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Chaos and hybrid projective synchronization of commensurate and incommensurate fractional-order Chen–Lee systems

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Abstract Recently, the fractional-order Chen–Lee system was proven to exhibit chaos by the presence of a positive Lyapunov exponent. However, the existence of chaos in fractional-order Chen–Lee systems has never been theoretically proven in the literature. Moreover, synchronization of chaotic fractional-order systems was extensively studied through numerical simulations in some of the literature, but a theoretical analysis is still lacking. Therefore, we devoted ourselves to investigating the theoretical basis of chaos and hybrid projective synchronization of commensurate and incommensurate fractional-order Chen–Lee systems in this paper. Based on the stability theorems of fractional-order systems, the necessary conditions for the existence of chaos and the controllers for hybrid projective synchronization were derived. The numerical simulations show coincidence with the theoretical results.

Keywords Chaos · Hybrid projective synchronization · Fractional-order system · Chen–Lee system · Commensurate · Incommensurate

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1 Introduction

Research into fractional-order systems has been very active in recent years. Systems with fractional-order dynamics were proven to exist in different disciplines [1], such as viscoelastic systems [2], electromagnetic waves [3], dielectric polarization [4], quantitative finance [5], and quantum evolution of complex systems [6]. Chaotic phenomena were found in several fractional-order systems, e.g., the fractional Lorenz system [7, 8], the fractional Duffing system [9–11], the fractional Chua system [12, 13], the fractional Chen system [14, 15], the fractional van der Pol system [16, 17], and the fractional Newton–Leipnik system [18, 19].

In 2004, Chen and Lee introduced a new chaotic system [20], which is now called the Chen–Lee system [21]. The Chen–Lee system relates to gyro motion originating from the anticontrol of chaos in a rigid body, and it can be implemented in electronic circuits [22]. Recently, Sheu et al. studied chaos of fractional-order Chen–Lee systems [23]. In their study, the existence of chaos in fractional-order Chen–Lee systems was proven numerically by the presence of a positive Lyapunov exponent, but the existence of chaos in fractional-order Chen–Lee systems has never been theoretically proven in the literature. In addition, synchronization of chaotic fractional-order systems was extensively studied using numerical simulations in some of the literature, but a theoretical analysis is still lacking. Therefore, we devoted ourselves to investigating the theoretical basis of chaos and hybrid projective synchronization of fractional-order Chen–Lee systems in this study.

Based on stability theorems [24–28] of commensurate and incommensurate fractional-order systems, the necessary conditions for the existence of chaos and the controllers for synchronization were derived. This paper is organized as follows. Section 2 introduces the definition of fractional derivative and the approximation method adopted in this study. Section 3 reviews the stability theorems of commensurate and incommensurate fractional-order systems. In Sect. 4, the necessary conditions for the existence of chaos of fractional-order Chen–Lee systems are derived, and numerical simulations are presented to confirm the theoretical conclusions. The hybrid projective synchronization of fractional-order Chen–Lee systems is achieved in Sect. 5, and the numerical results were coincident with the theoretical analysis. Finally, conclusions are drawn in Sect. 6.

2 Fractional derivative and approximation method

There are some definitions of fractional derivatives [1, 29], such as the Grünwald–Letnikov definition, the Riemann–Liouville definition, and the Caputo definition. Among these different definitions of fractional derivatives, the Caputo fractional derivate defined as follows was used in this study:

$$D_*^q x(t) = \begin{cases} \frac{1}{\Gamma(m-q)} \int_0^t (\tau - t)^{m-q-1} x^{(m)}(\tau)\, d\tau, \\ \quad m-1 < q < m, \\ \frac{d^m}{dt^m} x(t), \quad q = m, \end{cases} \tag{1}$$

where m is the first integer which is not less than q, $x^{(m)}$ is a general m-order derivative, and Γ is the gamma function. Due to the facts that initial values of integer-order derivatives are measurable in real applications and nonhomogeneous initial conditions are allowed, the Caputo fractional derivate was adopted.

