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Control of planar Bautin bifurcation

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Abstract On a versal deformation of the Bautin bifurcation it is possible to find dynamical systems that undergo Hopf or non-hyperbolic limit cycle bifurcations. Our paper concerns a nonlinear control system in the plane whose nominal vector field has a pair of purely imaginary eigenvalues. We find conditions to control the Hopf and Bautin bifurcation using the symmetric multilinear vector functions that appear in the Taylor expansion of the vector field around the equilibrium. The control law designed by us depends on two bifurcation parameters and four control parameters, which establish the stability of the equilibrium point and the orientation and stability of the limit cycles. Two examples are given.

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1 Introduction

During the recent years many publications in the field of dynamical systems focus on bifurcations control. Different control systems were designed in order to create different types of bifurcation and to manipulate the bifurcation characteristics such as the stability and orientation of limit cycles or the stability of equilibria [2].

Many authors developed control laws for linear or nonlinear systems with one or more uncontrollable modes.

Although there exist many papers in the literature concerning the control of the codimension one Hopf bifurcation [6, 9, 10], there are few results concerning the control of the generalized Hopf bifurcation of codimension two (Bautin) [1].

In this paper we study a nonlinear planar control system with two uncontrollable modes on the imaginary axis. We design a control law such that the resulting control system undergoes controllable Hopf or Bautin bifurcation. Our analysis is based on the normal form theory [5, 7]. We prove that our control law determines the orientation and stability of periodic solutions for both Hopf and Bautin bifurcations.

Consider the planar nonlinear control system

$$\dot{x} = F(x) + G(x)u, \tag{1}$$

where $x = (x_1, x_2)^T \in \mathbb{R}^2$ is the state vector, $F, G : \mathbb{R}^2 \to \mathbb{R}^2$ are smooth vector fields, $u = u(x; \mu, \beta)$, $\mu \in \mathbb{R}^2$, $\beta \in \mathbb{R}^3$, and $G(x)\,u$ is the control input.

Assume that $F(0) = 0$, $G(0) = 0$ and $J = dF(0)$ has purely imaginary eigenvalues $\lambda_{1,2} = \pm\omega_0 i$, $\omega_0 \geq 0$, and

$$M = dG(0) = \begin{pmatrix} \frac{\partial G_1}{\partial x_1} & \frac{\partial G_1}{\partial x_2} \\ \frac{\partial G_2}{\partial x_1} & \frac{\partial G_2}{\partial x_2} \end{pmatrix}_{|x=0} = (m_{ij})_{i,j=1,2}.$$

In the following $\mu = (\mu_1, \mu_2) \in \mathbb{R}^2$ is the Bautin bifurcation parameter, $\beta = (\beta_1, \beta_2, \beta_3) \in \mathbb{R}^3$ and $\operatorname{tr}(M)$ are the control parameters. Here β_1 determines the stability of the equilibrium point, β_2 determines the orientation and the stability of the limit cycle emerging at a Hopf bifurcation, whilst β_3 and $\operatorname{tr}(M)$ establish the orientation and the stability of the limit cycles emerging at the Bautin bifurcation.

Consider the following expression for the scalar function u:

$$u(x, \mu, \beta) = \beta_1\mu_1 + (\beta_2 + \mu_2)\left(x_1^2 + x_2^2\right) + \beta_3\left(x_1^2 + x_2^2\right)^2. \tag{2}$$

The aim of the paper is to find values of the control parameters $\beta_1, \beta_2, \beta_3$ and $\operatorname{tr}(M)$ such that system (1) undergoes controllable Hopf or Bautin bifurcations.

In order to do this, two well known theorems from the bifurcation theory [7] are used.

Theorem (Hopf) [7] *Suppose the planar system*

$$\dot{x} = f(x, \alpha), \quad x \in \mathbb{R}^2, \quad \alpha \in \mathbb{R}, \tag{3}$$

with smooth f, *has for all sufficiently small* $|\alpha|$ *the equilibrium* $x = 0$ *with eigenvalues*

$$\lambda_{1,2}(\alpha) = \mu(\alpha) \pm i\omega(\alpha), \tag{4}$$

where $\mu(0) = 0$, $\omega(0) = \omega_0 > 0$. *Let the following conditions be satisfied*:

(H.1) $l_1(0) \neq 0$, *where* $l_1(\alpha)$ *is the first Lyapunov coefficient*.

(H.2) $\mu'(0) \neq 0$.

Then, there are invertible coordinate and parameter changes and a time reparametrization transforming (3) *into the complex form*

$$\dot{z} = (b_1 + i)z + sz|z|^2 + O\left(|z|^4\right), \tag{5}$$

where $s = \operatorname{sign}(l_1(0)) = \pm 1$.

Theorem (Bautin) [7] *Suppose the planar system*

$$\dot{x} = f(x, \alpha), \quad x \in \mathbb{R}^2, \quad \alpha \in \mathbb{R}^2, \tag{6}$$

with smooth f, *has the equilibrium* $x = 0$ *with eigenvalues*

$$\lambda_{1,2}(\alpha) = \mu(\alpha) \pm i\omega(\alpha), \tag{7}$$

for all $\|\alpha\|$ *sufficiently small, where* $\omega(0) = \omega_0 > 0$. *For* $\alpha = 0$, *let the Bautin bifurcation conditions hold*:

$$\mu(0) = 0, \qquad l_1(0) = 0, \tag{8}$$

where $l_1(\alpha)$ *is the first Lyapunov coefficient. Assume that the following general conditions are satisfied*:

(B.1) $l_2(0) \neq 0$, *where* $l_2(\alpha)$ *is the second Lyapunov coefficient*.

(B.2) *the map* $\alpha \to (\mu(\alpha), l_1(\alpha))^T$ *is regular at* $\alpha = 0$.

Then, by the introduction of a complex variable, applying smooth invertible coordinate transformation that depend smoothly on the parameters, and performing smooth parameter and time changes, the system can be reduced to the complex form

$$\dot{z} = (b_1 + i)z + b_2 z|z|^2 + sz\left|z^4\right| + O\left(|z|^6\right), \tag{9}$$

where $s = \operatorname{sign}(l_2(0)) = \pm 1$.

2 Control of the Hopf bifurcation

In this section we find sufficient conditions such that system (1) undergoes a Hopf bifurcation that can be controlled. To this aim we shall use a complex variable form of a two-dimensional system, following the lines in [7].

Expanding system (1) around $x=0$ we get

$$\dot{x} = Ax + \mathcal{F}(x), \tag{10}$$

where

$$A = J + \beta_1\mu_1 M,$$
$$\mathcal{F}(x) = \frac{1}{2}B(x,x) + \frac{1}{3!}C(x,x,x) + \frac{1}{4!}D(x,x,x,x) + \frac{1}{5!}E(x,x,x,x,x) + O(\|x\|^6)$$

and B, C, D, E are multilinear symmetric forms, with

$$\begin{aligned}
B(x,x) &= d^2F(0)(x,x) + \beta_1\mu_1 d^2G(0)(x,x),\\
C(x,x,x) &= d^3F(0)(x,x,x) + \beta_1\mu_1 d^3G(0)(x,x,x) + 6(\beta_2+\mu_2)(x_1^2+x_2^2)Mx,\\
D(x,x,x,x) &= d^4F(0)(x,x,x,x) + \beta_1\mu_1 d^4G(0)(x,x,x,x) + 12(\beta_2+\mu_2)(x_1^2+x_2^2)d^2G(0)(x,x),\\
E(x,x,x,x,x) &= d^5F(0)(x,x,x,x,x) + \beta_1\mu_1 d^5G(0)(x,x,x,x,x)\\
&\quad + 20(\beta_2+\mu_2)(x_1^2+x_2^2)d^3G(0)(x,x,x) + 120\beta_3(x_1^2+x_2^2)^2 Mx.
\end{aligned} \tag{11}$$

In the following we find sufficient conditions for the parameters β_1, β_2 such that the conditions of the Hopf bifurcation theorem are fulfilled for system (1).

