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ORIGINAL PAPER

Generalized synchronization of spatiotemporal chaos in a weighted complex network

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Abstract The nodes of the network are composed of the spatiotemporal chaos systems. The relations between the nodes are built through a weighted connection and the nonlinear terms of the chaos systems themselves are taken as coupling functions. The structure of the coupling functions between the connected nodes and the range of the control gain are obtained based on Lyapunov stability theory. It is proven that generalized chaos synchronization of the weight complex network can be realized even if the coupling strength between the nodes is adopted as any weight value. Subsequently, the catalytic reaction diffusion system which has spatiotemporal chaos behavior is taken as example, and simulation results show the effectiveness of the synchronization principle.

Keywords Chaos synchronization · Weighted complex network · Generalized synchronization · Lyapunov stability theory

1 Introduction

The establishment of complex network models, discussions about their properties, and the study of network dynamics have attracted considerable attention since random graph [1], small-world network [2] and scale-free network [3] were presented by Erdös, Watts, and Barabási, respectively, and much progress has been made [4–13]. Especially, chaos synchronization in networks has become a focal point in network dynamics. Therefore, Timme et al. studied the network synchronization law of pulse-coupled dynamic systems [14]; Motter et al. analyzed the influence of coupling strength on the synchronizing ability of a complex network [15]; Lu et al. accomplished synchronization analysis of linearly coupled networks of discrete time systems [16]; Atay et al. investigated synchronization of complex network when delays exist among the nodes [17]; Lü et. al. constructed general complex dynamical networks, and realized the synchronization [18]; Han et al. discussed the changes on synchronization ability of coupled networks from ring networks to chain networks [19]; Yu et al. realized synchronization of stochastic delayed neural networks based on stability theory [20]; Henning et al. studied the synchronization property of complex network with nodes of Fitzhugh–Nagumo systems [21]; Hung et al. reported globally generalized synchronization in scale-free networks [22]. However, most existing synchronization researches involved in complex networks are unweighted networks, that is, they consider only connections between nodes and ignore the intensity difference and effects. In fact, intensity difference exists between nodes of networks and weight values should be considered when complex networks are investigated. Barrat et al. presented a weighted net-

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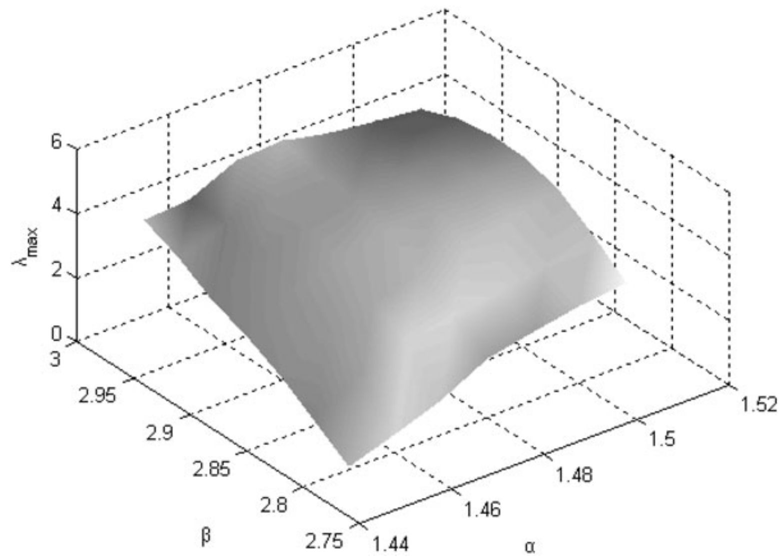


Fig. 1 The evolution of maximum Lyapunov exponent with parameters α and β

work model and analyzed its dynamic characteristics [23]; Li et al. fulfilled statistical analysis of a class of real networks [24]; Fang et al. constructed a harmonious unifying hybrid preferential model, and apply the design idea to a typical analysis of weighted network models [25]; Xiang et al. proposed feedback controllers and added effective control on weighted complex networks with heterogeneous delays [26]. Unluckily, chaos synchronization of weighted networks is less reported so far. Furthermore, an important premise in the existing network synchronization research is that the structure of the complex network is known, i.e., determined network nodes and determined coupling connection. In practice, the structure of the complex network is often partly known or totally unknown. Therefore, it is necessary to find a synchronization method that can be used in any complex network or even in any weighted complex network.

In this paper, a method of realizing generalized synchronization of spatiotemporal chaos in a weighted complex network is presented. Spatiotemporal chaos systems which show both spatial and temporal behavior are taken as nodes of the network, the nonlinear terms of the systems themselves are taken as coupling functions, and the relations between the nodes are built through a weighted connection. The structure of the coupling functions between the connected nodes and the range of the control gain are obtained based on Lyapunov stability theory. Simulation results show that generalized chaos synchronization can be fulfilled when the coupling strength between the nodes is given as any weight value. The catalytic reaction diffusion system which has spatiotemporal chaos behavior is taken as example, and simulation results show the effectiveness of the synchronization principle.