The Adams–Bashforth–Moulton predictor-corrector scheme [30, 31] and the frequency domain approximation method [32] are two conventional approximation methods for numerically solving fractional-order differential equations. In this study, the Adams–Bashforth–Moulton predictor-corrector scheme was chosen because of the reliability of its numerical results [33].

3 Stability theorem of fractional-order systems

The stability theorem of fractional-order systems has extensively been investigated in recent years [24–28]. Some important results of the stability theorem for fractional-order systems are reviewed in this section.

Theorem 1 [24, 27, 28] *Consider the following fractional-order system*:

$$\frac{d^q x}{dt^q} = f(x), \tag{2}$$

where $x \in R^n$, $f \in R^n$, q_i*'s are rational numbers between* 0 *and* 1, *and* $\frac{d^q}{dt^q} = [\frac{d^{q_1}}{dt^{q_1}}, \frac{d^{q_2}}{dt^{q_2}}, \dots, \frac{d^{q_n}}{dt^{q_n}}]^{\mathrm{T}}$ *denotes the fractional derivatives of order* q_i. *Assume that* $q_i = v_i/u_i$, $(u_i, v_i) = 1$, $u_i, v_i \in N$, *for* $i = 1, 2, \dots, n$, *and* M *is the lowest common multiple of the denominators* u_i *of* q_i. *The equilibrium points*, x^*, *of system* (2) *satisfy*

$$f(x^*) = 0. \tag{3}$$

(1) *If* $q_1 = q_2 = \cdots = q_n$, *system* (2) *is called a commensurate fractional-order system. The equilibrium points are asymptotically stable if*

$$|\arg(\lambda)| > q\pi/2 \tag{4}$$

for all eigenvalues, λ, *of the Jacobian matrix* $J = \partial f/\partial x|_{x^*}$.

(2) *If* q_i*'s are rational numbers between* 0 *and* 1, *but not identically equal to each other, system* (2) *is called an incommensurate fractional-order system. The equilibrium points are asymptotically stable if*

$$|\arg(\lambda)| > \pi/2M \tag{5}$$

for all root λ*'s of the following equation*

$$\det(\operatorname{diag}([\lambda^{Mq_1}, \lambda^{Mq_2}, \dots, \lambda^{Mq_n}]) - J) = 0. \tag{6}$$

4 Chaos in fractional-order Chen–Lee systems

The Chen–Lee system [20] which originated from the anticontrol of chaos in a rigid body was proposed by Chen and Lee in 2004. Since then, the Chen–Lee system has garnered much attention [34–36]. The study of Chen–Lee systems was extended from integer orders to fractional orders by Sheu et al. [23] in 2007. In their study, the existence of chaos in fractional-order Chen–Lee systems was numerically proven by the presence of a positive Lyapunov exponent, but the existence of chaos in fractional-order Chen–Lee system has never been theoretically proven in the literature.

It was found that chaotic scrolls are generated only around the saddle points of index 2 in a 3-D nonlinear integer-order system. It was also found that the saddle points of index 2 play a crucial role in generating chaotic scrolls in a 3-D fractional-order system [27]. Therefore, the necessary condition for a fractional-order system to display chaotic phenomena is instability of the equilibrium points.

Theorem 2 [27] *Consider the fractional-order system* (2).

(1) *The necessary condition for the existence of chaos in a commensurate fractional-order system is*

$$q\pi/2 - \min_i\{|\arg(\lambda_i)|\} \geq 0, \tag{7}$$

where λ*'s are eigenvalues of the Jacobian matrix* $J = \partial f/\partial x|_{x^*}$.

