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Oscillations of a beam with a time-varying mass

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Abstract This paper analyzes dynamical behavior of a simply supported Euler–Bernoulli beam with a time-varying mass on its surface. Though the system under consideration is linear, it exhibits dynamics similar to a nonlinear system behavior including internal resonances. The asymptotical solutions for the beam displacement has been found by combining the classical Galerkin method with the averaging method for equations in Banach spaces. The resonance conditions have been derived. It has been proposed a method for finding a number of possible resonances.Effect of the beam parameters on its dynamical behavior is investigated as well.

Keywords Time-varying mass · Beam · Internal resonances

1 Introduction

Systems with a time-varying mass are found in physics and engineering (robotic, conveyor systems, excavators, chemistry) and fluid-structure interaction problems [1]. Oscillations of electric transmission lines and cables of cable-stayed bridges with water rivulets on the cable surface can also be considered as time-varying dynamic systems [2]. Usually, the stay cables are mantled with polyurethane and have a nearly circular cross section. The cable can be considered as a beam with a small bending rigidity. In instable cases, water flowing round the cable causes generation of one or two rivulets forming separation points of the air flow around a dry cylinder oscillating at the same frequency as the cable [3]. When the rivulets are subjected to the action of various mechanical or structural factors, they display interesting dynamical phenomena such as wave propagation, wave steepening, and development of chaotic responses. Macroscopic thin rivulets are entities that should be taken into account in biophysics, physics, and engineering, as well as in natural settings. The rivulets can be composed of common liquids such as water or oil, rheologically complex materials such as polymers solutions or melts. In engineering, the rivulets serve in heat and mass transfer processes to limit fluxes and to protect surfaces, and they are applied in paints, adhesives, and membranes. As this takes place, the rivulets often move on beam surfaces. Work [4] presents the experimental study of the rivulet flow along the lower side of an inclined cylinder upon a liquid jet supply. It has been shown that in the range of the considered parameters the magnitude of the wetter surface of the cylinder is constant and it is not dependent on the flow rate. The rivulet width also does not depend on the flow rate. The data on the rivulet thickness and the wave characteristics have been obtained experimentally. It follows

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from the experiment that the wavy regime is a typical rivulet regime. The fairly small width of the rivulet leads to coincidence of the transverse dimensions of the wave with the liquid flow width and, therefore, quite certain types of waves are generated, namely periodic waves that are almost sinusoidal, soliton-like waves, and waves with two humps and a very smooth tail. The rivulet thickness in the steady-state region as a function of time has been obtained. On the basis of this results it can be concluded that the thickness of the rivulet and its mass in the transverse cross section change with time. The rivulets can be blown off the beam surface when the velocity of the wind and acceleration of the beam become large enough to do it. The system mass changes when the rivulet is blown off. Papers [2, 5] have shown that even a marginal change in the mass can lead to the cable instability. The aforementioned papers have considered a one-degree-of-freedom system with a time-varying mass. This paper analyzes the dynamical behavior of an Euler–Bernoulli beam with a time-varying mass on its surface. We consider an initial stage of the beam dynamics when the displacements are small. The influence of axial rigidity of the beam, 3-dimensional motion of the structure, and large displacement are not considered. The asymptotical solutions for the beam displacement has been found by combining the classical Galerkin method with the averaging method for equations in Banach spaces (the classical Krylov–Bogolyubov theorem and generalizations). The resonance conditions have been derived. A method which helps to find number of possible resonances is proposed. Effect of the beams parameters on its dynamical behavior is investigated as well.

2 Statement of problem

The following linear hyperbolic equation describing the Euler–Bernoulli beam dynamics has been used:

$$Du_{xxxx} - Ku_{xx} + \beta_0 u_t + \big(M(x,t)u_t\big)_t = 0, \tag{1}$$

where $u(x,t)$ is the beam displacement, $t \geq 0$, $x \in [0, L]$ and where $D = EI$ is the rigidity coefficient, E is the Young modulus, I is the beam inertia momentum, $K > 0$ is the longitudinal force coefficient, β_0 is the positive coefficient, the term $\beta_0 u_t$ denotes dissipative effects. The boundary conditions are as follows:

$$u(x,t) = u_{xx}(x,t) = 0, \quad x = 0, L. \tag{2}$$

The time-varying mass $M(x,t)$ is assumed to have the form

$$M(x,t) = m_0\big(1 + \delta\mu(x,t)\big), \tag{3}$$

where m_0 is the constant part of the beam mass. Experimental data obtained in [4] give the following expression for $\mu(x,t)$:

$$\mu = \sin\big(\gamma_0(x - \Omega' t)\big). \tag{4}$$

It should be noted that (1) can be transformed to a dimensionless form when the rescaling variables are introduced.

For the rescaling, the following relations are used: $m_0 = \rho A_0$, where ρ is the beam density, A_0 is the constant part of the beam cross-section area. Let us introduce $c_0^2 = E/\rho$, $\bar{c}^2 = K/(A_0\rho)$ and then, for the variables $\bar{u}, \bar{x}, \bar{t}$ we have

$$\bar{u} = u/L, \qquad \bar{t} = c_0 t/L, \qquad \bar{x} = x/L.$$

Then (1) takes the form

$$a\bar{u}_{\bar{x}\bar{x}\bar{x}\bar{x}} - b\bar{u}_{\bar{x}\bar{x}} + \beta\bar{u}_{\bar{t}} + \big(\bar{M}(\bar{x},\bar{t})\bar{u}_{\bar{t}}\big)_{\bar{t}} = 0,$$

where $a = I/A_0L^2$, $b = (\bar{c}/c_0)^2$, $\beta = \beta_0 L/(\rho c_0 A_0)$, $\bar{M} = 1 + \delta\bar{\mu}(\bar{x},\bar{t})$. Then, $x \in [0, \bar{L}]$, $\bar{L} = 1$, and $\bar{\mu} = \sin(\gamma\bar{x} - \Omega\bar{t})$ with $\gamma = \gamma_0 L$, $\Omega = \gamma_0\Omega' L/c_0$. For simplification, the bar is omitted and the final equation takes the form

$$au_{xxxx} - bu_{xx} + \beta u_t + \big(M(x,t)u_t\big)_t = 0, \quad x \in [0, 1], \tag{5}$$

where $M = 1 + \delta\mu$, $\mu = \sin(\gamma x - \Omega t)$.

The asymptotics of $u(x,t)$ for small δ can be found by combining the classical Galerkin method with the averaging method for equations in Banach spaces (the classical Krylov–Bogolyubov theorem and generalizations, see the Appendix).