2 Synchronization principle

Suppose a complex network consists of N nodes and the state equations of the nodes are spatiotemporal chaos systems, in which the n-dimensional variable of node i at time t is $x_i(r,t)$, and $x_i(r,t) = (x_{i1}(r,t), x_{i2}(r,t), \ldots, x_{in}(r,t)) \in R^n$. Each node can be described by the following equation if the coupling action is not considered:

$$\frac{\partial x_i(r,t)}{\partial t} = F\big(x_i(r,t)\big) \tag{1}$$

where r, t are spatial and temporal variables, and $F: R^n \to R^n$.

$F(x_i(r,t))$ can be properly separated as follows:

$$\begin{aligned}\frac{\partial x_i(r,t)}{\partial t} &= F\big(x_i(r,t)\big)\\ &= Ax_i(r,t) + H\big(x_i(r,t)\big)\end{aligned} \tag{2}$$

where A is a linear coefficients matrix of the system.

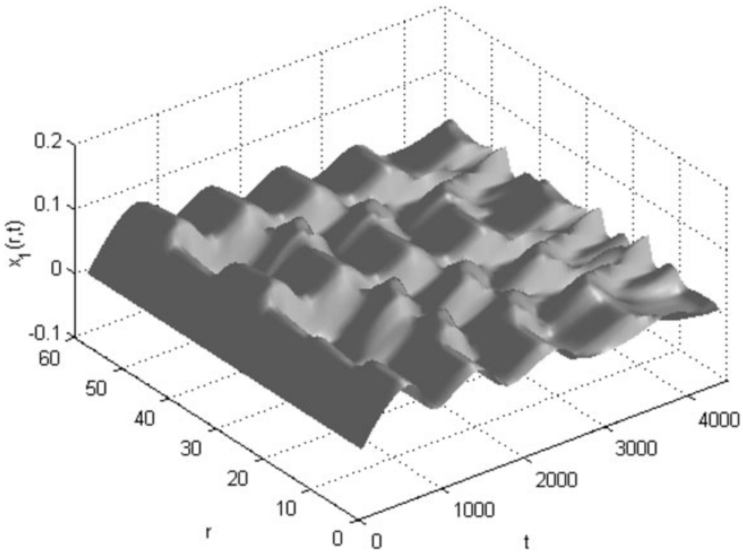


Fig. 2 The spatiotemporal evolution of variable $x_1(r,t)$

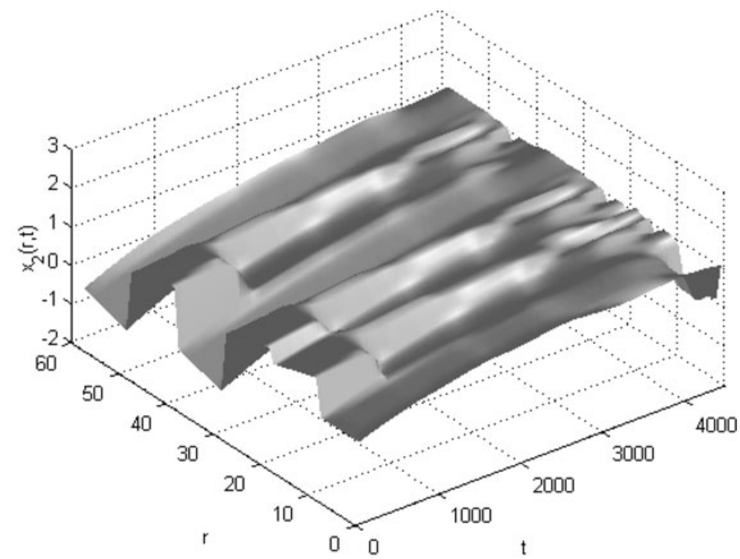


Fig. 3 The spatiotemporal evolution of variable $x_2(r,t)$

When coupling is considered, the state equation of node i can be described by

$$\begin{aligned}\frac{\partial x_i(r,t)}{\partial t} &= F\big(x_i(r,t)\big)\\ &\quad + J_i\big(x_1(r,t), x_2(r,t), \ldots, x_N(r,t)\big)\\ &= Ax_i(r,t) + H\big(x_i(r,t)\big)\\ &\quad + J_i\big(x_1(r,t), x_2(r,t), \ldots, x_N(r,t)\big)\\ &(i = 1, 2, \ldots, N) \qquad (3)\end{aligned}$$

where $J_i(x_1(r,t), x_2(r,t), \ldots, x_N(r,t))$ are coupling functions of the connected nodes to be determined.