Condition (H.1) *At* $\mu_1 = 0$, *in* (10) *we have*

$$\begin{aligned}
A &= J,\\
B(x,x) &= d^2F(0)(x,x),\\
C(x,x,x) &= d^3F(0)(x,x,x) + 6(\beta_2+\mu_2)H(x),
\end{aligned} \tag{12}$$

where

$$H(x) = (x_1^2+x_2^2)Mx. \tag{13}$$

In the previous assumptions, the Jacobian matrix A of system (1) at $x=0$ and $\mu_1=0$, possesses a pair of purely imaginary eigenvalues $\lambda_{1,2} = \pm i\omega_0$. Let $q \in \mathbb{C}^2$ be an eigenvector of A associated with the eigenvalue $\lambda_1 = i\omega_0$ and $p \in \mathbb{C}^2$ an eigenvector of A^T associated with $\bar{\lambda}_1$, such that $\langle p, q\rangle = 1$. Using the complex variable $z = \langle p, x\rangle$, we get $x = zq + \bar{z}\bar{q}$, and (10) at $\mu_1 = 0$ reads:

$$\dot{z} = \lambda_1 z + g(z,\bar{z}),$$

where $g(z,\bar{z}) = \langle p, \mathcal{F}(zq+\bar{z}\bar{q})\rangle$. Function g can be written as a formal Taylor series in two complex variables as:

$$g(z,\bar{z}) = \sum_{k+l\geq 2} \frac{1}{k!l!} g_{kl} z^k \bar{z}^l,$$

where

$$g_{kl} = \frac{\partial^{k+l}}{\partial^k z \partial^l \bar{z}} g(z,\bar{z})_{|z=0} = \frac{\partial^{k+l}}{\partial^k z \partial^l \bar{z}} \langle p, \mathcal{F}(zq+\bar{z}\bar{q})\rangle_{|z=0}.$$

The first Lyapunov coefficient at the bifurcation point can be expressed as in [7, p. 99],

$$l_1(0) = \frac{1}{2\omega_0^2}\operatorname{Re}(i g_{20} g_{11} + \omega_0 g_{21}), \tag{14}$$

where [7, pp. 93–94],

$$\begin{aligned}
g_{20} &= \langle p, B(q,q)\rangle,\\
g_{11} &= \langle p, B(q,\bar{q})\rangle,\\
g_{21} &= \langle p, C(q,q,\bar{q})\rangle.
\end{aligned} \tag{15}$$

For system (1) we have

$$\begin{aligned}
g_{20} &= \langle p, d^2F(0)(q,q)\rangle,\\
g_{11} &= \langle p, d^2F(0)(q,\bar{q})\rangle,\\
g_{21} &= \langle p, d^3F(0)(q,q,\bar{q})\rangle + 6(\beta_2+\mu_2)\langle p, H(q,q,\bar{q})\rangle.
\end{aligned} \tag{16}$$

Taking into account the expression of the quadratic form H given by (13), we get

$$H(x,x,y) = \frac{1}{3}(x_1^2+x_2^2)My + \frac{2}{3}(x_1y_1+x_2y_2)Mx, \tag{17}$$

where $x = (x_1,x_2)^T$, $y = (y_1,y_2)^T \in \mathbb{C}^2$.

Thus, at $\mu_1 = 0$, we obtain:

$$l_1(0, \mu_2) = \delta + \frac{3}{\omega_0}(\beta_2 + \mu_2)\,\mathrm{Re}\big(\langle p, H(q, q, \bar{q})\rangle\big), \tag{18}$$

with

$$\delta = \frac{1}{2\omega_0^2}\,\mathrm{Re}\big(i\langle p, d^2F(0)(q, q)\rangle\langle p, d^2F(0)(q, \bar{q})\rangle + \omega_0\langle p, d^3F(0)(q, q, \bar{q})\rangle\big) \tag{19}$$

depending only on F. Remark that δ is the first Lyapunov coefficient corresponding to the vector field F. For every $\mu_2 \in \mathbb{R}$, there exists a unique value

$$\beta_{2\mathrm{crit}}(\mu_2) = \frac{-\delta\omega_0}{3\,\mathrm{Re}(\langle p, H(q, q, \bar{q})\rangle)} - \mu_2 \tag{20}$$

of β_2 such that $l_1(0, \mu_2) = 0$, provided that $\mathrm{Re}(\langle p, H(q, q, \bar{q})\rangle) \neq 0$. Considering $\beta_2 \neq \beta_{2\mathrm{crit}}(\mu_2)$, condition (H.1) is satisfied.

Remark If the matrix J has the particular form

$$J = \begin{pmatrix} 0 & -\omega_0 \\ \omega_0 & 0 \end{pmatrix}$$

we can choose the eigenvectors $q = (1, -i)^T$, $p = (\frac{1}{2}, -\frac{i}{2})^T$, thus

$$H(q, q, \bar{q}) = \frac{4}{3}Mq.$$

and the first Lyapunov coefficient in (18) reads

$$l_1(0, \mu_2) = \delta + \frac{2}{\omega_0}(\beta_2 + \mu_2)\mathrm{tr}(M). \tag{21}$$

In this particular case we get

$$\beta_{2\mathrm{crit}}(\mu_2) = -\frac{\delta\omega_0}{2\mathrm{tr}(M)} - \mu_2, \tag{22}$$

provided that $\mathrm{tr}(M) \neq 0$.

Condition (H.2) *Since for $\mu_1 \neq 0$ we have*

$$\mathrm{Re}\,\lambda_{1,2}(\mu_1) = \frac{1}{2}\beta_1\mu_1\mathrm{tr}(M),$$

it follows that

$$\frac{d\,\mathrm{Re}\,\lambda_{1,2}}{d\mu_1}|_{\mu_1=0}. = \frac{1}{2}\beta_1\mathrm{tr}(M), \tag{23}$$

thus $\frac{d\,\mathrm{Re}\,\lambda}{d\mu_1}|_{\mu_1=0} \neq 0$ *iff* $\beta_1 \neq 0$, *provided* $\mathrm{tr}(M) \neq 0$.

We have proved the following result.