3 Preliminary a priori energy estimate

Assume that $0 \leq \delta < 1$. Then the function $M^{1/2}$ is correctly defined, since $M > 0$. Let us introduce a functional E associated with (5):

$$E[u] = \frac{1}{2}\big(a\|u_{xx}\|^2 + b\|u_x\|^2 + \|M^{1/2}u_t\|^2\big). \tag{6}$$

Here we use the standard notation $\|v\|^2 = \int_0^L v^2\,dx$. Let us derive an estimate for E having important physical and mathematical consequences. This functional can be interpreted as an energy.

If $\delta = 0$ and $\beta = 0$, then $dE[u]/dt = 0$ on solutions of (5), i.e., the energy conserves. In a general case, let us multiply (5) by u_t and then one obtains

$$\int_0^L \left(au_{xxxx}u_t - bu_{xx}u_t + Mu_tu_{tt} + M_tu_t^2\right)dx = -\beta\int_0^L u_t^2\,dx.$$

We can rewrite it as

$$\int_0^L \left(au_{xxxx}u_t - bu_{xx}u_t + Mu_tu_{tt} + \frac{1}{2}M_tu_t^2\right)dx = -\int_0^L \left(\beta u_t^2 + \frac{1}{2}M_tu_t^2\right)dx.$$

Integrating it by parts, and taking into account that boundary conditions must be satisfied at any time, one has

$$\frac{d}{dt}\frac{1}{2}\int_0^L \left(au_{xx}^2 + bu_x^2 + Mu_t^2\right)dx = -\int_0^L \left(\beta u_t^2 + \frac{1}{2}M_tu_t^2\right)dx.$$

Notice that the left-hand side of this relation is equal to $\frac{dE[u]}{dt}$. Thus, one obtains the main energetic estimate

$$\frac{dE[u]}{dt} \le \left(\frac{1}{2}\sup_{x\in[0,L],\ t>0}\left|M_t(x,t)\right| - \beta\right)\|u_t\|^2. \tag{7}$$

Now it is obvious that, if the dissipation is sufficiently large, then, all solutions of (5) are stationary for large times. In fact, there holds the following proposition.

Proposition 3.1 *If* $2\beta > \sup_{x,t}|M_t(x,t)|$, *then solution oscillations are decreasing*:

$$\|u_t\| \to 0 \quad as\ t \to \infty.$$

This proposition follows from the standard arguments. In fact, integrating (7) over $[0,t]$ one has $E(t) = E[u(\cdot,t)]$, it satisfies $E(t) \le E(0) - r\int_0^t \|u_s\|^2\,ds$. Designate

$$X(t) = \int_0^t \|u_s\|^2\,ds.$$

Since $M > 0$, there is a constant r_1 such that $r_1\|u_t\|^2 \le E(t)$. Therefore,

$$r_1\frac{dX(t)}{dt} = r_1\|u_t\|^2 \le E(0) - rX$$

that entails $X(t) \le E(0)/r + C_0\exp(-rt)$.

Since $X(t)$ is bounded as $t \to \infty$, $\|u_t\|$ converges to 0. Which proves our assertion. The assertion shows that the most interesting cases are the following (1) β and δ are of the same order, or (2) $\beta \ll \delta$. Indeed, according to Proposition 3.1, in the case when $\beta \gg \delta$, the solution time oscillations vanish for large times: $\|u_t\| \to 0$ as $t \to \infty$. A new parameter $\kappa = \beta/(2\delta)$ is introduced for the two cases under consideration. Then another important estimate can be found for $0 < \kappa < C$. Assume $|M_t| < 2c_1\delta$ (c_1 is a positive constant). Then using (7), the following estimate is obtained

$$\left|\frac{dE}{dt}\right| \le (c_1\delta - \beta)\|u_t\|.$$

Since $\beta \ge 0$ and $\|u_t\|^2 \le E$, the expression for $E(t)$ has the form

$$E(t) \le E(0)\exp(c_1\delta t), \quad t > 0. \tag{8}$$

This inequality shows that the oscillations can increase with time t in an exponential manner, however, this growth is weak ($\log E(t) \approx \text{const}\cdot\delta t$). Below, it is shown that, actually, such a behavior is possible, if the frequency Ω and the other parameters are adjusted in a special way to fulfill a resonance condition (see Sect. 5 below). Otherwise, the oscillations are bounded or they are exponentially fading. The estimate (8) shows that the solutions exist, they are unique and lie in the corresponding Sobolev space, i.e., the problem is well posed mathematically.

4 Eigenfunctions of non-perturbed problem

The boundary conditions (2), where $L = 1$ are considered. Let us set $\beta = 0$ (dissipation is removed) and $\delta = 0$. Let us denote by $\mathcal{L}$ the linear operator that defines the left-hand side of (5). If a solution of (5) has the form $\psi(x)\exp(\lambda t)$, the parameter λ is a purely imaginary number: $\lambda = i\omega$, $\omega \in \mathbf{R}$. Indeed, if

$$a\psi_{xxxx} - b\psi_{xx} = \mathcal{L}\psi = -\lambda^2\psi,$$

then $-\lambda^2\|\psi\|^2 = a\|\psi_{xx}\|^2 + b\|\psi_x\|^2 \geq 0$. The complex eigenfunctions $\tilde{\psi}_n(x)$ of the linear operator $\mathcal{L}$ have the form $\exp(kx)$, $k \in \mathbf{C}$. By $\mathcal{L}\tilde{\psi}_n = \omega_n^2\tilde{\psi}_n$, one has

$$k_\pm^2 = (2a)^{-1}\Big(b \pm \sqrt{b^2 + 4a\omega_n^2}\Big). \tag{9}$$

Then $k_+ = k_1 \in \mathbf{R}$ and $k_- = ik_2,\ k_2 \in \mathbf{R}$, thus, the real eigenfunctions ψ have the form

$$\psi = C_1 \sinh k_1 x + B_1 \cosh k_1 x + C_2 \sin k_2 x + B_2 \cos k_2 x,$$

and

$$\psi_{xx} = k_1^2(C_1 \sinh k_1 x + B_1 \cosh k_1 x) - k_2^2(C_2 \sin k_2 x + B_2 \cos k_2 x).$$

Substituting $x = 0$ one has $B_1 + B_2 = 0$ and $k_1^2 B_1 - k_2^2 B_2 = 0$, that gives $B_1 = B_2 = 0$. Now, to define the frequency ω_n we resolve the system

$$C_1 \sinh k_1 + C_2 \sin k_2 = 0, \tag{10}$$

$$k_1^2 C_1 \sinh k_1 - k_2^2 C_2 \sin k_2 = 0, \tag{11}$$

Since nontrivial solutions are sought, this gives $\sin k_2 \sinh k_1 = 0$, thus

$$k_2(n) = k(n) = \pi n, \qquad \omega_n^2 = a\pi^4 n^4 + b\pi^2 n^2. \tag{12}$$

Therefore, the eigenfunctions take the form

$$\psi_n(x) = C_n \sin(\pi n x), \quad n = 1, 2, \ldots. \tag{13}$$

Constants C_n are chosen such that $\|\psi_n\| = 1$. Note that $C_n = \sqrt{2}$.