(2) *The necessary condition for the existence of chaos in a incommensurate fractional order system is*

$$\pi/2M - \min_i\{|\arg(\lambda_i)|\} \geq 0, \tag{8}$$

where λ_i*'s are roots of the following equation*:

$$\det\left(\operatorname{diag}\left(\left[\lambda^{Mq_1}, \lambda^{Mq_2}, \ldots, \lambda^{Mq_n}\right]\right) - J\right) = 0. \tag{9}$$

The fractional-order Chen–Lee system [23] is described as

$$\begin{cases} \frac{d^{q_1}x_1}{dt^{q_1}} = -x_2x_3 + ax_1, \\ \frac{d^{q_2}x_2}{dt^{q_2}} = x_1x_3 + bx_2, \\ \frac{d^{q_3}x_3}{dt^{q_3}} = (1/3)x_1x_2 + cx_3, \end{cases} \tag{10}$$

where $a = 5$, $b = -10$, $c = -3.8$, and the initial condition is $(0.2, 0.2, 0.2)$. The five equilibrium points, x^*, of the fractional-order Chen–Lee system are

$$E_1 = (0, 0, 0),$$
$$E_2 = (\sqrt{3bc}, \sqrt{-3ac}, \sqrt{-ab}),$$
$$E_3 = (\sqrt{3bc}, -\sqrt{-3ac}, -\sqrt{-ab}),$$
$$E_4 = (-\sqrt{3bc}, -\sqrt{-3ac}, \sqrt{-ab}),$$
$$E_5 = (-\sqrt{3bc}, \sqrt{-3ac}, -\sqrt{-ab}),$$

and the Jacobian matrix evaluated at x^* is

$$J = \begin{bmatrix} a & -x_3^* & -x_2^* \\ x_3^* & b & x_1^* \\ (1/3)x_2^* & (1/3)x_1^* & c \end{bmatrix}. \tag{11}$$

The eigenvalues corresponding to equilibrium point E_1 are $(5, -10, -3.8)$; thus, E_1 is a saddle point of index 1. The eigenvalues corresponding to equilibrium points E_2, E_3, E_4, and E_5 are identical to $(-13.177, 2.1885 \pm 7.2723i)$, thus E_2, E_3, E_4, and E_5 are all saddle points of index 2. Since the characteristic equations of the Jacobian matrix evaluated at equilibrium points E_2, E_3, E_4, and E_5 are identical, the equilibrium point E_2 was used to represent the other saddle points of index 2 in estimating the necessary conditions of the existence of chaos.

(1) Commensurate case: $(q_1, q_2, q_3) = (0.9, 0.9, 0.9)$.

The characteristic equation of the Jacobian matrix (11) becomes

$$\lambda^3 - (a + b + c)\lambda + 4abc = 0.$$

Substitute the parameters a, b, c into above equation and obtain:

$$\lambda^3 + 8.8\lambda + 760 = 0.$$

It can be seen that the necessary condition (7) is satisfied:

$$0.9\pi/2 - \min_i\{|\arg(\lambda_i)|\} = 0.1352 > 0.$$

Based on Theorem 2, the Chen–Lee system with commensurate fractional orders $(q_1, q_2, q_3) = (0.9, 0.9, 0.9)$ exhibits a chaotic attractor. The maximal Lyapunov exponent calculated by the method presented in [37] is 2.323, and the phase diagrams are shown in Fig. 1. The numerical simulations confirm the theoretical conclusion.

(2) Incommensurate case: $(q_1, q_2, q_3) = (0.95, 0.9, 0.85)$.

Equation (9) becomes

$$\lambda^{M(q_1+q_2+q_3)} - a\lambda^{M(q_2+q_3)} - b\lambda^{M(q_1+q_3)} - c\lambda^{M(q_1+q_2)} + 4abc = 0.$$

Substitute the parameters a, b, c, q_1, q_2, q_3 into the above equation and obtain