Theorem 1 *Consider the planar nonlinear control system*

$$\dot{x} = F(x) + G(x)\,u(x, \mu, \beta), \tag{24}$$

where $x \in \mathbb{R}^2$, $F, G : \mathbb{R}^2 \to \mathbb{R}^2$ are smooth vector fields, $F(0) = 0$, $G(0) = 0$, $J = dF(0) = \left(\begin{smallmatrix} 0 & -\omega_0 \\ \omega_0 & 0 \end{smallmatrix}\right)$, and $\mathrm{tr}(dG(0)) \neq 0$.Then there exist $\beta_1, \beta_2 \in \mathbb{R}$ such that with the control law $G(x)\,u(x, \mu, \beta)$, where

$$u(x, \mu, \beta) = \beta_1\mu_1 + (\beta_2 + \mu_2)\big(x_1^2 + x_2^2\big) + \beta_3\big(x_1^2 + x_2^2\big)^2, \tag{25}$$

system (24) undergoes a Hopf bifurcation at $\mu_1 = 0$,provided that $\beta_2 \neq \beta_{2\mathrm{crit}}(\mu_2)$. In addition it is possible to control the stability and the direction of the limit cycles emerging near the origin by selecting positive or negative β_1 and $\beta_2 > \beta_{2\mathrm{crit}}(\mu_2)$ or $\beta_2 < \beta_{2\mathrm{crit}}(\mu_2)$, where $\beta_{2\mathrm{crit}}(\mu_2)$ is given by (22).

Remark According to (21) and (22), it follows that

$$l_1(0, \mu_2) > 0 \quad \Leftrightarrow \quad \begin{cases} \beta_2 > \beta_{2\mathrm{crit}}, & \text{if } \mathrm{tr}(M) > 0, \\ & \text{or} \\ \beta_2 < \beta_{2\mathrm{crit}}, & \text{if } \mathrm{tr}(M) < 0, \end{cases}$$

while

$$l_1(0, \mu_2) < 0 \quad \Leftrightarrow \quad \begin{cases} \beta_2 < \beta_{2\mathrm{crit}}, & \text{if } \mathrm{tr}(M) > 0, \\ & \text{or} \\ \beta_2 > \beta_{2\mathrm{crit}}, & \text{if } \mathrm{tr}(M) < 0. \end{cases}$$

Using Theorem 1, and selecting appropriate values for the control parameters β_1, β_2, all the possible Hopf bifurcation diagrams, considering μ_1 as the bifurcation parameter, can be obtained. See Fig. 1. In this figure the cases (i) and (iii) correspond to $l_1 < 0$, thus to a supercritical Hopf bifurcation, while the cases (ii) and (iv) correspond to $l_1 > 0$, thus to a subcritical Hopf bifurcation.

3 Control of the Bautin bifurcation

In this section we find the critical value of the control parameter β_3 such that system (1) undergoes all kind of Bautin bifurcation.

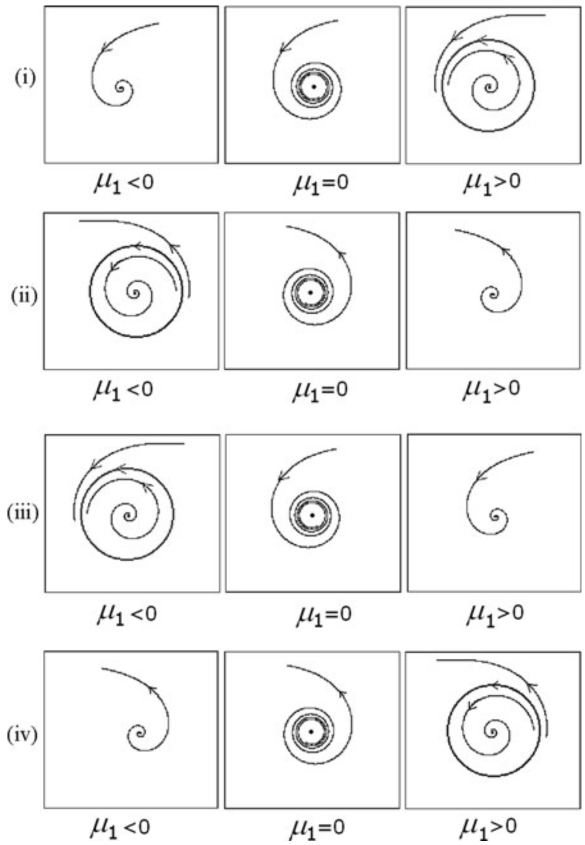

Fig. 1 Hopf bifurcation diagrams: (**i**) $\beta_1 > 0, \beta_2 < \beta_{2\text{crit}}(\mu_2)$, $\text{tr}(M) > 0$ or $\beta_1 < 0, \beta_2 > \beta_{2\text{crit}}(\mu_2)$, $\text{tr}(M) < 0$; (**ii**) $\beta_1 > 0, \beta_2 > \beta_{2\text{crit}}(\mu_2)$, $\text{tr}(M) > 0$ or $\beta_1 < 0$, $\beta_2 < \beta_{2\text{crit}}(\mu_2)$, $\text{tr}(M) < 0$; (**iii**) $\beta_1 < 0, \beta_2 < \beta_{2\text{crit}}(\mu_2)$, $\text{tr}(M) > 0$ or $\beta_1 > 0, \beta_2 > \beta_{2\text{crit}}(\mu_2)$, $\text{tr}(M) < 0$; (**iv**) $\beta_1 < 0$, $\beta_2 > \beta_{2\text{crit}}(\mu_2)$, $\text{tr}(M) > 0$ or $\beta_1 > 0, \beta_2 < \beta_{2\text{crit}}(\mu_2)$, $\text{tr}(M) < 0$

Assume that the first Lyapunov coefficient is zero at $\mu_2 = 0$, that is $\beta_2 = \beta_{2\text{crit}}(0)$.

Condition (B.1) *In order to compute the second Lyapunov coefficient at* $(\mu_1, \mu_2) = (0, 0)$, *we use the following formula* [7, p. 310]:

$$
\begin{aligned}
&12 l_2(0,0)\\
&= \frac{1}{\omega_0}\operatorname{Re} g_{32} + \frac{1}{\omega_0^2}\operatorname{Im}\Big[g_{20}\bar g_{31} - g_{11}(4g_{31} + 3\bar g_{22})\\
&\quad - \frac{1}{3}g_{02}(g_{40} + \bar g_{13}) - g_{30}g_{12}\Big]\\
&\quad + \frac{1}{\omega_0^3}\bigg\{\operatorname{Re}\bigg[g_{20}\bigg(\bar g_{11}(3g_{12} - \bar g_{30})\\
&\quad + g_{02}\Big(\bar g_{12} - \frac{1}{3}g_{30}\Big) + \frac{1}{3}\bar g_{02}g_{03}\bigg)\\
&\quad + g_{11}\bigg(\bar g_{02}\Big(\frac{5}{3}\bar g_{30} + 3g_{12}\Big)\\
&\quad + \frac{1}{3}g_{02}\bar g_{03} - 4g_{11}g_{30}\bigg)\bigg]\\
&\quad + 3\operatorname{Im}(g_{20}g_{11})\operatorname{Im} g_{21}\bigg\}\\
&\quad + \frac{1}{\omega_0^4}\big\{\operatorname{Im}\big[g_{11}\bar g_{02}\big(\bar g_{20}^2 - 3\bar g_{20}g_{11} - 4g_{11}^2\big)\big]\\
&\quad + \operatorname{Im}(g_{20}g_{11})\big[3\operatorname{Re}(g_{20}g_{11}) - 2|g_{02}|^2\big]\big\}. \qquad (26)
\end{aligned}
$$