5 Asymptotic solutions and main evolution equation

In the non-perturbed case $\delta = 0$ the solutions of (5) can be expressed through the eigenfunctions from Sect. 3:

$$u = \sum_{n \in \mathbf{Z}} X_n \psi_n(x) \exp(i\omega_n t),$$

where X_n are constant coefficients determining complex amplitudes of oscillations such that $X_{-n} = X_n^*$ (the star denotes complex conjugation). Here, it is formally set that $\omega_n = -\omega_{-n}$.

A slow rescaling time $\tau = \delta t$ is introduced for small $\delta > 0$. Assuming that amplitudes X_n are unknown functions of the slow time τ, the asymptotic ansatz is used. This known mathematical idea admits a simple physical interpretation: a small perturbation generates slow oscillations of coefficients X_n. That situation is quite standard: the straightforward perturbation method does not work here due to occurrence of secular terms, and for this reason the two time scale approach applied, as it was done in many previous works, for example [6]. As a result, the principal term of our asymptotical solution can be written as

$$u = \sum_{n=1}^{+\infty}\big(X_n(\tau)\psi_n(x)\exp(i\omega_n t) + X_n^*(\tau)\psi_n(x)\exp(-i\omega_n t)\big). \tag{14}$$

Substitute this solution into (5) that results in the complicated equation where only the main terms of the order δ are taken into account (the terms of the order 1 are mutually annihilated). Thus,

$$\big(X_n(\tau)\psi_n(x)\exp(i\omega_n t)\big)_{tt} = \psi_n(x)\exp(i\omega_n t)\Big(-\omega_n^2 X_n(\tau) + 2\delta i\omega_n \frac{dX_n}{d\tau}\Big) + O(\delta)^2, \tag{15}$$

$$\beta\big(X_n(\tau)\psi_n(x)\exp(i\omega_n t)\big)_t = -2\kappa\delta i\omega_n X_n(\tau)\psi_n(x)\exp(i\omega_n t) + O(\delta)^2, \tag{16}$$

and the $O(1)$-order terms in (15) annihilate with the same $O(1)$-order term generated by spatial derivatives in (5). It is seen from (15), (16) that the main non-vanishing terms are of order $O(\delta)$. They appear from (15), (16) and the varying in time mass contribution $\delta(\mu(x,t)u_t)_t$, where u_t can be approximated by

$$u_t = \sum_{n=1}^{+\infty} i\omega_n\psi_n(x)\big(X_n(\tau)\exp(i\omega_n t) - X_n^*(\tau)\exp(-i\omega_n t)\big) + O(\delta). \tag{17}$$

Let us denote by $\langle f, g\rangle$ the scalar product in $L_2[0,1]$: $\langle f, g\rangle = \int_0^1 f(x)g(x)\,dx$. Then applying the standard Galerkin procedure to (15), (16), (17) the following infinite system is obtained

$$\frac{dX_n}{d\tau} = -\kappa X_n + \sum_{m \in \mathbf{Z}, m \neq 0} A_{nm}(t)X_m, \quad t = \tau/\delta, \tag{18}$$

where $n \in \mathbf{Z},\ n \neq 0$ and the entries

$$A_{nm}(t) = (2\omega_n)^{-1}\omega_m \exp(-i\omega_n t) \big\langle \big(\sin(\gamma x - \Omega t)\exp(i\omega_m t)\big)_t \psi_m(x), \psi_n(x)\big\rangle \tag{19}$$

give the contribution of the time-varying mass. Remind that $\Omega = \gamma_0 \Omega' L/c_0$.

In general case these entries are

$$A_{nm}(t) = (2\omega_n)^{-1}\omega_m \exp(-i\omega_n t) \big\langle \big(\mu(x,t)\exp(i\omega_m t)\big)_t \psi_m(x), \psi_n(x)\big\rangle. \tag{20}$$

To estimate these varying in time mass terms, the following lemma is used. For sufficiently general $\mu(x,t)$, see the following lemma.

Lemma 5.1 *Suppose the function $\mu(x,t)$ satisfies*

$$\sup_{t,x}\big(\big|\mu(x,t)\big| + |\mu_x| + |\mu_{xx}|\big) < C$$

and similarly for μ_t:

$$\sup_{x,t}\big(|\mu_t + |\mu_{xt}| + |\mu_{xxt}|\big) < C_1.$$

Then the entries A_{mn} admit the estimate

$$|A_{nm}| < C_1\big[\big(1+|m-n|\big)^{-2} + \big(1+|m+n|\big)^{-2}\big], \tag{21}$$

Proof In fact, $|\omega_n| > c|n|$ for large n, where $c > 0$. The identity $2\sin a \sin b = \cos(a-b) - \cos(a+b)$ is used. Then

$$\begin{aligned} |S_{mn}| &= \left|\int_0^1 \sin(\pi m x)\sin(\pi n x)\mu(x,t)\,dx\right| \\ &= \frac{1}{2}\left|\int_0^1 \big(\cos\big(\pi(m-n)x\big) - \cos\big(\pi(m+n)x\big)\big)\mu(x,t)\,dx\right|. \end{aligned}$$

If $m \neq n$ and $m \neq -n$, the last integral is integrated by parts and then,

$$|S_{mn}| < C\big[\big(1+|m-n|\big)^{-2} + \big(1+|m+n|\big)^{-2}\big]$$

under our assumptions to μ estimate (18). For $m = n$ or $m = -n$, this estimate follows from assumptions on μ. □

This lemma allows us to show that the main evolution (15) defines a well-posed Cauchy problem in an appropriate Banach space. Let us define this Banach space $\mathcal{B}$. It consists of complex valued sequences $X = \{X_n\}$, $n \in \mathbf{Z}$ such that

$$X_{-n} = X_n^*, \qquad \sup_n\big|X_n n^2\big| < \infty.$$

The norm in this space is defined by

$$|X| = \sup_n\big|X_n n^2\big|.$$

Lemma 5.2 *The operator $\mathcal{A}X$ defined by*

$$(\mathcal{A}X)_n = \sum_{m\in\mathbf{Z}} A_{mn}X_m$$

is a bounded in $\mathcal{B}$ linear operator.

