_2, we substitute the expressions for A and k in terms of C into the above expression, and look for values of C for which $M = 0$. These values give the amplitude of the limit cycle of the nonzero ε system, when such a limit cycle exists. We denote this special amplitude by C_2. The $M = 0$ condition, which determines C_2, can be written concisely as

$$H_2 = a_2 \text{I}_1(k_2) + b_2 C_2^2 \text{I}_2(k_2) = 0 \tag{40}$$

where

$$\text{I}_1(k_2) = \frac{1}{3k_2^2}\left[\left(2k_2^2 - 1\right)\text{E}(k_2) + \left(1 - k_2^2\right)\text{K}(k_2)\right]$$
$$\text{I}_2(k_2) = \frac{1}{15k_2^4}\left[2\left(k_2^4 - k_2^2 + 1\right)\text{E}(k_2) - \left(k_2^4 - 3k_2^2 + 2\right)\text{K}(k_2)\right]$$

where k_2 denotes the value of the modulus corresponding to C_2 and $\text{E}(k)$ is the complete elliptic integral of the second kind. The expressions for I_1 and I_2 were obtained by evaluating the definite integrals appearing in the expression for M [4].

Hence, for each set of parameter values, (40) can be solved numerically to give the value of C_2 and then the approximate expression for Y is given by

$$Y(t_2) \approx C_2 \text{cn}(A_2 t_2, k_2) \tag{41}$$

where

$$A_2^2 = \alpha_2 + \beta_2 C_2^2, \qquad k_2^2 = \frac{\beta_2 C_2^2}{2A_2^2},$$
$$\alpha_2 = 1 - 4\gamma_2^2 \frac{a_3}{b_3} g_2, \qquad \beta_2 = -4\gamma_2^2 \frac{a_3}{b_3} g_2^2$$

and this solution has a period T_2 expressed as

$$T_2 = \frac{4\text{K}(k_2)}{A_2}$$

Note that for certain parameter values, (40) could have more than one solution. Based on the knowledge of the sequence of bifurcations that occur for such a system [7], we know that the stable limit cycle always has a larger amplitude than any unstable limit cycle that might exist simultaneously. So, C_2 is taken to be the solution to (40) with the largest value. Also, for certain parameter values, there could be no solutions to (40) and that would mean that no limit cycle solution exists for the Y equation.

Substituting (41) into the equation governing ξ, we obtain

$$\frac{\partial^2 \xi}{\partial t_2^2} - \frac{\gamma_1}{\omega_2}\left(1 + g_1 X^2\right)\frac{dY}{dt_2} = 0$$

$$\Rightarrow \quad \frac{\partial \xi}{\partial t_2} = \frac{\gamma_1}{\omega_2}\left(1 + g_1 X^2\right)Y + c_3(t_1)$$
$$\Rightarrow \quad \frac{\partial \xi}{\partial t_2} = \frac{\gamma_1}{\omega_2}\left(1 + g_1 X^2\right)Y \tag{42}$$

We have set c_3 to be identically zero in order to satisfy the condition that ξ has a zero average over t_2. Now, we can use this result, along with (41), to evaluate the definite integral that appears in the X equation:

$$\begin{aligned}
\left\langle \xi \frac{dY}{dt_2} \right\rangle_2 &= \frac{1}{T_2}\int_0^{T_2} \xi \frac{dY}{dt_2}\, dt_2 \\
&= \frac{1}{T_2}\left[(\xi Y)\big|_0^{T_2} - \int_0^{T_2} \frac{d\xi}{dt_2} Y\, dt_2 \right] \\
&= -\frac{1}{T_2}\int_0^{T_2} \frac{\gamma_1}{\omega_2}\left(1 + g_1 X^2\right) Y^2\, dt_2 \\
&= -\frac{\gamma_1}{\omega_2}\left(1 + g_1 X^2\right) \\
&\quad \times \frac{1}{T_2}\int_0^{T_2} C_2^2 \mathrm{cn}^2(A_2 t_2, k_2)\, dt_2 \\
\Rightarrow \quad \left\langle \xi \frac{dY}{dt_2} \right\rangle_2 &= \frac{\gamma_1}{\omega_2}\left(1 + g_1 X^2\right) F
\end{aligned}$$

$$\text{where } F = \frac{4C_2^2}{A_2^2 k_2^2 T_2}\left[\left(1 - k_2^2\right)\mathrm{K}(k_2) - \mathrm{E}(k_2)\right] \tag{43}$$

The expression for F was obtained by evaluating the definite integral of cn^2 [4]. Note that we have set the constant term generated by the integration by parts to zero. This is because Y is assumed periodic in t_2 with period T_2, then this would be also true for ξ as seen from (42).