Theorem *According to the system* (3), *when the coupling functions are taken as*

$$\begin{aligned}J_k = &- H\big(x_k(r,t)\big) + \delta_k\big[H\big(x_1(r,t)\big) + J_1\big]\\ &- m\big[x_k(r,t) - \delta_k x_1(r,t)\big]\\ &(k = 2, 3, \ldots, N) \qquad (4)\end{aligned}$$

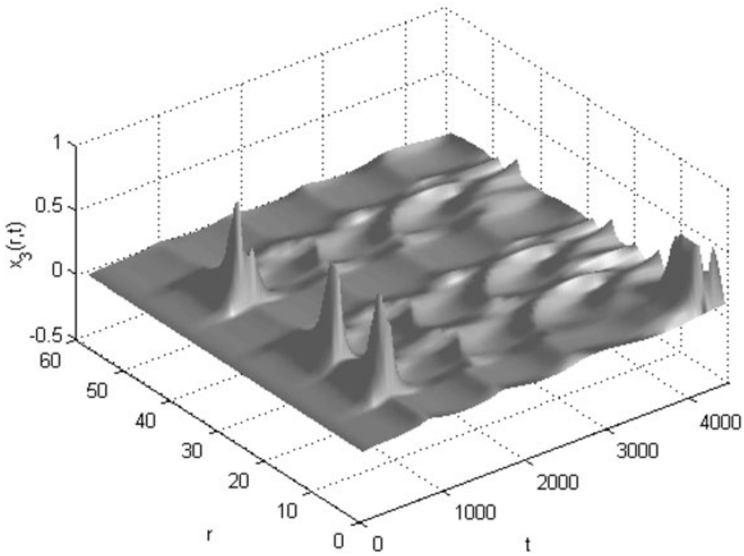


Fig. 4 The spatiotemporal evolution of variable $x_3(r,t)$

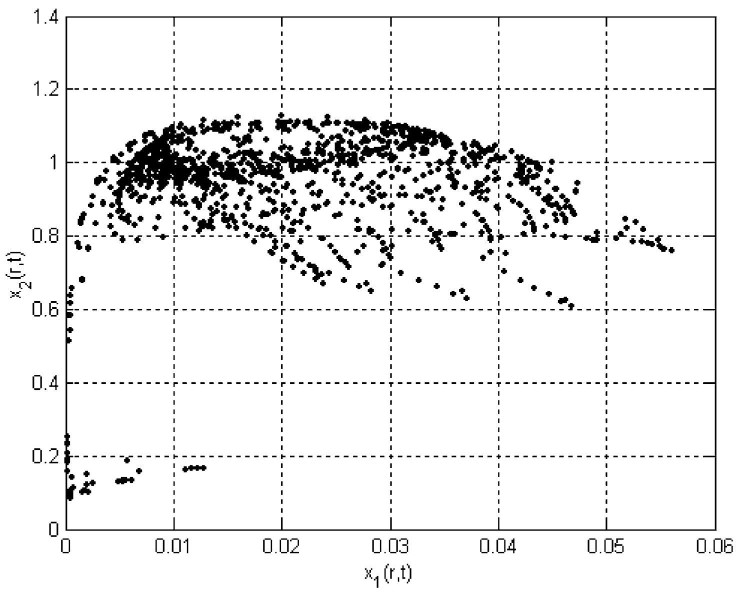


Fig. 5 The phase map of system (14)

where δ_k is the relative weight of the coupling strength between the nodes of the network, $\delta_k = \prod_{i=1}^{k-1}(-\frac{1}{\xi_i})$, ξ_i is the scaling factor of the generalized synchronization, m is the control gain. Then generalized synchronization between all nodes of any weighted complex network can be fulfilled.

Proof The errors between the state variables of the network are defined as

$$e_i(r,t) = x_i(r,t) + \xi_i x_{i+1}(r,t) \quad (i = 1, 2, \ldots, N-1) \tag{5}$$

From (5), we obtain

$$\frac{\partial e_i(r,t)}{\partial t} = \frac{\partial x_i(r,t)}{\partial t} + \xi_i \frac{\partial x_{i+1}(r,t)}{\partial t} = Ae_i + \Delta H(x_i, x_{i+1}) + \Delta J_i \tag{6}$$

where $\Delta H(x_i, x_{i+1}) = H(x_i(r,t)) + \xi_i H(x_{i+1}(r,t))$, $\Delta J_i = J_i(x_1(r,t), x_2(r,t), \ldots, x_N(r,t)) +$