The proof uses (21) and the asymptotics $\omega_n = O(n^2)$ as $n \to \infty$. Note that

$$\left|\sum_{m\in\mathbf{Z}} A_{mn}X_m\right| \le cn^{-2}|X| \sum_{m\in\mathbf{Z}}\big(\big(1+|n-m|\big)^{-2} + \big(1+|m+n|\big)^{-2}\big), \quad c > 0.$$

Since $\sum(1+|m|)^{-2} < C_2$ then,

$$\left|\sum_{m\in\mathbf{Z}} A_{mn}X_m\right| < n^{-2}C_3|X|.$$

Therefore, $\mathcal{A}$ is a bounded operator.

Due to this Lemma, evolution equation (18) can be investigated by averaging theorems (see the Appendix). It was obtained, that $X(t)$ is close to the following averaging solutions defined by averaging system (18):

$$\frac{dX_n}{d\tau} = -\kappa X_n + \sum_{m\in\mathbf{Z}} \bar{A}_{mn}X_m, \tag{22}$$

where

$$\bar{A}_{mn} = \lim_{T\to\infty} T^{-1}\int_0^T A_{mn}(t)\,dt. \tag{23}$$

It is not difficult to show by (19) that $\bar{A}_{mn} \neq 0$ only if the following condition holds.

Resonance condition: *There are n, m such that*

$$|\omega_n \pm \omega_m \pm \Omega| < a_0\delta, \tag{24}$$

where a_0 is a positive constant independent of δ. Introducing a detuning parameter $\phi = \delta^{-1}|\omega_n \pm \omega_m \pm \Omega|$, we can rewrite the resonance condition in the form $|\phi| < a_0$. In the numerical simulations (see the next section), we set $a_0 = 1$. All the resonances belong to two different types: "+" resonances, when the sign in (24) is plus and "−" resonances when the sign is minus. If the resonance condition holds for some $n = n_r = N$ and $m_r = m(N)$, a weak exponential time growth of $|X|$ and the energy E is possible:

$$E(t) = \exp(\tilde{c}\delta t)E(0), \tag{25}$$

where $\tilde{c}$ is a positive constant. Note that $\tilde{c} < c_2$ since inequality (8) always holds for the energy E. If the dissipation is absent ($\kappa = 0$), the coefficient $\tilde{c}\delta$ is defined by an eigenvalue $\lambda_0(R)$ of some matrix $\bar{\mathbf{A}}$. This eigenvalue λ_0 has the maximal positive real part $\mathrm{Re}\,\lambda_0$. The entries of a finite dimensional, constant in time matrix $\bar{\mathbf{A}}$ are defined by

$$\bar{A}_{n_r,m_r} = \lim_{T\to+\infty} T^{-1}\int_0^T A_{m_r n_r}(t)\,dt. \tag{26}$$

In the following sections, $\bar{\mathbf{A}}$ is computed for some particular cases.

6 Resonance analysis

6.1 General properties of resonances

Under boundary conditions (2) for each Ω', the resonance is absent, or there is either a finite set of natural numbers m_r, n_r satisfying the resonance condition (24) with the sign "−", or a finite set of natural numbers satisfying "+" resonance condition (24). In fact, $\omega_n^2 = a'n^4 + b'n^2$ that follows from (12) (here $a' = a\pi^4, b' = b\pi^2$ are positive constants). Then for large n, m

$$|\omega_n \pm \omega_m| > C_3|n + m|, \quad C_3 > 0, \tag{27}$$

where C_3 is a constant independent of n, m. Our assertion for "+" is obvious, since $\omega_n \to \infty$ at $n \to \infty$. Therefore, it can be concluded that *for each Ω the resonance condition holds only for some finite set $\mathcal{R}(\Omega)$ of natural n, m*. This means that the problem of amplitude growing reduces to a finite dimensional problem on eigenvalues of a finite dimensional matrix.

The number of resonances increases to ∞ as rigidity $D \to 0$ for a fixed length L (after the rescaling this means that $a \to 0$). In fact, the limit case $a = 0$ gives $b(n \pm m) = \pm\Omega'$. If there are such n, m, then all $n' = n + N, m' = m + N$ also satisfy the resonance condition, thus we have obtained an infinite set of n_r, m_r (for the case if we take the sign "−" in the left-hand side of (24)). Thus, the case of small dimensionless rigidities a needs a special investigation, because it is impossible to take into account all these resonances. We will consider this problem in the next publication.

For some frequencies Ω, the resonances do not exist. Indeed, $\min_{n,m}|\omega_n - \omega_m| = \min_n|\omega_{n+1} - \omega_n| = \rho(a, b) > 0$. Then if $\Omega < \rho$, the resonances are absent. If $D = a = 0$, then $\rho = b^{1/2}$. The case of the sign + is simpler: since $\omega_n, \omega_m > 0$, results in $\min_{n,m}|\omega_n + \omega_m| \ge 2\omega_1$.

Moreover, if the dissipation exists, i.e., $\kappa > 0$, it is possible that the resonances exist but the amplitude $|X|$ does not increase. In fact, the growth of X is defined by the eigenvector of $\bar{A}$ with the eigenvalue λ_0 (see above). If $\mathrm{Re}\,\lambda_0 < \kappa$, then $|X(t)|$ and $E(t)$ are decreasing. The resonances can hardly be investigated better by an analytical approach, but computer simulations should be applied. Some analytic results for resonances were obtained in the framework of the KAM-theory [7, 8] that cannot be applied to our problem. The KAM theory concerns with weakly perturbed nonlinear Hamiltonian equations in finite dimensional spaces; Hamiltonian perturbations do not depend on time. This paper studies a perturbed Hamiltonian equation in an infinite dimensional case; perturbations depend on time and they are not Hamiltonian. Suppose that the frequency Ω is a random parameter uniformly distributed in some interval $0, \omega_{\max}$. Let us choose a small δ and let us compute a probability that the resonance condition (24) holds for some m, n. Here, we suppose that n, m run over some interval, for example, $m = 1, 2, \ldots, N_{\max}$, $n = 1, 2, \ldots, N_{\max}$. To compute this probability and to find the number of resonances, we choose a random Ω and check the resonance condition. This allows us to obtain an estimate of the probability that the relation (24) holds with the help of the Monte-Carlo method. The numerical results confirm the above analysis. The probability P to be in the dangerous domain is a decreasing function of the rigidity. Figure 1 illustrates the relation between the number of possible resonances and