$$\lambda^{54} + 3.8\lambda^{37} + 10\lambda^{36} - 5\lambda^{35} + 760 = 0.$$

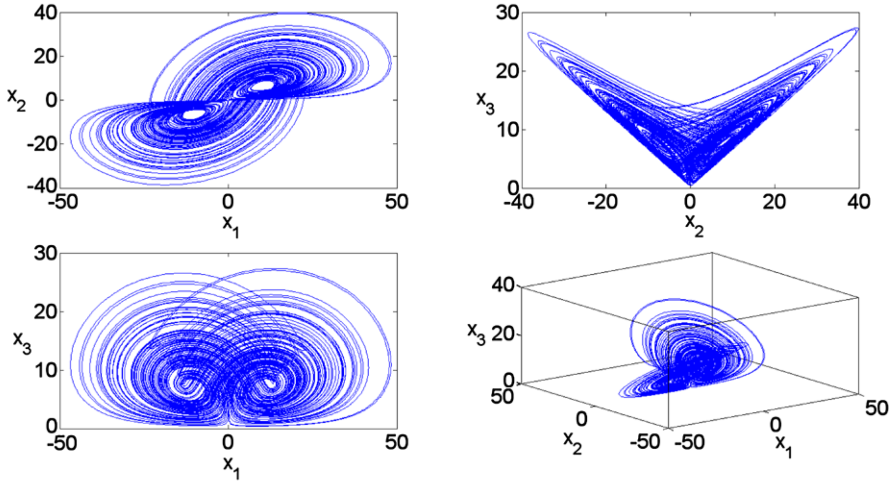

Fig. 1 Phase diagrams of the Chen–Lee system with commensurate fractional orders $(q_1, q_2, q_3) = (0.9, 0.9, 0.9)$

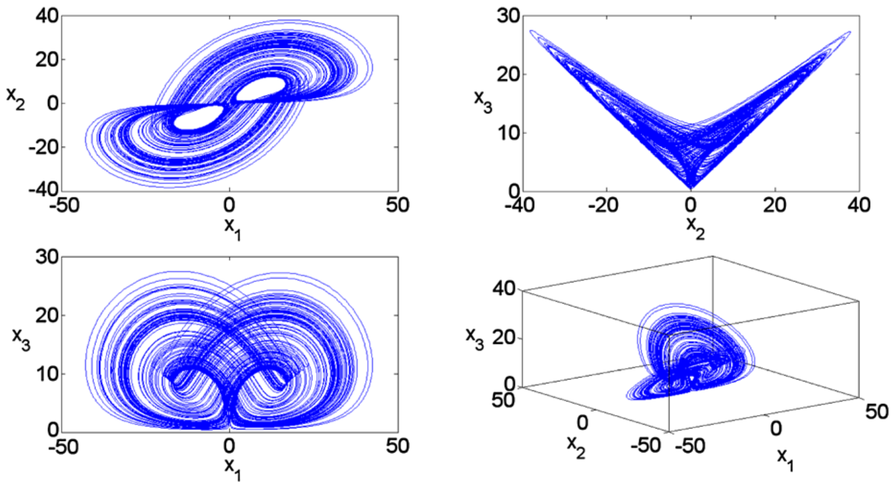

Fig. 2 Phase diagrams of the Chen–Lee system with incommensurate fractional orders $(q_1, q_2, q_3) = (0.95, 0.9, 0.85)$

It can be seen that the necessary condition (8) is satisfied:

$$\pi/40 - \min_i\{|\arg(\lambda_i)|\} = 0.0072 > 0.$$

Based on Theorem 2, the Chen–Lee system with incommensurate fractional orders $(q_1, q_2, q_3) = (0.95, 0.9, 0.85)$ exhibits a chaotic attractor. The maximal Lyapunov exponent is 1.705, and the phase diagrams are shown in Fig. 2. The numerical simulations confirm the theoretical conclusion.

5 Hybrid projective synchronization of fractional-order Chen–Lee systems

Consider the master and slave fractional-order chaotic systems described by

$$\frac{d^q x}{dt^q} = f(x) \tag{12}$$

and

$$\frac{d^q y}{dt^q} = f(y) + u, \tag{13}$$

where $x, y \in R^n$ are master- and slave-state vectors, $f: R^n \to R^n$ is a nonlinear vector function, $u \in R^n$ is a controller vector, and $\frac{d^q}{dt^q} = [\frac{d^{q_1}}{dt^{q_1}}, \frac{d^{q_2}}{dt^{q_2}}, \ldots, \frac{d^{q_n}}{dt^{q_n}}]^{\mathrm{T}}$ denotes fractional derivatives of order q_i which are rational numbers between 0 and 1. Define the error state vector as