Fig. 6 The spatiotemporal evolution of error variable $e_{11}(r,t)$

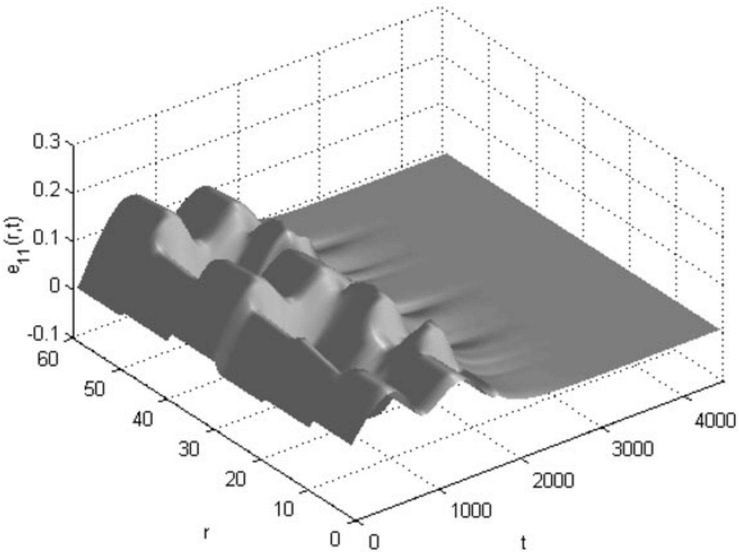
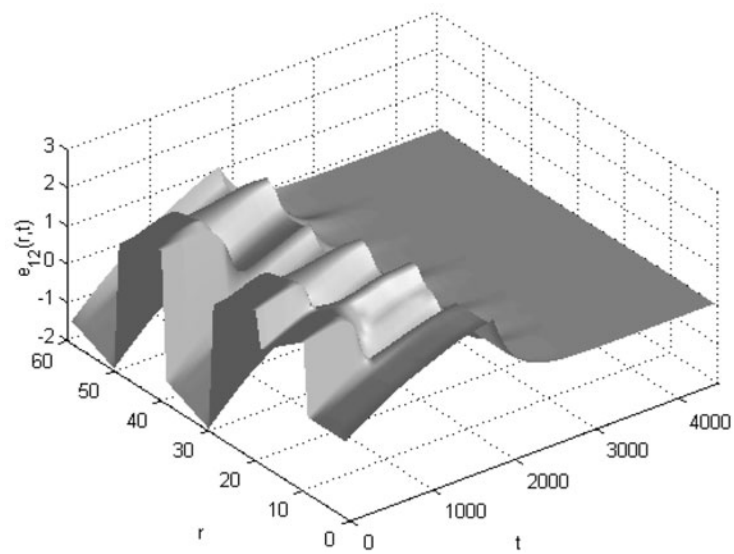


Fig. 7 The spatiotemporal evolution of error variable $e_{12}(r,t)$



$\xi_i J_{i+1}(x_1(r,t), x_2(r,t), \ldots, x_N(r,t))$. If

$$\Delta J_i = -\Delta H(x_i, x_{i+1}) - me_i(r,t) \quad (i = 1, 2, \ldots, N-1) \tag{7}$$

then

$$\begin{aligned} J_k &= \left(-\frac{1}{\xi_{k-1}}\right)\left[J_{k-1} + \Delta H(x_{k-1}, x_k) + me_{k-1}(r,t)\right] \\ &= \sum_{i=1}^{k-1}\prod_{j=i}^{k-1}\left(-\frac{1}{\xi_j}\right)\left[\Delta H(x_i, x_{i+1}) + me_i(r,t)\right] \\ &\quad + \prod_{i=1}^{k-1}\left(-\frac{1}{\xi_i}\right)J_1 \\ &= -H\big(x_k(r,t)\big) + \prod_{i=1}^{k-1}\left(-\frac{1}{\xi_i}\right)\left[H(x_1(r,t)) + J_1\right] \\ &\quad - m\left[x_k(r,t) - \prod_{i=1}^{k-1}(-\frac{1}{\xi_i})x_1(r,t)\right] \\ &= -H\big(x_k(r,t)\big) + \delta_k\left[H\big(x_1(r,t)\big) + J_1\right] \end{aligned}$$