The coefficients g_{kl} *used in the above formula can be written in terms of the functions* C, D, E. *Thus, from*

$$
\frac{1}{3!}\langle p, \mathcal{C}(zq + \bar z\bar q, zq + \bar z\bar q, zq + \bar z\bar q)\rangle = \sum_{k+l=3}\frac{1}{k!l!}g_{kl}z^k\bar z^l,
$$

we obtain:

$$
\begin{aligned}
g_{30} &= \langle p, C(q,q,q)\rangle, & g_{21} &= \langle p, C(q,q,\bar q)\rangle,\\
g_{12} &= \langle p, C(q,\bar q,\bar q)\rangle, & g_{03} &= \langle p, C(\bar q,\bar q,\bar q)\rangle.
\end{aligned}
$$

From $\frac{1}{4!}\langle p, \mathcal{D}(zq + \bar z\bar q, zq + \bar z\bar q, zq + \bar z\bar q, zq + \bar z\bar q)\rangle = \sum_{k+l=4}\frac{1}{k!l!}g_{kl}z^k\bar z^l$, *we obtain*:

$$
\begin{aligned}
g_{40} &= \langle p, D(q,q,q,q)\rangle, & g_{31} &= \langle p, D(q,q,q,\bar q)\rangle,\\
g_{22} &= \langle p, D(q,q,\bar q,\bar q)\rangle, & g_{13} &= \langle p, D(q,\bar q,\bar q,\bar q)\rangle.
\end{aligned}
$$

Similarly we get $g_{32} = \langle p, E(q,q,q,\bar q,\bar q)\rangle$.

From (12) *we have*

$$
\begin{aligned}
D &= d^4F(0) + 12\beta_2 S,\\
E &= d^5F(0) + 20\beta_2 K + 120\beta_3 R,
\end{aligned}
$$

where $S(x,y,z,t)$, $K(x,y,z,t,u)$ *and* $R(x,y,z,t,u)$ *are multilinear symmetric forms such that*:

$$
\begin{aligned}
S(x,x,x,x) &= \big(x_1^2 + x_2^2\big)d^2G(0),\\
K(x,x,x,x,x) &= \big(x_1^2 + x_2^2\big)d^3G(0),\\
R(x,x,x,x,x) &= \big(x_1^2 + x_2^2\big)^2 Mx,
\end{aligned}
$$

respectively. Applying the above formula, the coefficients g_{kl} *can be split as*:

$g_{30} = \gamma_{30} + 6\beta_2\rho_{30}$, *where*

$$\gamma_{30} = \langle p, d^3F(0)(q,q,q)\rangle, \quad \rho_{30} = \langle p, H(q,q,q)\rangle;$$

$g_{21} = \gamma_{21} + 6\beta_2\rho_{21}$, *where*

$$\gamma_{21} = \langle p, d^3F(0)(q,q,\bar{q})\rangle, \quad \rho_{21} = \langle p, H(q,q,\bar{q})\rangle;$$

$g_{12} = \gamma_{12} + 6\beta_2\rho_{12}$, *where*

$$\gamma_{12} = \langle p, d^3F(0)(q,\bar{q},\bar{q})\rangle, \quad \rho_{12} = \langle p, H(q,\bar{q},\bar{q})\rangle;$$

$g_{03} = \gamma_{03} + 6\beta_2\rho_{03}$, *where*

$$\gamma_{03} = \langle p, d^3F(0)(\bar{q},\bar{q},\bar{q})\rangle, \quad \rho_{03} = \langle p, H(\bar{q},\bar{q},\bar{q})\rangle;$$

$g_{13} = \gamma_{13} + 12\beta_2\rho_{13}$, *where*

$$\gamma_{13} = \langle p, d^4F(0)(q,\bar{q},\bar{q},\bar{q})\rangle,$$
$$\rho_{13} = \langle p, S(q,\bar{q},\bar{q},\bar{q})\rangle;$$

$g_{40} = \gamma_{40} + 12\beta_2\rho_{40}$, *where*

$$\gamma_{40} = \langle p, d^4F(0)(q,q,q,q)\rangle,$$
$$\rho_{40} = \langle p, S(q,q,q,q)\rangle;$$

$g_{22} = \gamma_{22} + 12\beta_2\rho_{22}$, *where*

$$\gamma_{22} = \langle p, d^4F(0)(q,q,\bar{q},\bar{q})\rangle,$$
$$\rho_{22} = \langle p, S(q,q,\bar{q},\bar{q})\rangle;$$

$g_{31} = \gamma_{31} + 12\beta_2\rho_{31}$, *where*

$$\gamma_{31} = \langle p, d^4F(0)(q,q,q,\bar{q})\rangle,$$
$$\rho_{31} = \langle p, S(q,q,q,\bar{q})\rangle;$$

$g_{32} = \gamma_{32} + 20\beta_2\rho_{32} + 120\beta_3\tau_{32}$, *where*

$$\gamma_{32} = \langle p, d^5F(0)(q,q,q,\bar{q},\bar{q})\rangle,$$
$$\rho_{32} = \langle p, K(q,q,q,\bar{q},\bar{q})\rangle,$$

and $\tau_{32} = \langle p, R(q,q,q,\bar{q},\bar{q})\rangle$.

Substituting into (26) *the expressions of the coefficients* g_{kl} *we get*

$$l_2(0,0) = \delta' + \frac{1}{12\omega_0}\mathrm{Re}(120\beta_3\tau_{32}), \tag{27}$$

where δ' *depends only on* F *and* G. *Hence*

$$l_2(0,0) = \delta' + \frac{10}{\omega_0}\beta_3\,\mathrm{Re}(\tau_{32}). \tag{28}$$

Consequently, there exists a unique value

$$\beta_{3\mathrm{crit}} = -\frac{\delta'\omega_0}{10\,\mathrm{Re}(\tau_{32})} \tag{29}$$

of β_3 *such that* $l_2(0,0) = 0$, *provided that* $\mathrm{Re}(\tau_{32}) \neq 0$.

Considering $\beta_3 \neq \beta_{3\mathrm{crit}}$, *condition* (*B*.1) *is satisfied.*

Lemma 1 If $Q : (\mathbb{R}^2)^5 \to \mathbb{R}$ is a multilinear symmetric form with

$$Q(x,x,x,x,x) = \sum_{j=0}^{5} c_{5-j,j}\, x_1^{5-j} x_2^{j},$$

then

$$\begin{aligned}
Q(x,x,x,t,t) &= c_{50}x_1^3t_1^2 + \frac{2}{5}c_{41}x_1^3t_1t_2 + \frac{1}{10}c_{32}x_1^3t_2^2 \\
&\quad + \frac{3}{5}c_{41}x_1^2x_2t_1^2 + \frac{6}{10}c_{32}x_1^2x_2t_1t_2 \\
&\quad + \frac{3}{10}c_{23}x_1^2x_2t_2^2 + \frac{3}{10}c_{32}x_1x_2^2t_1^2 \\
&\quad + \frac{6}{10}c_{23}x_1x_2^2t_1t_2 + \frac{3}{5}c_{14}x_1x_2^2t_2^2 \\
&\quad + \frac{1}{10}c_{23}x_2^3t_1^2 + \frac{2}{5}c_{14}x_2^3t_1t_2 + c_{05}x_2^3t_2^2.
\end{aligned}$$