$$e = y - \alpha x, \tag{14}$$

where $\alpha = \mathrm{diag}([\alpha_1, \alpha_2, \ldots, \alpha_n])$, and α_i's are scaling factors. Then the error dynamics are obtained:

$$\frac{d^q e}{dt^q} = f(y) - \alpha f(x) + u. \tag{15}$$

By appropriately designing the controller vector, the error dynamics can be rewritten as

$$\frac{d^q e}{dt^q} = A(x)e. \tag{16}$$

Now the hybrid projective synchronization problem between the master and slave systems has been transformed into a stability problem of a zero solution of the error dynamics. Therefore, the conclusions of Theorem 1 can be applied in this section.

Theorem 3 [24, 27, 28]

(1) *If $q_1 = q_2 = \cdots = q_n$, hybrid projective synchronization will be achieved if*

$$q\pi/2 - \min_i \{ |\arg(\lambda_i)| \} < 0 \tag{17}$$

for all eigenvalues, λ, of Jacobian matrix J of system (16) evaluated at the origin.

(2) *If q_i's are rational numbers between 0 and 1, but not identically equal to each other, hybrid projective synchronization will be achieved if*

$$\pi/2M - \min_i \{ |\arg(\lambda_i)| \} < 0 \tag{18}$$

for all root λ's of the following equation:

$$\det\big(\mathrm{diag}\big([\lambda^{Mq_1}, \lambda^{Mq_2}, \ldots, \lambda^{Mq_n}]\big) - J\big) = 0. \tag{19}$$

The master and slave fractional-order Chen–Lee systems are respectively described by

$$\begin{cases} \frac{d^{q_1} x_1}{dt^{q_1}} = -x_2 x_3 + a x_1, \\ \frac{d^{q_2} x_2}{dt^{q_2}} = x_1 x_3 + b x_2, \\ \frac{d^{q_3} x_3}{dt^{q_3}} = (1/3) x_1 x_2 + c x_3, \end{cases} \tag{20}$$

and

$$\begin{cases} \frac{d^{q_1} y_1}{dt^{q_1}} = -y_2 y_3 + a y_1 + u_1, \\ \frac{d^{q_2} y_2}{dt^{q_2}} = y_1 y_3 + b y_2 + u_2, \\ \frac{d^{q_3} y_3}{dt^{q_3}} = (1/3) y_1 y_2 + c y_3 + u_3, \end{cases} \tag{21}$$

where $a = 5$, $b = -10$, $c = -3.8$, and the initial conditions are $(x_1(0), x_2(0), x_3(0)) = (0.2, 0.2, 0.2)$ and $(y_1(0), y_2(0), y_3(0)) = (0.3, 0.3, 0.3)$. Define the hybrid projective synchronization error as $e = y - \alpha x$, where $\alpha = \mathrm{diag}([\alpha_1, \alpha_2, \alpha_3])$, and α_i's are scaling factors. Then the error dynamics are obtained:

$$\begin{cases} \frac{d^{q_1} e_1}{dt^{q_1}} = -e_2 e_3 - \alpha_3 x_3 e_2 - \alpha_2 x_2 e_3 \\ \qquad + (\alpha_1 - \alpha_2\alpha_3) x_2 x_3 + a e_1 + u_1, \\ \frac{d^{q_2} e_2}{dt^{q_2}} = e_1 e_3 + \alpha_3 x_3 e_1 + \alpha_1 x_1 e_3 \\ \qquad + (\alpha_1\alpha_3 - \alpha_2) x_1 x_3 + b e_2 + u_2, \\ \frac{d^{q_3} e_3}{dt^{q_3}} = (1/3)(e_1 e_2 + \alpha_2 x_2 e_1 + \alpha_1 x_1 e_2) \\ \qquad + (1/3)(\alpha_1\alpha_2 - \alpha_3) x_1 x_2 + c e_3 + u_3. \end{cases} \tag{22}$$