After substituting (43), the X equation becomes

$$\begin{aligned}
&\frac{d^2X}{dt_1^2} + X + \left(a_1 + b_1 X^2\right)\frac{dX}{dt_1} \\
&- 2\gamma_1^2 g_1 X\left(1 + g_1 X^2\right) F = 0
\end{aligned}$$

We rewrite this as

$$\frac{d^2X}{dt_1^2} + \alpha_1 X + \beta_1 X^3 + \left(a_1 + b_1 X^2\right)\frac{dX}{dt_1} = 0 \tag{44}$$

where

$$\alpha_1 = 1 - 2\gamma_1^2 g_1 F, \qquad \beta_1 = -2\gamma_1^2 g_1^2 F$$

This equation has the same form as (39) and so it will admit a similar solution to that given in (41). An approximate solution for X can then be written as

$$X(t_1) \approx C_1 \mathrm{cn}(A_1 t_1, k_1) \tag{45}$$

where

$$A_1^2 = \alpha_1 + \beta_1 C_1^2, \qquad k_1^2 = \frac{\beta_1 C_1^2}{2A_1^2},$$
$$\alpha_1 = 1 - 2\gamma_1^2 g_1 F, \qquad \beta_1 = -2\gamma_1^2 g_1^2 F$$
$$F = \frac{4C_2^2}{A_2^2 k_2^2 T_2}\left[\left(1 - k_2^2\right)\mathrm{K}(k_2) - \mathrm{E}(k_2)\right]$$

C_1 is determined by numerically solving the following equation:

$$H_1 = a_1 \mathrm{I}_1(k_1) + b_1 C_1^2 \mathrm{I}_2(k_1) = 0 \tag{46}$$

with

$$\mathrm{I}_1(k_1) = \frac{1}{3k_1^2}\left[\left(2k_1^2 - 1\right)\mathrm{E}(k_1) + \left(1 - k_1^2\right)\mathrm{K}(k_1)\right]$$
$$\begin{aligned}
\mathrm{I}_2(k_1) = \frac{1}{15k_1^4}\Big[&2\left(k_1^4 - k_1^2 + 1\right)\mathrm{E}(k_1) \\
&- \left(k_1^4 - 3k_1^2 + 2\right)\mathrm{K}(k_1)\Big]
\end{aligned}$$

The corresponding period of X in t_1 is

$$T_1 = \frac{4\mathrm{K}(k_1)}{A_1}$$

Again, we note here that the X equation will admit such a limit cycle for a certain set of values of the parameters, only if these values allow a solution to (46).

Note that if parameters in the Y equation are such that the only stable steady state solution is an equilibrium point, then Y takes on a constant value. In this case, the term including $\frac{dY}{dt_2}$ in (34) vanishes, and Y has no influence on X, in which case the form of the equation governing X reduces to that of (26) on Z, in which case the X steady state solution becomes analogous to (36). That is, the limit cycle of X becomes very close to a harmonic oscillation $X \approx 2\sqrt{\frac{-a_1}{b_1}}\cos t_1$ which corresponds to the dotted X solution in Fig. 6.

References

1. Blekhman, I.I.: Vibrational Mechanics–Nonlinear Dynamic Effects, General Approach, Application. World Scientific, Singapore (2000)

2. Belhaq, M., Sah, S.: Horizontal fast excitation in delayed van der Pol oscillator. Commun. Nonlinear Sci. Numer. Simul. **13**, 1706–1713 (2008)
3. Bourkha, R., Belhaq, M.: Effect of fast harmonic excitation on a self-excited motion in van der Pol oscillator. Chaos Solitons Fractals **34**(2), 621 (2007)
4. Byrd, P., Friedman, M.: Handbook of Elliptic Integrals for Engineers and Scientists, 2nd edn. Springer, Berlin (1971)
5. Coppola, V.T., Rand, R.H.: Averaging using elliptic functions: approximation of limit cycles. Acta Mech. **81**, 125–142 (1990)
6. Fahsi, A., Belhaq, M.: Effect of fast harmonic excitation on frequency-locking in a van der Pol-Mathieu-Duffing oscillator. Commun. Nonlinear Sci. Numer. Simul. **14**(1), 244–253 (2009)
7. Guckenheimer, J., Holmes, P.: Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields. Springer, New York (1983)
8. Jensen, J.S.: Non-trivial effects of fast harmonic excitation. PhD dissertation, DCAMM Report, S83, Dept. Solid Mechanics, Technical University of Denmark (1999)
9. Nadim, F., Manor, Y., Nusbaum, M.P., Marder, E.: Frequency regulation of a slow rhythm by a fast periodic input. J. Neurosci. **18**(13), 5053–5067 (1998)
10. Barkin, Yu.V., Vilke, V.G.: Celestial mechanics of planet shells. Astron. Astrophys. Trans. **23**(6), 533–553 (2004)
11. Nayfeh, A.H., Chin, C.M.: Nonlinear interactions in a parametrically excited system with widely spaced frequencies. Nonlinear Dyn. **7**, 195–216 (1995)
12. Rand, R.H.: Lecture notes in nonlinear vibrations. Version 52 (2005). http://audiophile.tam.cornell.edu/randdocs
13. Sah, S., Belhaq, M.: Effect of vertical high-frequency parametric excitation on self-excited motion in a delayed van der Pol oscillator. Chaos Solitons Fractals **37**(5), 1489–1496 (2008)
14. Thomsen, J.J.: Vibrations and Stability, Advanced Theory, Analysis and Tools. Springer, Berlin (2003)
15. Thomsen, J.J.: Slow high-frequency effects in mechanics: problems, solutions, potentials. Int. J. Bifurc. Chaos **15**, 2799–2818 (2005)
16. Tuwankotta, J.M., Verhulst, F.: Hamiltonian systems with widely separated frequencies. Nonlinearity **16**, 689–706 (2003)