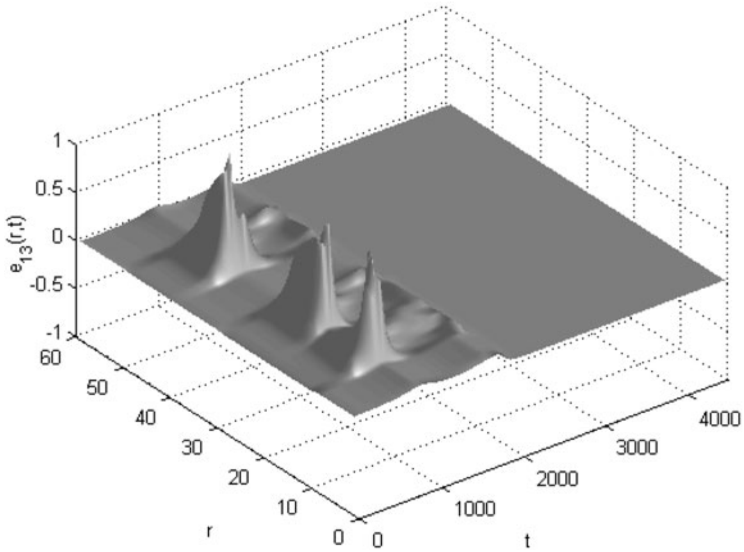


Fig. 8 The spatiotemporal evolution of error variable $e_{13}(r,t)$

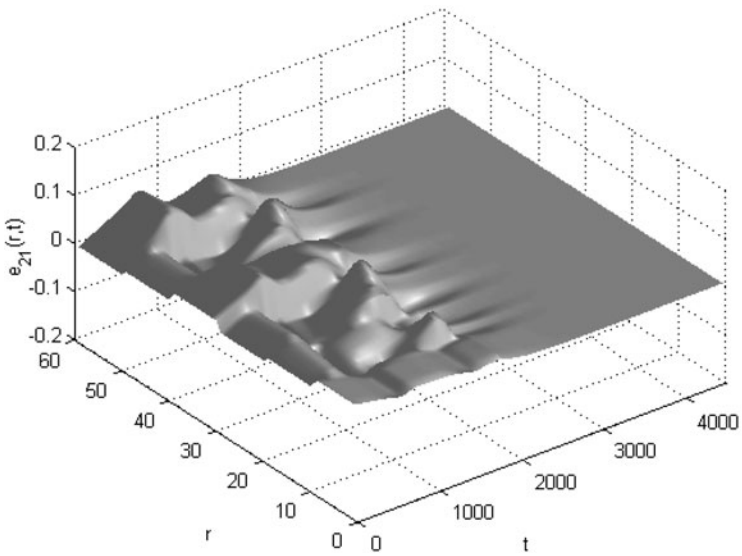


Fig. 9 The spatiotemporal evolution of error variable $e_{21}(r,t)$

$$- m\left[x_k(r,t) - \delta_k x_1(r,t)\right]$$
$$(k = 2, 3, \ldots, N) \tag{8}$$

where

$$\delta_k = \prod_{i=1}^{k-1}\left(-\frac{1}{\xi_i}\right) \tag{9}$$

It is obviously seen, δ_k is the relative weight of the coupling strength between the nodes of the network.

Constructing the Lyapunov function according to the weighted complex network

$$V(r,t) = \frac{1}{2}\sum_{i=1}^{N-1} e_i^2(r,t) \tag{10}$$

Considering (6) and (7), we can obtain the derivative form of $V(r,t)$

$$\frac{\partial V(r,t)}{\partial t} = \sum_{i=1}^{N-1} e_i(r,t)\frac{\partial e_i(r,t)}{\partial t}$$

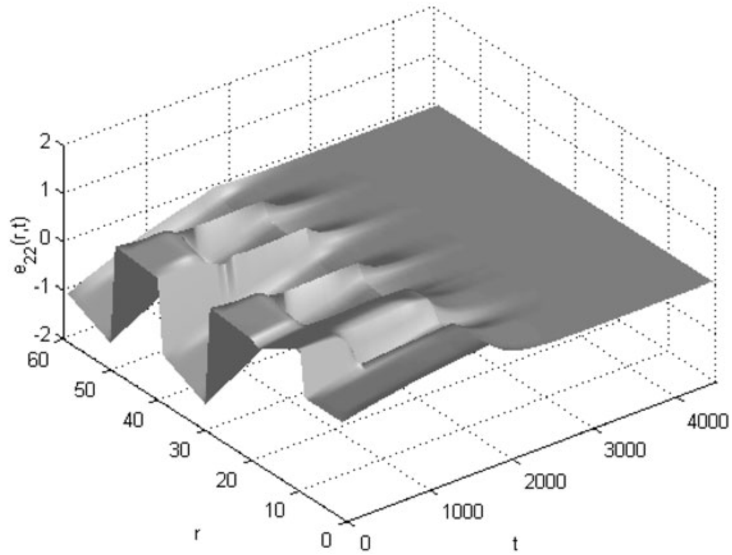


Fig. 10 The spatiotemporal evolution of error variable $e_{22}(r,t)$

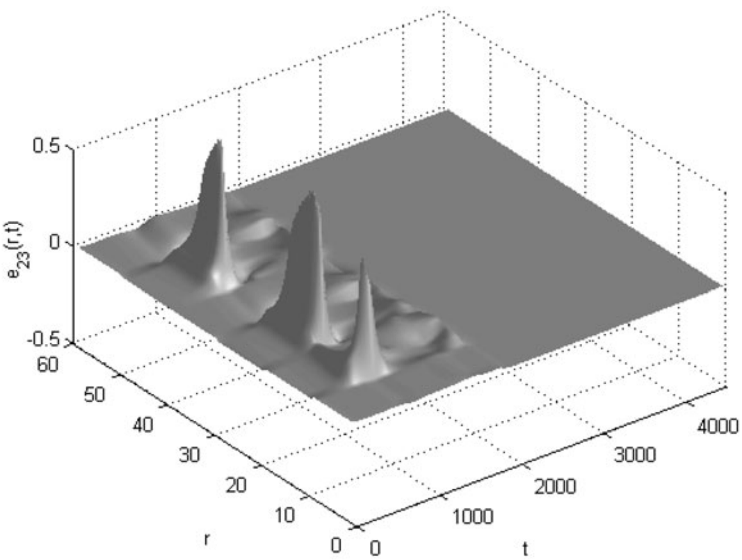


Fig. 11 The spatiotemporal evolution of error variable $e_{23}(r,t)$

$$= (A - m)\sum_{i=1}^{N-1} e_i^2(r,t) \tag{11}$$

From (11), if

$$m \geq A \tag{12}$$

then

$$\frac{\partial V(r,t)}{\partial t} \leq 0 \tag{13}$$

According to Lyapunov stability theory [27], the whole network realizes generalized synchronization.