Proof It follows from direct computations. □

Remark In the particular case when $J = dF(0) = \begin{pmatrix} 0 & -\omega_0 \\ \omega_0 & 0 \end{pmatrix}$, we can choose $q = (1, -i)^T$, $p = (\frac{1}{2}, -\frac{i}{2})^T$. Thus from (22) we have

$$\beta_{2\mathrm{crit}}(0) = -\frac{\omega_0\delta}{2\mathrm{tr}(M)}. \tag{30}$$

Using Lemma 1, we get

$$R(q,q,q,\bar{q},\bar{q}) = \begin{pmatrix} \frac{8}{5}(m_{11} - i m_{12}) \\ \frac{8}{5}(m_{21} - i m_{22}) \end{pmatrix},$$

consequently

$$\tau_{32} = \frac{4}{5}(m_{11} + m_{22}) + \frac{4}{5}i(m_{21} - m_{12}) \tag{31}$$

and the second Lyapunov coefficient in (28) reads:

$$l_2(0,0) = \delta' + \frac{8}{\omega_0}\beta_3\mathrm{tr}(M). \tag{32}$$

In this case we have

$$\beta_{3\text{crit}} = -\frac{\delta' \omega_0}{8\operatorname{tr}(M)}, \tag{33}$$

if $\operatorname{tr}(M) \neq 0$.

Condition (B.2) *The map*

$$(\mu_1, \mu_2) \to \big(\operatorname{Re}\lambda(\mu_1, \mu_2), l_1(\mu_1, \mu_2)\big)$$

is regular at $(\mu_1, \mu_2) = (0, 0)$ *provided*

$$\Delta = \begin{vmatrix} \frac{\partial}{\partial \mu_1}\operatorname{Re}\lambda(\mu_1, \mu_2) & \frac{\partial}{\partial \mu_2}\operatorname{Re}\lambda(\mu_1, \mu_2) \\ \frac{\partial}{\partial \mu_1} l_1(\mu_1, \mu_2) & \frac{\partial}{\partial \mu_2} l_1(\mu_1, \mu_2) \end{vmatrix} \neq 0 \tag{34}$$

at $(\mu_1, \mu_2) = (0, 0)$. *Since*

$$\operatorname{Re}\lambda(\mu_1, \mu_2) = \frac{1}{2}\beta_1 \mu_1 \operatorname{tr}(M), \tag{35}$$

we have $\frac{\partial}{\partial \mu_2}\operatorname{Re}\lambda(\mu_1, \mu_2) = 0$, *hence only the term* $\frac{\partial}{\partial \mu_2} l_1(\mu_1, \mu_2)$ *have to be evaluated.*

The first Lyapunov coefficient corresponding to the Hopf bifurcation in (1) *can be written in terms of the derivatives of the functions in the right hand side in* (1) *as* [5]

$$l_1(\mu_1, \mu_2) = \frac{R_1}{16\omega_0^2} + \frac{R_2}{16\omega_0}, \tag{36}$$

where R_1 *contains only terms from the second order derivatives of the vector field* $F(x) + G(x)u$, *and*

$$R_2 = \frac{\partial^3 F_1}{\partial x_1^3} + \frac{\partial^3 F_1}{\partial x_1 \partial x_2^2} + \frac{\partial^3 F_2}{\partial x_1^2 \partial x_2} + \frac{\partial^3 F_2}{\partial x_2^3}, \tag{37}$$

where by F_1, F_2 *we denote the components of the two dimensional vector field* $F(x) + G(x)u$. *Taking into account* (12) *we obtain* $\frac{\partial R_1}{\partial \mu_2} = 0$, *and*

$$R_2 = 8(\beta_2 + \mu_2)\operatorname{tr}(M), \tag{38}$$

thus $\frac{\partial R_2}{\partial \mu_2} = 8\operatorname{tr}(M)$. *It follows that*

$$\Delta = \frac{1}{4\omega_0}\beta_1 \big(\operatorname{tr}(M)\big)^2. \tag{39}$$

Consequently the transversality condition (*B*.2) *is fulfilled, provided*

$$\beta_1 \operatorname{tr}(M) \neq 0. \tag{40}$$

We have proved the following result.

Theorem 2 Consider the planar nonlinear control system

$$\dot{x} = F(x) + G(x)u(x, \mu, \beta), \tag{41}$$

where $x \in \mathbb{R}^2$, $F, G : \mathbb{R}^2 \to \mathbb{R}^2$ are smooth vector fields, $F(0) = 0$, $G(0) = 0$, $J = dF(0)$ having purely imaginary eigenvalues, $M = dG(0)$ and $\operatorname{tr}(M) \neq 0$. Then there exist β_1, $\beta_2, \beta_3 \in \mathbb{R}$, where $\beta_2 = \beta_{2\text{crit}}(0)$ is given by (30), such that with the control law $G(x)u(x, \mu, \beta)$, where

$$u(x, \mu, \beta) = \beta_1 \mu_1 + (\beta_2 + \mu_2)\big(x_1^2 + x_2^2\big) + \beta_3\big(x_1^2 + x_2^2\big)^2, \tag{42}$$

system (41) undergoes a Bautin bifurcation at $\mu_1 = \mu_2 = 0$, for $\beta_3 \neq \beta_{3\text{crit}}$. In addition it is possible to control the stability and the direction of the limit cycles emerging near the origin by selecting positive or negative β_1, $\operatorname{tr}(M)$ and $\beta_3 > \beta_{3\text{crit}}$ or $\beta_3 < \beta_{3\text{crit}}$, where $\beta_{3\text{crit}}$ is given by (33).

Remark According to (32) and (33), it follows that

$$l_2(0,0) > 0 \quad \Leftrightarrow \quad \begin{cases} \beta_3 > \beta_{3\text{crit}}, & \text{if } \operatorname{tr}(M) > 0, \\ & \text{or} \\ \beta_3 < \beta_{3\text{crit}}, & \text{if } \operatorname{tr}(M) < 0, \end{cases}$$

while

$$l_2(0,0) < 0 \quad \Leftrightarrow \quad \begin{cases} \beta_3 < \beta_{3\text{crit}}, & \text{if } \operatorname{tr}(M) > 0, \\ & \text{or} \\ \beta_3 > \beta_{3\text{crit}}, & \text{if } \operatorname{tr}(M) < 0. \end{cases}$$

Using Theorem 2, and selecting appropriate values for the control parameters β_1, β_3 and $\operatorname{tr}(M)$, all the possible Bautin bifurcation diagrams, considering (μ_1, μ_2) as the bifurcation parameter, can be obtained. See Fig. 2–9. In these figures, the curve C_{NH} contains values of parameters corresponding to non-hyperbolic limit cycle bifurcation. Remark that the curve C_{NH} is tangent to the μ_2 semi-axis, corresponding to $l_1(0, \mu_2) > 0$ as $l_2(0, 0) < 0$ or to the μ_2 semi-axis, corresponding to $l_1(0, \mu_2) < 0$ as $l_2(0, 0) > 0$. In addition, as $\mu_1 = \mu_2 = 0$, the first Lyapunov coefficient is zero, for all Figs. 2–9 we have $\beta_2 = \beta_{2\text{crit}}(0)$. Remark that in Figs. 2, 4, 6, 8 we have $l_2 > 0$, while in Figs. 3, 5, 7, 9 we have $l_2 < 0$.