If the controllers are designed as

$$\begin{cases} u_1 = e_2 e_3 + (\alpha_2\alpha_3 - \alpha_1) x_2 x_3 - k_1 e_1, \\ u_2 = -e_1 e_3 + (\alpha_2 - \alpha_1\alpha_3) x_1 x_3 - k_2 e_2, \\ u_3 = -(1/3) e_1 e_2 + (1/3)(\alpha_3 - \alpha_1\alpha_2) x_1 x_2 - k_3 e_3, \end{cases} \tag{23}$$

then the error dynamics can be written as

$$\begin{cases} \frac{d^{q_1} e_1}{dt^{q_1}} = (a - k_1) e_1 - \alpha_3 x_3 e_2 - \alpha_2 x_2 e_3, \\ \frac{d^{q_2} e_2}{dt^{q_2}} = \alpha_3 x_3 e_1 + (b - k_2) e_2 + \alpha_1 x_1 e_3, \\ \frac{d^{q_3} e_3}{dt^{q_3}} = (1/3)\alpha_2 x_2 e_1 + (1/3)\alpha_1 x_1 e_2 + (c - k_3) e_3. \end{cases} \tag{24}$$

The Jacobian matrix of system (24) at the origin is

$$J = \begin{bmatrix} a - k_1 & -\alpha_3 x_3 & -\alpha_2 x_2 \\ \alpha_3 x_3 & b - k_2 & \alpha_1 x_1 \\ (1/3)\alpha_2 x_2 & (1/3)\alpha_1 x_1 & c - k_3 \end{bmatrix}. \tag{25}$$

(1) Commensurate case: $(q_1, q_2, q_3) = (0.9, 0.9, 0.9)$.

The characteristic equation of the Jacobian matrix (25) becomes

$$\begin{aligned} &\lambda^3 + (k_1 + k_2 + k_3 - a - b - c)\lambda^2 \\ &\quad + \big[(k_1 - a)(k_2 - b) + (k_1 - a)(k_3 - c) \\ &\quad + (k_2 - b)(k_3 - c) \\ &\quad - (1/3)\alpha_1^2 x_1^2 + (1/3)\alpha_2^2 x_2^2 + \alpha_3^2 x_3^2\big]\lambda \\ &\quad + (k_1 - a)(k_2 - b)(k_3 - c) \\ &\quad + (2/3)\alpha_1\alpha_2\alpha_3 x_1 x_2 x_3 - (1/3)(k_1 - a)\alpha_1^2 x_1^2 \\ &\quad + (1/3)(k_2 - b)\alpha_2^2 x_2^2 + (k_3 - c)\alpha_3^2 x_3^2 = 0. \end{aligned}$$

In this case, the scaling factors are set to $\alpha_1 = -1$, $\alpha_2 = 1$, and $\alpha_3 = -1$. Design $k_1 = k_2 = k_3 = 10$

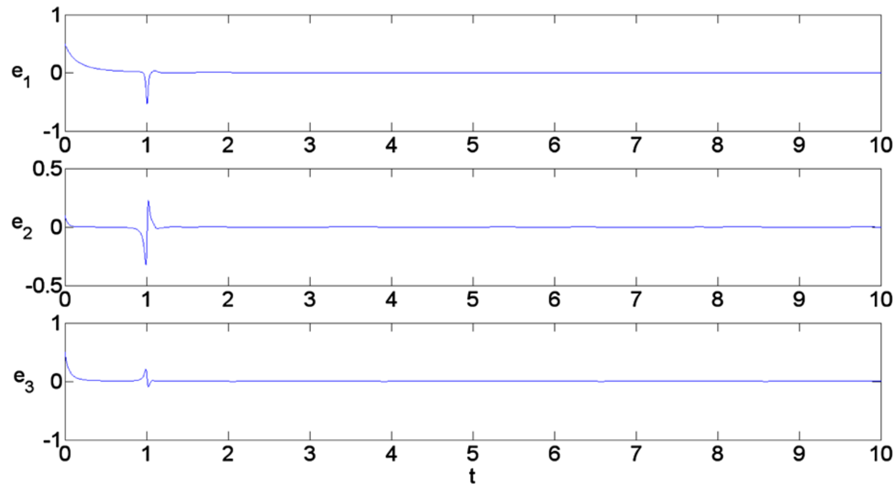

Fig. 3 Time history of hybrid projective synchronization errors for fractional-order Chen–Lee systems with $(q_1, q_2, q_3) = (0.9, 0.9, 0.9)$ and $(\alpha_1, \alpha_2, \alpha_3) = (-1, 1, -1)$