From (9), it can be seen that the relationship between the scaling factors ξ_i and the relative weight of the coupling strength between the nodes of the network δ_k are built. In other words, no matter what value δ_k is given, generalized synchronization of any weighted complex network can be fulfilled. □

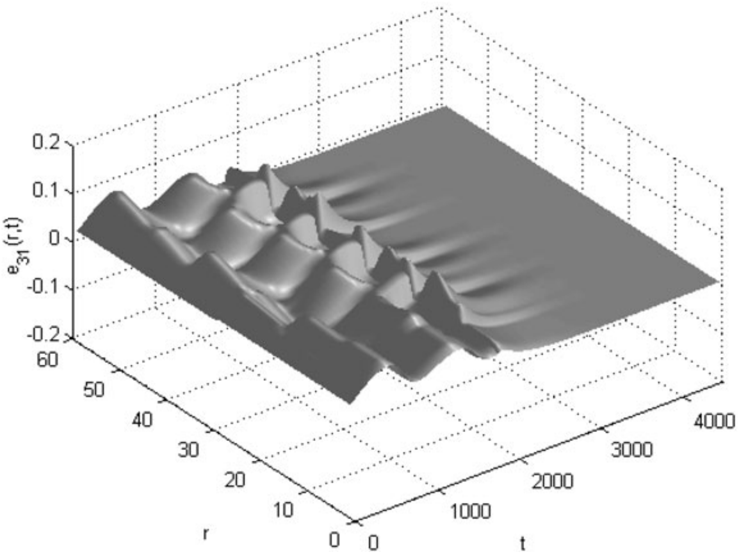


Fig. 12 The spatiotemporal evolution of error variable $e_{31}(r,t)$

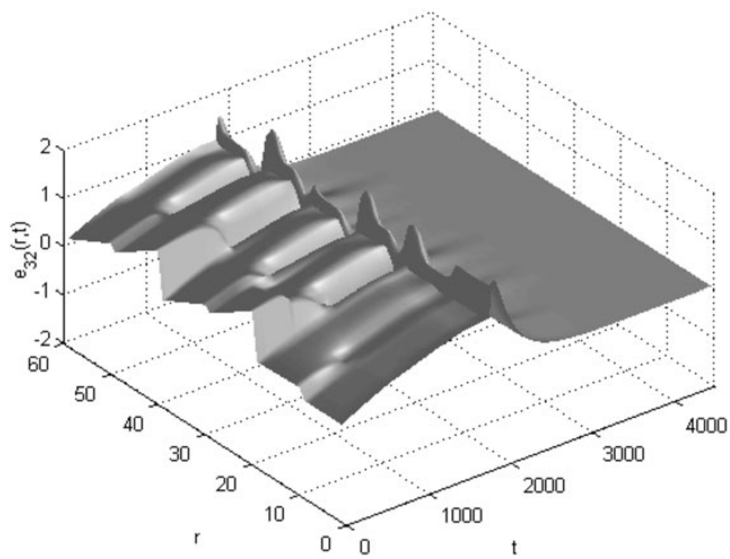


Fig. 13 The spatiotemporal evolution of error variable $e_{32}(r,t)$

3 Application example

The catalytic reaction diffusion system with rich spatiotemporal chaos patterns is taken as example to test the effectiveness of the principle above.

Lynch studied the properties of the catalytic reaction diffusion system with three variables, and found that the system shows spatiotemporal chaos behavior when the parameters are in a certain range. The catalytic reaction diffusion system can be describes as follows [28]:

$$\begin{cases} \dfrac{\partial x_1(r,t)}{\partial t} \\ \quad = 1 - x_1(r,t) - a_1 x_1(r,t) x_3^2(r,t) + d_1 \nabla^2 x_1(r,t) \\ \dfrac{\partial x_2(r,t)}{\partial t} \\ \quad = \beta - x_2(r,t) - a_2 x_2(r,t) x_3^2(r,t) + d_2 \nabla^2 x_2(r,t) \\ \dfrac{\partial x_3(r,t)}{\partial t} = 1 - (1+a_3) x_3(r,t) \\ \qquad + \alpha\big[a_1 x_1(r,t) + a_2 x_2(r,t)\big] x_3^2(r,t) \\ \qquad + d_3 \nabla^2 x_3(r,t) \end{cases} \tag{14}$$