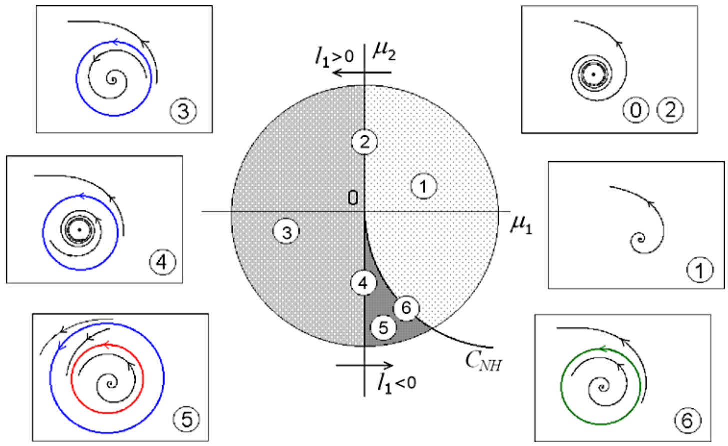

Fig. 2 Bautin bifurcation diagram: $\beta_1 > 0$, $\beta_3 > \beta_{3\mathrm{crit}}$, $\mathrm{tr}(M) > 0$

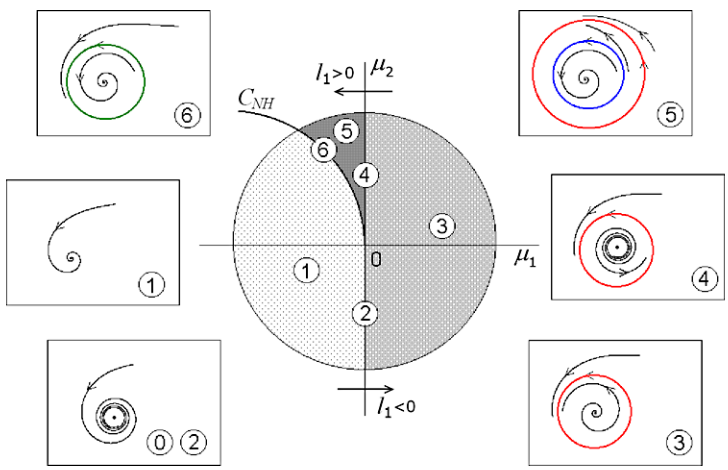

Fig. 3 Bautin bifurcation diagram: $\beta_1 > 0$, $\beta_3 < \beta_{3\mathrm{crit}}$, $\mathrm{tr}(M) > 0$

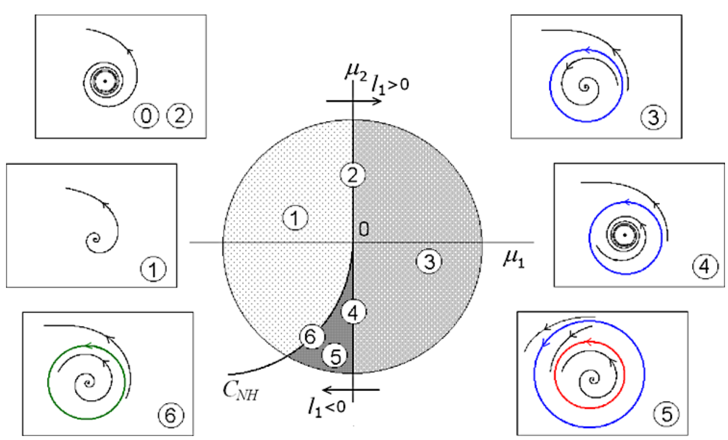

Fig. 4 Bautin bifurcation diagram: $\beta_1 < 0$, $\beta_3 > \beta_{3\mathrm{crit}}$, $\mathrm{tr}(M) > 0$

4 Applications

Example 1 Consider the planar system

$$\dot{x} = F(x), \tag{43}$$

where

$$F(x) = \begin{pmatrix} -x_2 - x_1 x_2^2 \\ x_1 \end{pmatrix}.$$

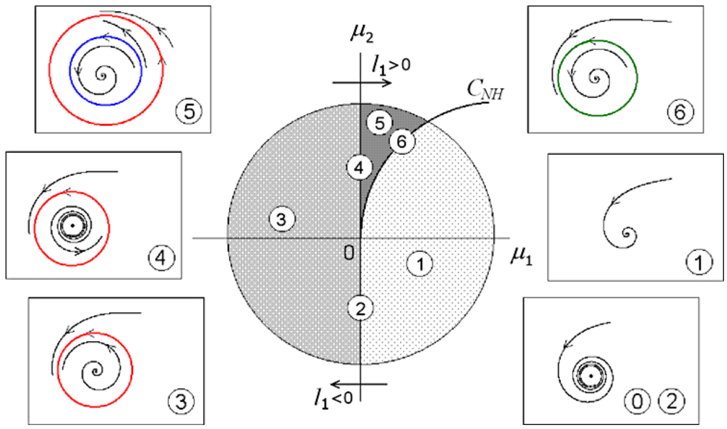

Fig. 5 Bautin bifurcation diagram: $\beta_1 < 0$, $\beta_3 < \beta_{3\text{crit}}$, $\text{tr}(M) > 0$

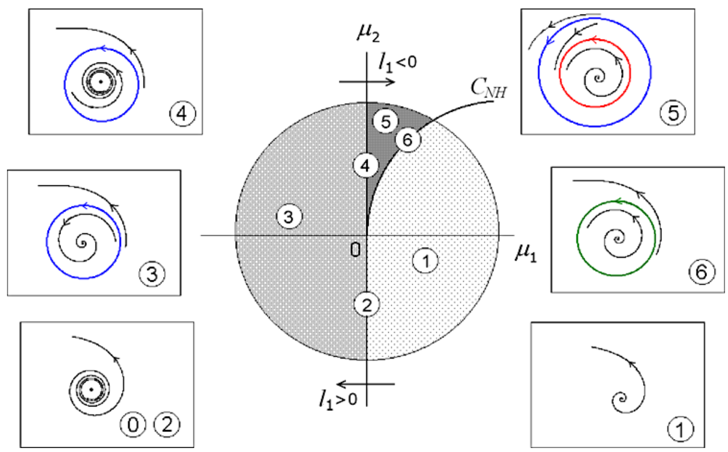

Fig. 6 Bautin bifurcation diagram: $\beta_1 < 0$, $\beta_3 < \beta_{3\text{crit}}$, $\text{tr}(M) < 0$

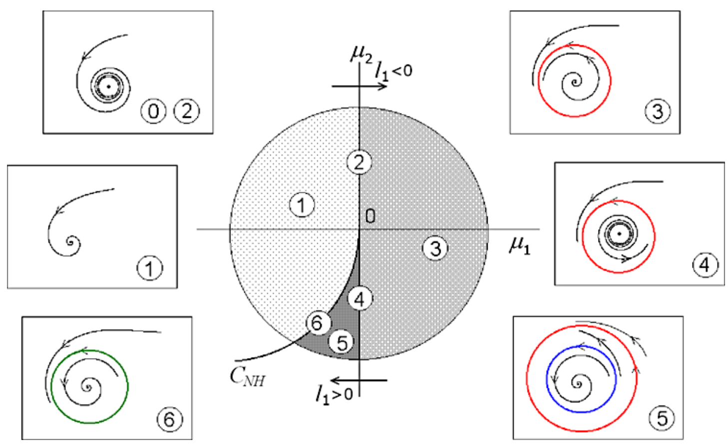

Fig. 7 Bautin bifurcation diagram: $\beta_1 < 0$, $\beta_3 > \beta_{3\text{crit}}$, $\text{tr}(M) < 0$