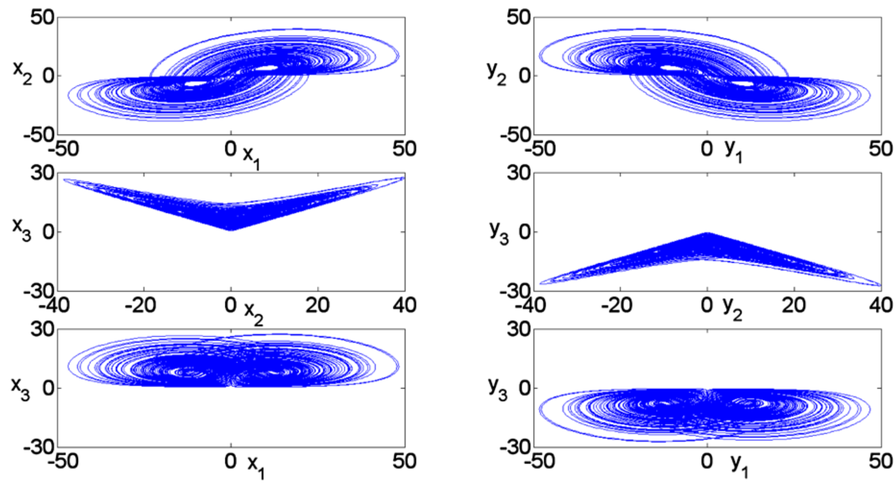

Fig. 4 Phase diagrams for master and slave fractional-order Chen–Lee systems with $(q_1, q_2, q_3) = (0.9, 0.9, 0.9)$ and $(\alpha_1, \alpha_2, \alpha_3) = (-1, 1, -1)$ when hybrid projective synchronization is achieved

and substitute the parameters a, b, c, α_1, α_2, α_3 into the above equation, it can be seen that the necessary condition (17) is satisfied:

$$0.9\pi/2 - \min_i\{|\arg(\lambda_i)|\} < 0.$$

Based on Theorem 3, the zero solution of the error dynamics is asymptotically stable. In other words, the hybrid projective synchronization between the master and slave Chen–Lee systems with commensurate fractional orders $(q_1, q_2, q_3) = (0.9, 0.9, 0.9)$ is achieved. The numerical simulations shown in Figs. 3 and 4 are coincident with this conclusion.

(2) Incommensurate case: $(q_1, q_2, q_3) = (0.95, 0.9, 0.85)$.

Equation (19) becomes

$$\begin{aligned}
&\lambda^{54} + (k_3 - c)\lambda^{37} + (k_2 - b)\lambda^{36} + (k_1 - a)\lambda^{35} \\
&\quad + \left[(k_2 - b)(k_3 - c) - \alpha_1^2 x_1^2\right]\lambda^{19} \\
&\quad + \left[(k_1 - a)(k_3 - c) - \alpha_2^2 x_2^2\right]\lambda^{18} \\
&\quad + \left[(k_1 - a)(k_2 - b) - \alpha_3^2 x_3^3\right]\lambda^{17} \\
&\quad + (2/3)\alpha_1\alpha_2\alpha_3 x_1 x_2 x_3 \\
&\quad + (k_1 - a)(k_2 - b)(k_3 - c) \\
&\quad - (1/3)(k_1 - a)\alpha_1^2 x_1^2 \\
&\quad + (1/3)(k_2 - b)\alpha_2^2 x_2^2 + (k_3 - c)\alpha_3^2 x_3^2 = 0.
\end{aligned}$$