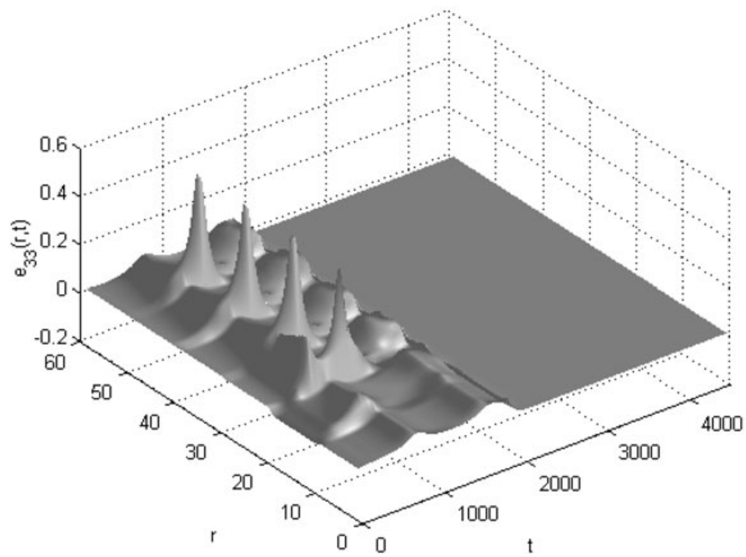


Fig. 14 The spatiotemporal evolution of error variable $e_{33}(r,t)$

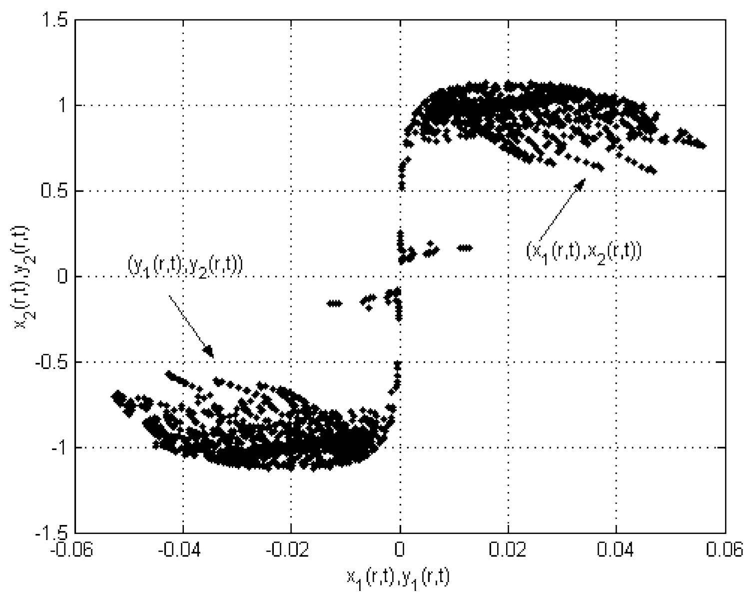


Fig. 15 The phase map of node 1 and 2 when synchronization

where the parameters $a_1 = 18000$, $a_2 = 400$, $a_3 = 80$. The diffusion coefficients $d_1 = 1.0$, $d_2 = 1.0$, $d_3 = 0.01$.

The propagation of the maximum Lyapunov exponent of the system according to parameters α and β is shown in Fig. 1. It is shown from Fig. 1 the Lyapunov exponent is positive when the parameters $\alpha = 1.5$ and $\beta = 2.93$, which means the system is in chaos. The evolution figurers of system state variables are shown in Figs. 2–4, and Fig. 5 is the phase map.

The synchronization of weighted complex network is numerical simulated when nodes number is taken as $N = 4$. The dynamic equation of a single node is the catalytic reaction diffusion system described by (14), and the variables of the four nodes are $(x_1(r,t), x_2(r,t), x_3(r,t))$, $(y_1(r,t), y_2(r,t), y_3(r,t))$, $(z_1(r,t), z_2(r,t), z_3(r,t))$ and $(w_1(r,t), w_2(r,t)$,