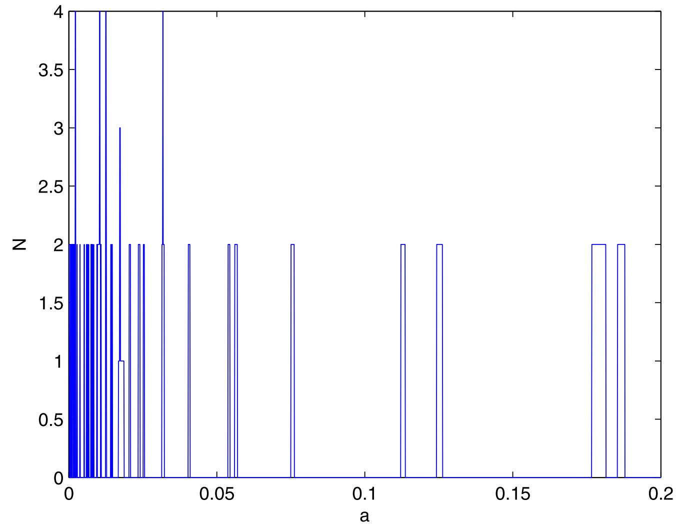

Fig. 1 Number of the possible resonances for parameters $b = 5$, $\Omega = 30$, $\delta = 0.1$

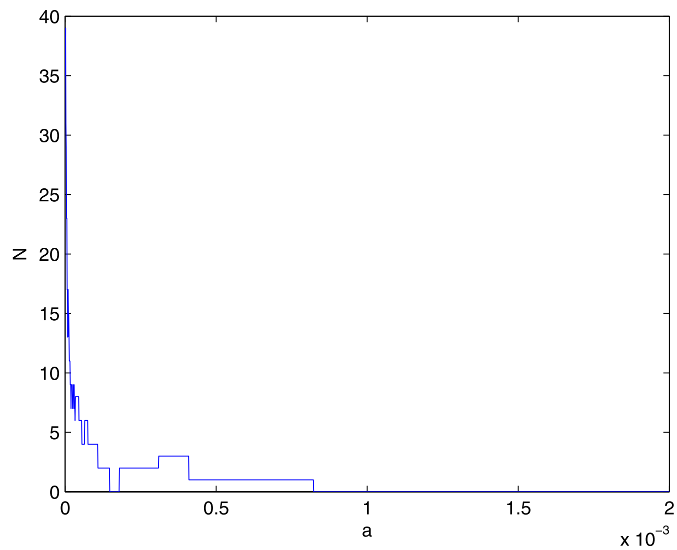

Fig. 2 Number of resonances for $b = 0.001$, $\delta = 0.1$, $\Omega = 0.5$ parameters

a. The diagram in Fig. 1 shows that number of possible resonances grows with the decreasing bending rigidity value a. The diagram has been plotted for parameters $b = 5$, $\Omega = 30$, $\delta = 0.1$. Figure 2 shows the number of resonances for $b = 0.001$, $\delta = 0.1$, $\Omega = 0.5$ parameters. The calculations performed for the other parameters have also revealed that with the increase of the beam mass variation value δ the number of possible resonances grows providing that the other beam parameters are fixed. By analogous numerical simulations, *admissible* values of the rigidity a (D) have been computed. We say that a is admissible, if there are no resonances. Simulations show that the set of admissible values has a complicated form for small a. For larger a, the density of admissible values become sufficiently close to 1 and with some critical a_c, all the values are admissible: no resonances for $a > a_c$. The last property can be formulated as follows.

Lemma 6.1 *For each Ω, there is a value a_c of the parameter a such that for all $a > a_c$ a resonance condition cannot be fulfilled whenever n, m.*

The lemma can be derived by the following simple arguments. Above, the function $\rho(a, b)$ has been intro-

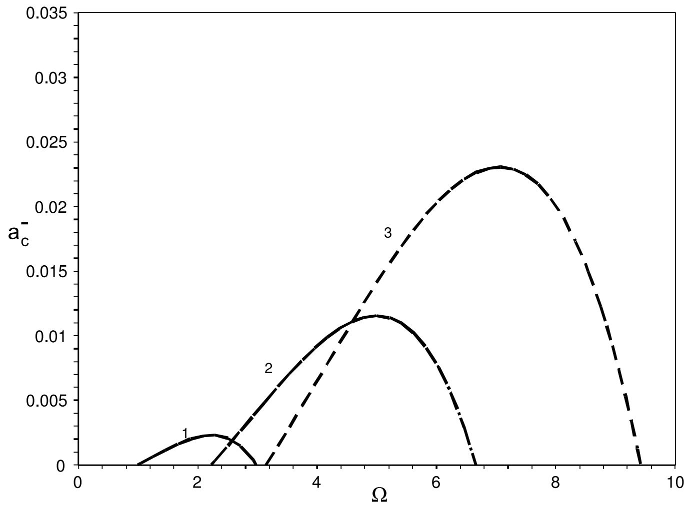

Fig. 3 $a_c^-(\Omega)$ found numerically for different b

duced. The minimum of this function increases with a. For some a_c this minimum is larger than Ω.

In the next section, it will be shown that the solution behavior may be quite different for the resonances of different types. The critical values a_c depend on the sign in (24), thus there are two critical curves $a_c^{\pm}$.

For a_c^-, one has a rather complicated relation:

$$a_c^- = \frac{-r_b + \sqrt{r_b^2 - 4r_a r_c}}{2r_a},$$

where

$$r_a = 144\pi^4, \qquad r_b = 72b\pi^6 - 40\Omega^2,$$

$$r_c = \left(3b\pi^2 - \Omega^2\right)^2 - 4\Omega^2 b\pi^2.$$

The critical value a_c^+ results in a simpler formula:

$$a_c^+ = 0.25\left(\Omega^2\pi^{-4} - b\pi^{-2}\right).$$

$a_c^+(\Omega)$ found numerically for different b has been plotted in Fig. 3. The relation for a_c can be obtained as follows. Let us notice that the difference $d_{n,m}(a) = \omega_n - \omega_m$, where $n > m$, is a increasing function of the both arguments n, m and also this difference is a strictly increasing function of a. Thus, for a fixed a, the minimal possible value $\omega_n - \omega_m$ attains at $n = 2, m = 1$: $d_{\min}(a) = d_{2,1}(a)$. The minimal possible value of a, where the minus resonance is possible, is defined, therefore, by relation $d_{\min}(a) = d_{2,1}(a) = \Omega$ that gives a quadratic equation for $a = a_c^-$. Resolving this equation, we obtain the above mentioned formula. Based on diagrams presented in Fig. 3, the following conclusions can be drawn. Critical value a_c^- increases with the growth of the beam mass variation frequency. The curve 1 in Fig. 3 corresponds to parameter $b = 0.1$, the dashed curve 2 corresponds to parameter $b = 0.5$ and the dotted curve 3 corresponds to $b = 1$. Also, value $a = a_c^-$ increases with the longitudinal force growth (i.e. b parameter). All values of $a = a_c^-$ belongs to the bounded domain. The smaller b parameter is, the narrower is the frequency band where admissible values of beam rigidity exist. The critical value a_c^+ results in a simpler formula:

$$a_c^+ = 0.25\left(\Omega^2\pi^{-4} - b\pi^{-2}\right).$$

Based on this formula, the following conclusions can be drawn. Critical a_c^+ value increases with the growth of the beam mass variation frequency Ω. At the longitudinal force growth (i.e. b parameter), a_c^+ value insignificantly decreases. Finally, b parameter at which $a_c^+ = 0$ diminishes with Ω decreasing. For analyzing the resonance case solutions, some auxiliary relations are derived.

6.2 Some auxiliary relations

When $\mu(x,t)$ is defined by (4), the entries A_{nm} can be computed explicitly by (19). When the rescaling variables are introduced, one has $\mu = \sin(\gamma x - \Omega t)$, where $\gamma = \gamma_0 L$, $\Omega = \gamma\Omega'\bar{c}/L$.

Let us introduce the matrices $\mathbf{B}, \tilde{\mathbf{B}}$ by

$$\tilde{B}_{nm} = \int_0^1 \sin(\gamma x)\psi_m(x)\psi_n(x)\,dx,$$
$$B_{nm} = \int_0^1 \cos(\gamma x)\psi_m(x)\psi_n(x)\,dx, \tag{28}$$

where ψ_n are defined by (13). These entries can be computed explicitly. A long but quite straightforward calculation gives

$$B_{nm} = (-1)^{n-m}\frac{\gamma\sin\gamma}{4}\left(\frac{1}{\pi(n-m)^2+\gamma^2} - \frac{1}{\pi(n+m)^2+\gamma^2}\right),$$
$$\tilde{B}_{nm} = (-1)^{n-m}\frac{\gamma\cos\gamma}{4} \times\left(\frac{1}{\pi(n+m)^2+\gamma^2} - \frac{1}{\pi(n-m)^2+\gamma^2}\right). \tag{29}$$

The matrices $\mathbf{B}, \tilde{\mathbf{B}}$ are symmetric and the entries B_{nm}, $\tilde{B}_{nm}$ change the signs under change $m \to -m$ or $n \to -n$. Some auxiliary complex coefficients are introduced in the form

$$R_{nm} = \frac{1}{2}(\omega_m + \Omega)(i\tilde{B}_{nm} - B_{nm}), \quad i = \sqrt{-1}, \tag{30}$$

$$\tilde{R}_{nm} = \frac{1}{2}(\omega_m - \Omega)(i\tilde{B}_{nm} + B_{nm}). \tag{31}$$

Using these relations and (20), one obtains

$$A_{nm} = (2\omega_n)^{-1}\omega_m\big(R_{nm}\exp(-i\omega_n t + i\omega_m t + i\Omega t) + \tilde{R}_{nm}\exp(-i\omega_n t + i\omega_m t - i\Omega t)\big). \tag{32}$$

Equation (32) is used to obtain averaged entries $\bar{A}_{nm}$.

6.3 Analysis of "−" resonances, $n \neq m$ and detuning parameter $\phi = 0$

Since the entries A_{nm} are decreasing in $|n|, |m|$, the most essential resonances correspond to small $|m|, |n|$. If $m \neq n$, the simplest case is $m = 2, n = 1$. Let us assume

$$\omega_m - \omega_n - \Omega = 0, \quad \omega_m > \omega_n,\ \Omega > 0. \tag{33}$$

Only contributions of the entries $A_{n'm'}$ with $m' = \pm m, n' = \pm n$ can be taken into account, the rest entry contributions vanish after the averaging procedure. Let us consider the averages of these entries under the condition (33). If δ is small enough, for a generic choice of the beam parameters, only a single resonance (33) exists. Equations (22) take the form

$$\frac{dX_n}{d\tau} = -\kappa X_n + \bar{A}_{nm}X_m,$$
$$\frac{dX_m}{d\tau} = -\kappa X_m + \bar{A}_{mn}X_n, \tag{34}$$

plus analogous complex conjugated equations for X_{-n}, X_{-m}. Here,

$$\bar{A}_{nm} = (2\omega_n)^{-1}\omega_m\tilde{R}_{nm},$$
$$\bar{A}_{mn} = (2\omega_m)^{-1}\omega_n R_{mn}. \tag{35}$$

The eigenvalues λ_1, λ_2 of the 2×2-matrix $\bar{\mathbf{A}}$ determines the solution large time behavior. These eigenvalues can be found by the relation

$$(\kappa + \lambda)^2 = \bar{A}_{nm}\bar{A}_{mn}. \tag{36}$$

Using relations (35), (29), and (30), it can be found that

$$\bar{A}_{nm}\bar{A}_{mn} = -\frac{1}{4}\omega_n\omega_m\big(\tilde{B}_{nm}^2 + B_{nm}^2\big) = -\sigma^2. \tag{37}$$

Finally, from (37), the following results of the solution behavior can be obtained. One has $\lambda_j = \pm i\sigma - \kappa$, i.e. here we observe damping oscillations if $\kappa > 0$ (a small dissipation exists) and pure oscillations, if $\kappa = 0$. This analytical result has been confirmed by the numerical simulations for the case $n = 1, m = 2$.

6.4 Analysis of "−" resonances for detuning parameter $\phi \neq 0$

Here, the − resonances with the detuning parameter $\phi \neq 0$ are considered, i.e.:

$$\omega_m - \omega_n - \Omega = \delta\phi, \quad \omega_m > \omega_n,\ \Omega > 0. \tag{38}$$

Here again, only contributions of the entries $A_{n'm'}$ with $m' = \pm m, n' = \pm n$ are taken into account, their remaining contributions vanish as a result of averaging. Let us denote $\Omega_0 = \omega_m - \omega_n$, then $\Omega - \Omega_0 = \delta\phi$.