It possesses the non-hyperbolic equilibrium $x = 0$, and $J = dF(0) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

Consider the control $u(x, \mu, \beta)G(x)$, with $u(x, \mu, \beta)$ given by (2), and $G(x) = sx$, $s \neq 0$. We have $G(0) = 0$, $M = dG(0) = \begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix}$ and $\text{tr}(M) = 2s \neq 0$. Thus the hypothesis of Theorems 1 and 2 are fulfilled for the nonlinear control system (1), which in our case is written as

$$\dot{x}_1 = -x_2 - x_1 x_2^2 + s x_1 u(x, \mu, \beta), \tag{44}$$

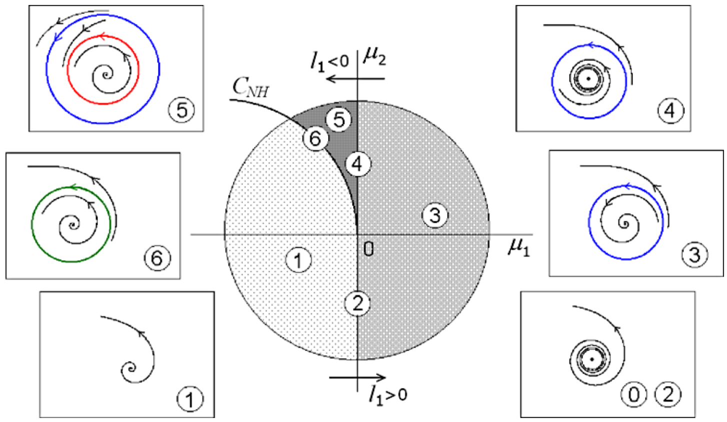

Fig. 8 Bautin bifurcation diagram: $\beta_1 > 0$, $\beta_3 < \beta_{3\text{crit}}$, $\text{tr}(M) < 0$

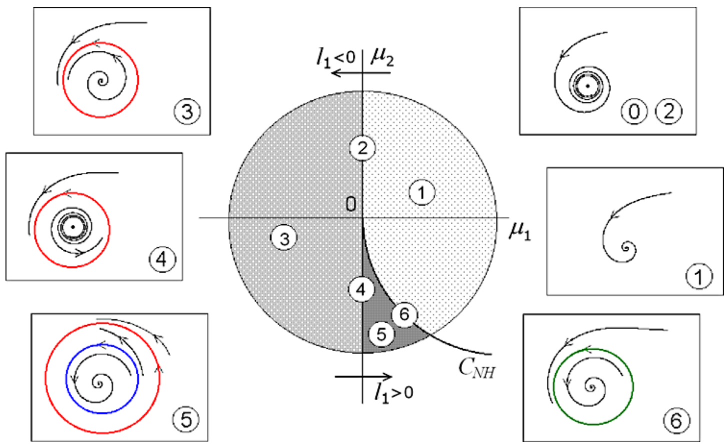

Fig. 9 Bautin bifurcation diagram: $\beta_1 > 0$, $\beta_3 > \beta_{3\text{crit}}$, $\text{tr}(M) < 0$

$\dot{x}_2 = x_1 + s x_2 u(x, \mu, \beta)$

with bifurcation parameter $\mu = (\mu_1, \mu_2)$.

Following the lines in Sect. 2, the first Lyapunov coefficient at $\mu_1 = 0$ has the expression

$$l_1(0, \mu_2) = -\frac{1}{2} + 4s(\beta_2 + \mu_2), \tag{45}$$

and we obtain the following critical value for β_2:

$$\beta_{2\text{crit}}(\mu_2) = \frac{1}{8s} - \mu_2. \tag{46}$$

For $\beta_2 = \beta_{2\text{crit}}(0) = \frac{1}{8s}$, applying the formula in Sect. 3, we find

$l_2(0, 0) = 16 s \beta_3$

and $\beta_{3\text{crit}} = 0$.

According to our results concerning the Hopf bifurcation control in Sect. 2, by choosing different values for β_1, β_2, s, we can manipulate the behavior of the system (44) in order to obtain all the diagrams from Fig. 1. For instance, if $\beta_1 = 1$, $\beta_2 = -1$, $s = 1$, $\beta_3 = 0.3$, $\mu_2 = 0.5$, three characteristic phase portraits similar to those from Fig. 1(i) are obtained in Fig. 10(i)–(iii). In this figure we plotted, using [3, 4], the trajectory through the initial point $(0.2, 0.25)$, for three different values of μ_1, showing the supercritical Hopf bifurcation. Our study is a local one, thus the Hopf bifurcation is emphasized only in a neighborhood of the point $(x_1, x_2, \mu_1) = (0, 0, 0)$. For parameter values of μ_1 far of the bifurcation value $\mu_1 = 0$, the limit cycle does not exist any longer, since it disappeared by another type of bifurcation (Fig. 10(iv)).

According to the results concerning the Bautin bifurcation control in Sect. 3, by choosing appropriate values for β_1, β_3, s, we can manipulate the behavior of the system (44) in order to obtain all the bifurcation diagrams from Figs. 2–9. For instance, in order to obtain a bifurcation diagram similar to that from Fig. 3, we first choose a positive $\text{tr}(M)$, such as $s = 1$. Then

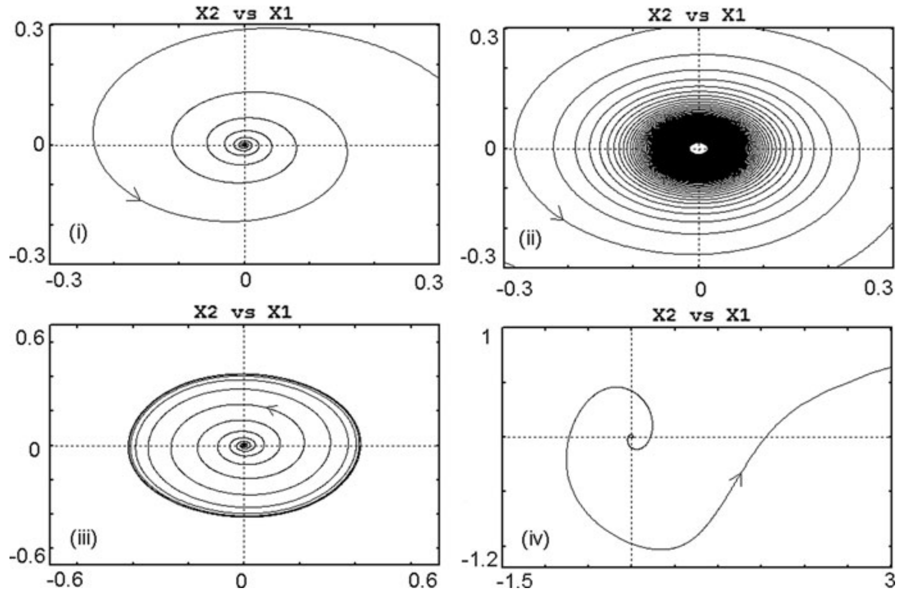

Fig. 10 The trajectory through the initial point $(x_1, x_2) = (0.2, 0.25)$ and parameters **(i)** $\mu_1 = -0.1$; **(ii)** $\mu_1 = 0$; **(iii)** $\mu_1 = 0.1$; **(iv)** $\mu_1 = 0.5$