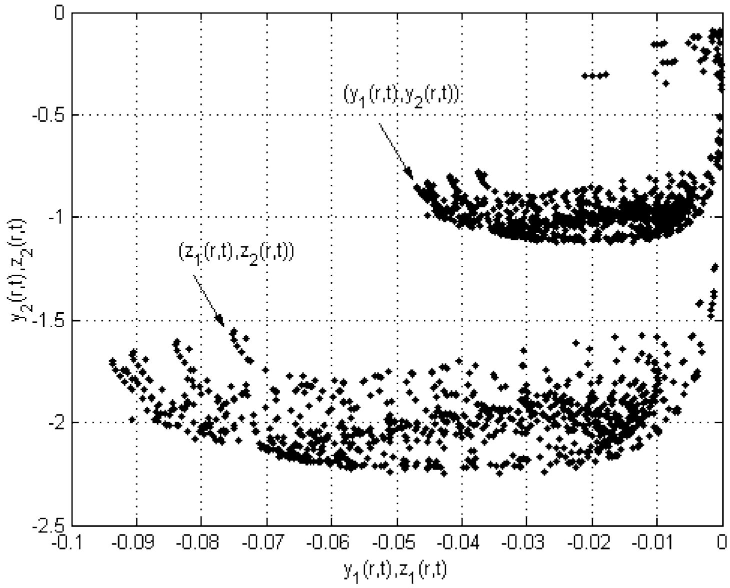


Fig. 16 The phase map of node 2 and 3 when synchronization

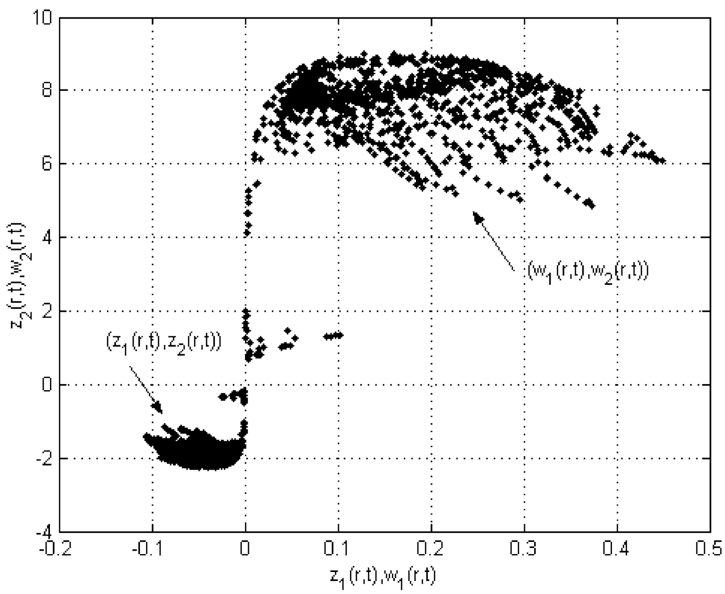


Fig. 17 The phase map of node 3 and 4 when synchronization

$w_3(r,t))$, respectively. The linear coefficients matrix of the state equations of the nodes is taken as follows:

$$A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -(1+a_3) \end{bmatrix} \tag{15}$$

In our simulation for network synchronization, the control gain is taken as $m = 0$, and the coupling function $J_1 = 0$, which means the first node of the network is taken as the target system, the others are response systems. Complex network is constructed according to (3), the chosen of J_k based on Theorem (4).

The scaling factors are arbitrarily taken as $\xi_1 = 1$, $\xi_2 = -\frac{1}{2}$, $\xi_3 = \frac{1}{4}$, which means the relative weight of the coupling strength between the nodes of the network are $\delta_2 = -1$, $\delta_3 = -2$, $\delta_4 = 8$, based on definition in (9). The network coupling is added at time series 2000, and simulation results are shown in Figs. 6–

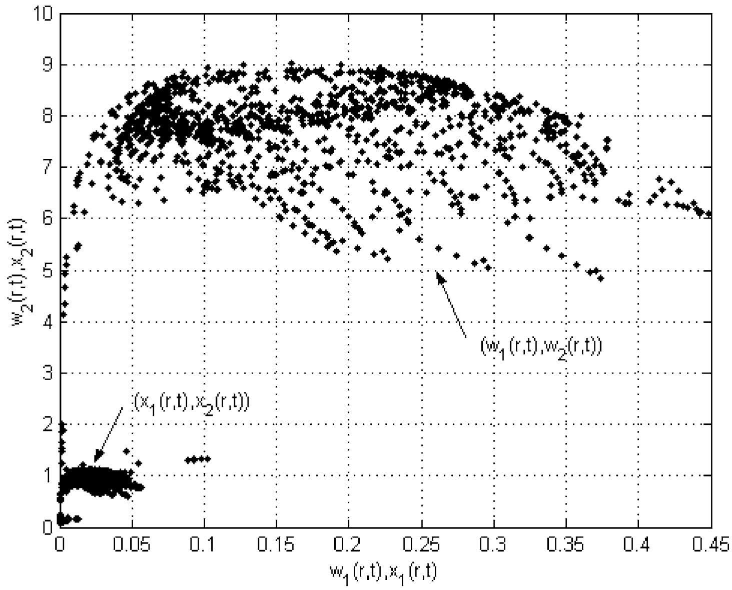


Fig. 18 The phase map of node 4 and 1 when synchronization

18. From Figs. 6–14, it is seen that all the error signals of the systems in the network approach to zero smoothly and rapidly in a short series of time after network coupling is added, and generalized synchronization is realized. Correspondingly, the amplitude of node 2 is the same as that of node 1 with opposite signs in Fig. 15; the amplitude of node 3 is two times of node 2 with the same sign in Fig. 16; the amplitude of node 4 is four times of node 3 with opposite signs in Fig. 17; the amplitude of node 1 is one eighth of node 4 with the same sign in Fig. 18.

4 Conclusions

Generalized synchronization of spatiotemporal chaos in a weighted complex network is studied in this paper. Spatiotemporal chaos systems are taken as nodes of the network, the nonlinear terms of the systems themselves are coupling functions, and the relations between the nodes are built through a weighted connection. The structure of the coupling functions between the connected nodes and the range of the feedback gain is obtained based on Lyapunov stability theory. The catalytic reaction diffusion systems are taken as examples, and simulation results show that if the range of the control gain m is bigger than that of the linear coefficients matrix of the system A, generalized chaos synchronization of the weighted complex network can be fulfilled when the coupling strength between the nodes is given as any weight value.

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