If δ is small enough, then (28) takes the form

$$\frac{dX_n}{d\tau} = -\kappa X_n + \bar{A}_{nm}\exp(-i\phi\tau)X_m,$$
$$\frac{dX_m}{d\tau} = -\kappa X_m + \bar{A}_{mn}\exp(i\phi\tau)X_n. \tag{39}$$

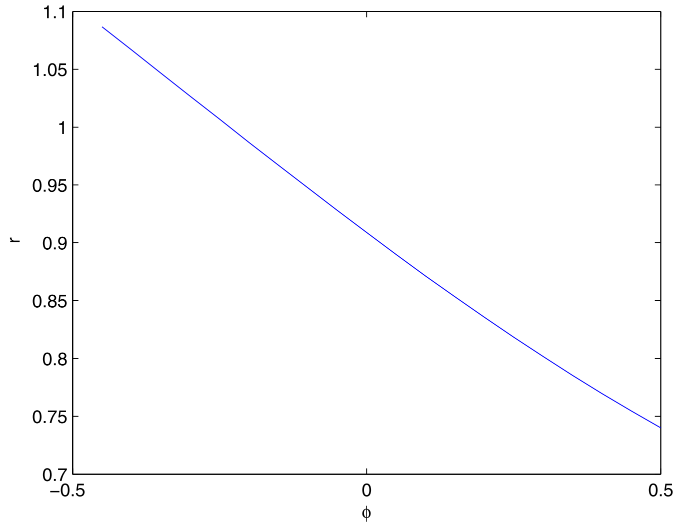

Fig. 4 The detuning parameter influence on solution decreasing rate r

Let us introduce α_{nm}, β_{nm} by $\bar{A}_{nm} = \alpha_{nm} + i\beta_{nm}$, $\bar{A}_{mn} = \alpha_{mn} + i\beta_{mn}$. By separating the imaginary and real parts, we obtain the following system:

$$\begin{aligned}
\frac{d\,\mathrm{Re}\,X_n}{d\tau} &= -\kappa\,\mathrm{Re}\,X_n + \big(\alpha_{nm}\cos(\phi\tau) \\
&\quad + \beta_{nm}\sin(\phi\tau)\big)\,\mathrm{Re}\,X_m \\
&\quad + \big(\alpha_{nm}\sin(\phi\tau) - \beta_{nm}\cos(\phi\tau)\big)\,\mathrm{Im}\,X_m, \\
\frac{d\,\mathrm{Im}\,X_n}{d\tau} &= -\kappa\,\mathrm{Im}\,X_n + \big(-\alpha_{nm}\sin(\phi\tau) \\
&\quad + \beta_{nm}\cos(\phi\tau)\big)\,\mathrm{Re}\,X_m \\
&\quad - \big(\alpha_{nm}\cos(\phi\tau) + \beta_{nm}\sin(\phi\tau)\big)\,\mathrm{Im}\,X_m, \\
\frac{d\,\mathrm{Re}\,X_m}{d\tau} &= -\kappa\,\mathrm{Re}\,X_m + \big(\alpha_{mn}\cos(\phi\tau) \\
&\quad - \beta_{mn}\sin(\phi\tau)\big)\,\mathrm{Re}\,X_n \\
&\quad + \big(-\alpha_{nm}\sin(\phi\tau) - \beta_{nm}\cos(\phi\tau)\big)\,\mathrm{Im}\,X_n, \\
\frac{d\,\mathrm{Im}\,X_m}{d\tau} &= -\kappa\,\mathrm{Re}\,X_m + \big(\alpha_{mn}\sin(\phi\tau) \\
&\quad + \beta_{mn}\sin(\phi\tau)\big)\,\mathrm{Re}\,X_n \\
&\quad - \big(\alpha_{mn}\cos(\phi\tau) - \beta_{mn}\sin(\phi\tau)\big)\,\mathrm{Im}\,X_n.
\end{aligned}$$

This system has been investigated numerically by the standard programs in MATLAB 2009a for the case when $n = 1, m = 2$. Results obtained for small detuning parameter values confirm the analytical conclusions, decreasing oscillations have been observed when the damping coefficient $\kappa > 0$, for large κ decreasing solutions without oscillations have been obtained. These analytical results are confirmed by the numerical simulations for $n = 1, m = 2$. Figure 4 illustrates the detuning parameter influence on solution decreasing rate r defined by

$$r = -\lim_{t\to+\infty} t^{-1}\log\big|X(t)\big|,$$

$$|X|^2 = |\mathrm{Re}\,X_n|^2 + |\mathrm{Im}\,X_n|^2 + |\mathrm{Re}\,X_m|^2 + |\mathrm{Im}\,X_m|^2.$$

The parameter r defines the solution amplitude for large times by $|X| = c\exp(-rt)$.

6.5 Analysis of "+" doubling resonances $2\omega_n = \Omega$

Similarly to Sect. 6.4, the (22) in this case take the form:

$$\begin{aligned}
\frac{dX_n}{d\tau} &= -\kappa X_n + \bar{A}_{n,-n}X_{-n}, \\
\frac{dX_{-n}}{d\tau} &= -\kappa X_{-n} + \bar{A}_{-nn}X_n,
\end{aligned} \tag{40}$$

where

$$\begin{aligned}
\bar{A}_{n,-n} &= -\frac{1}{2}\tilde{R}_{n,-n}, \\
\bar{A}_{-n,n} &= -\frac{1}{2}R_{-n,n}.
\end{aligned} \tag{41}$$

The eigenvalues λ_i can be found from the relation

$$(\kappa+\lambda)^2 = \frac{1}{4}\omega_n^2\big(\tilde{B}_{n,-n}^2 + B_{n,-n}^2\big) = \tilde{\sigma}^2 > 0.$$

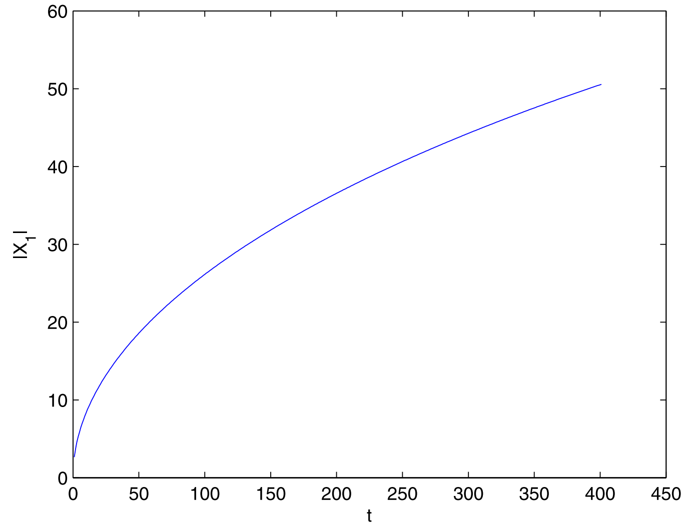

Fig. 5 An exponential growth of the solution for "+" doubling resonances

Finally, the following behavior has been obtained: $\lambda_i = \pm\tilde{\sigma} - \kappa$, i.e. here the exponentially decreasing solutions at $\kappa > \tilde{\sigma}$ and exponentially increasing solutions at $\kappa < \tilde{\sigma}$ are observed. These results are consistent with the numerical simulations that also demonstrate a solution exponential growth for sufficiently small κ (for $n = m = 1$). The case of nonzero detuning parameter values is investigated by simulations in MATLAB 2009a, similar to Sect. 6.4. The results of computations presented in Fig. 5 have shown a fast growth of the solution.