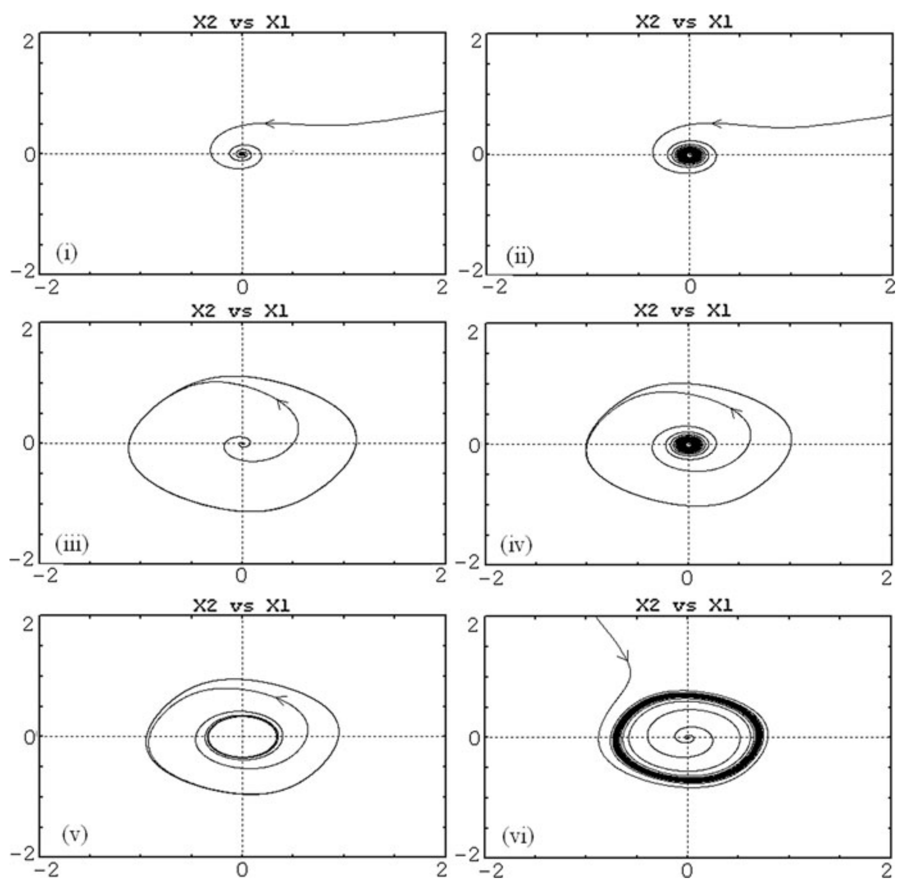

Fig. 11 The trajectory through the initial point $(x_1, x_2) = (0.5, 0.5)$ and parameters **(i)** $(\mu_1, \mu_2) = (-0.2, -1)$, region 1 in Fig. 3; **(ii)** $(\mu_1, \mu_2) = (0, -1)$, region 3; **(iii)** $(\mu_1, \mu_2) = (0.5, 1)$, region 3; **(iv)** $(\mu_1, \mu_2) = (0, 1)$, region 4; **(v)** $(\mu_1, \mu_2) = (-0.2, 1)$, region 5; **(vi)** $(\mu_1, \mu_2) = (-0.5, 1)$, region 1, close to the non-hyperbolic limit cycle bifurcation curve C_{NH}

we compute the critical values $\beta_{2crit}(0) = 0.125$ and $\beta_{3crit} = 0$. After that, we choose $\beta_2 = \beta_{2crit}(0)$ and values $\beta_1 > 0$ and $\beta_3 < \beta_{3crit}$. For example, for the control parameter values $\beta_1 = 0.5$, $\beta_3 = -1$, six characteristic phase portraits similar to those from Fig. 3 are obtained in Fig. 11. In this figure we plotted, the trajectory through the initial point $(x_1, x_2) = (0.5, 0.5)$, for six different values of (μ_1, μ_2).

Example 2 Consider the system

$$\dot{x} = F(x), \tag{47}$$

where

$$F(x) = \begin{pmatrix} -x_2 + x_1^2 + \frac{1}{3}x_1^3 \\ x_1 \end{pmatrix},$$

which possesses the non-hyperbolic equilibrium $x = 0$, and $J = dF(0) = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$.

Consider the control $u(x, \mu, \beta)G(x)$, with $u(x, \mu, \beta)$ given by (2), and

$$G(x) = \begin{pmatrix} sx_1 + x_1x_2^2 \\ sx_2 + x_2^3 \end{pmatrix}, \quad s \neq 0.$$

We have $G(0) = 0$, $M = dG(0) = \left(\begin{smallmatrix} s & 0 \\ 0 & s \end{smallmatrix}\right)$ and $\mathrm{tr}(M) = 2s \neq 0$. Thus the hypothesis of Theorems 1 and 2 are fulfilled for the nonlinear control system

$$\dot{x} = F(x) + u(x, \mu, \beta)G(x), \tag{48}$$

with bifurcation parameter $\mu = (\mu_1, \mu_2)$. The first Lyapunov coefficient at $\mu_1 = 0$ has the expression:

$$l_1(0, \mu_2) = \frac{1}{2} + 4s(\beta_2 + \mu_2), \tag{49}$$

and the critical value for β_2 is

$$\beta_{2\mathrm{crit}}(\mu_2) = -\frac{1}{8s} - \mu_2. \tag{50}$$

For $\beta_2 = \beta_{2\mathrm{crit}}(0) = -\frac{1}{8s}$, we find

$$l_2(0, 0) = 16s\beta_3 - \frac{1}{s} + \frac{1}{3}. \tag{51}$$

and $\beta_{3\mathrm{crit}} = \frac{1}{16s}(\frac{1}{s} - \frac{1}{3})$.

Choosing appropriate values for the control parameters β_1, β_2, β_3, $s = \frac{1}{2}\mathrm{tr}(M)$, all the four types of Hopf and all eight types of Bautin bifurcation diagrams can be obtained.

5 Conclusions

In this paper we consider a nonlinear planar system with a non-hyperbolic equilibrium at the origin, which possesses a pair of imaginary uncontrollable modes. We design a control law such that the resulting control system undergoes controllable Hopf or Bautin bifurcation. By two control parameters (β_1, β_2) the stability and orientation of the limit cycle in the Hopf case can be controlled by feedback, obtaining four possible cases. Similarly, the characteristics of the limit cycles in the Bautin case, as $\beta_2 = \beta_{2\mathrm{crit}}$, can be controlled by three control parameters (β_1, β_3 and $\mathrm{tr}(M)$), obtaining all the eight possible cases. Two examples emphasizing the validity of our theoretical results are given. A similar study can be done in order to control more degenerate Hopf bifurcations, computing Lyapunov coefficients of order greater than 2 by using formulae in [8].

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