6.6 Internal resonances

More complicated resonances are possible at a special parameters choice, where we have an interaction of three modes. These resonances occur if $\omega_n - \omega_m = \pm\Omega$ and $\omega_n + \omega_{m'} = \Omega$.

To understand this situation, let us consider the simplest case, where $n = 1, m = 2, m' = 1$. Then one has $|\omega_1 - \omega_2| = 2\omega_1$ that gives $\omega_2^2 = 9\omega_1^2$. Using relations (12), one finds

$$7\pi^2 a = 5b\pi^2.$$

Therefore, to have such a resonance, we have to adjust a and b in a special way.

Such resonances are more seldom than the simplest cases investigated above, and here we can apply only numerical simulations. We do not consider such resonances here. Notice, that if rigidity a is small (for example, $a = O(\delta)$), then we should take into account coexistence of many resonances $\omega_n - \omega_m = \Omega$, $\omega_{n'} + \omega_{m'} = \Omega$, where $n' - m' = n - m$, and this complicated case needs other methods.

7 Conclusions

It has been shown that the problem of the Euler–Bernoulli beam with a time-varying mass reduces to a finite dimensional problem on eigenvalues of a finite dimensional matrix.

All the resonances $\omega_n + \omega_m = \Omega$ can be studied by the way described in the previous section and it is finally obtained:

I. All the "+" resonances are dangerous: if the dissipation is small enough, they lead to growing oscillations.

II. The "−" resonances result in slow harmonic modulations of fast oscillation amplitudes. These fast oscillations decrease slowly if a dissipation exists. This decrease depends on detuning parameter values.

This paper has developed the asymptotic approach for describing effects induced in an elastic beam with a small time-varying mass. If dimensionless rigidity a/b is not too small, the influence of the varying in time mass can be computed by the known averaging and asymptotic methods. It has been shown that the problem is, in a sense, well computable. The number of resonances in the system is limited and they can be analyzed as follows: firstly, a finite matrix is calculated; secondly, the spectral analysis of the symmetric matrix is performed. However, in some cases, for a special choice of the beam parameters, a slow growth of oscillations is observed. In the simplest case, when only a single resonance exists, the problem admits a complete analytical solution (see Sect. 6). The calculations performed for the beam parameters have also revealed that with the increase of the beam mass variation value δ, the number of possible resonances grows providing that the other beam parameters are fixed. If the rigidity is large enough, the resonances do not exist and the oscillations are bounded. More complicated resonances are possible at a special choice of parameters, where we have an interaction of three modes-internal resonances.

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Appendix: Krylov–Bogolyubov theorem

The Krylov–Bogolyubov method of averaging is useful in the nonlinear oscillation theory for studding oscillation processes. The method is based on the exact differential equation approximation by an averaged equation. Various averaging schemes (Gauss, Fatou, Delone-Hill, etc.) were widely applied in the celestial mechanics well in advance of the work of Krylov and Bogolyubov. These two authors developed a general algorithm and proved that the solutions of the averaged system approximate the exact solutions ([9, 10]). Bogolyubov developed the rigorous theory of the method with a comprehensive explanation of the general averaging principle (see [11–13]). He showed that the averaging method was correct when the transformation of variables allowing us to eliminate the time from the equations existed. He also established a relationship between the solutions of the exact and averaged equations over an infinite time interval. These results were later extended by Mitropol'skii and others (see [14–16]). The standard form of the system of equations, where this method of averaging can be applied, is:

$$\frac{dX}{dt} = \delta F(t, X), \quad X \in B, \tag{42}$$

where t is the time and δ is the small positive parameter. The main assumptions to F are as follows: F is a smooth function of X satisfying

$$\lim_{T\to+\infty} T^{-1} \int_0^T F(t, X)\, dt = F_0(X).$$

For example, F might be a periodic or almost-periodic function of t. The approximation of the system (42) solution is given by

$$X \approx X^0, \tag{43}$$

where X^0 is the solution of the "averaged" equation

$$\frac{dX^0}{dt} = \delta F_0\big(X^0\big), \quad X \in B. \tag{44}$$

An asymptotic series can be constructed

$$X = X^0 + \delta X^1 + \cdots, \tag{45}$$

where X^0 are functions chosen so, that expression (45) should satisfy (42) up to quantities of the order $O(\delta^m)$ so that the functions should satisfy some recurrent conditions. There is an iterative procedure to find the functions X^m.

Unfortunately, results obtained in [9] and in [16] can mainly be applied to the finite dimensional case. However, the aforementioned method can be used for the infinite dimensional problem under consideration due to the very strong theorem about averaging methods correctness (see the book [17]):

Theorem 8.1 [17] *Suppose S is a ball in the Banach space B. Assume that $F(t, X)$ is bounded for $X \in S$, continuous in t and satisfies*

$$\big|F(t, X_1) - F(t, X_2)\big| \le c|X_1 - X_2|. \tag{46}$$

Moreover, for each X, the time average

$$\lim_{T\to+\infty} T^{-1} \int_0^T F(t, X)\, dt = F_0(X)$$

exists. If $Y(t)$, $t \in [0, T_0]$ is a solution of the averaged equation

$$\frac{dY}{dt} = F_0(Y), \tag{47}$$

such that $Y \in S$ for all $t \in [0, T_0]$, then for each $\eta > 0$ there is an $\delta_0 > 0$ such that for $0 < \delta < \delta_0$ solution

$X(t,\delta)$ of (42) with initial data $X(0,\delta)=Y(0)$ admits the estimate

$$\left|X(t,\delta)-Y(t)\right|\le\eta\quad(0\le t\le T/\delta).$$

It is clear that in our case the estimate (46) holds, since $\mathcal{A}$ is a bounded linear operator. Moreover, Y lies in a ball due to a priori estimate $|Y(t)|\le C|Y(0)|\exp(ct)$. Thus, this theorem can be applied to our problem.

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