In this case, the scaling factors are set to $\alpha_1 = -1$, $\alpha_2 = 2$, and $\alpha_3 = -0.5$. Design $k_1 = k_2 = k_3 = 10$ and substitute the parameters a, b, c, α_1, α_2, α_3

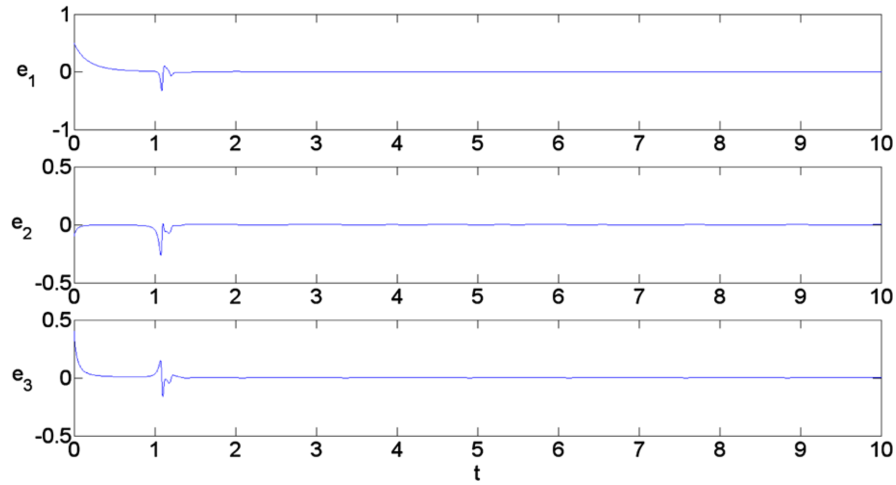

Fig. 5 Time history of hybrid projective synchronization errors for fractional-order Chen–Lee systems with $(q_1, q_2, q_3) = (0.95, 0.9, 0.85)$ and $(\alpha_1, \alpha_2, \alpha_3) = (-1, 2, -0.5)$

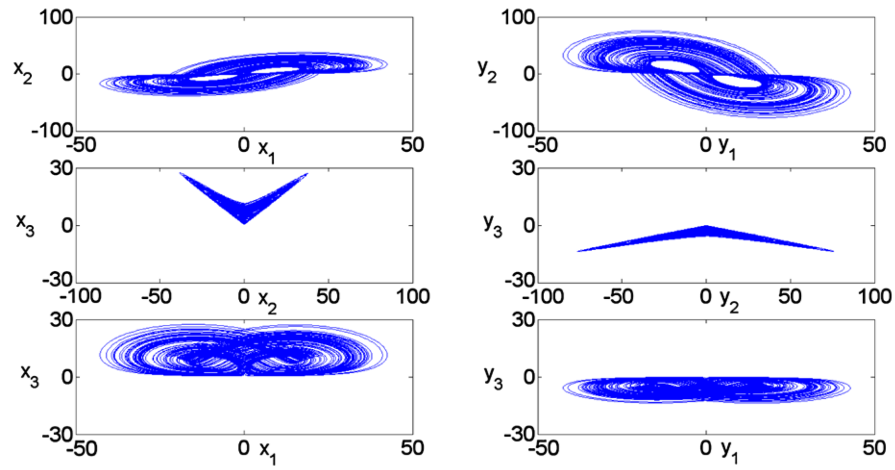

Fig. 6 Phase diagrams for master and slave fractional-order Chen–Lee systems with $(q_1, q_2, q_3) = (0.95, 0.9, 0.85)$ and $(\alpha_1, \alpha_2, \alpha_3) = (-1, 2, -0.5)$ when hybrid projective synchronization is achieved

into above equation, it can be seen that the necessary condition (18) is satisfied:

$$\pi/40 - \min_i\{|\arg(\lambda_i)|\} < 0.$$

Based on Theorem 3, the zero solution of the error dynamics is asymptotically stable. In other words, the hybrid projective synchronization between the master and slave Chen–Lee systems with incommensurate fractional orders $(q_1, q_2, q_3) = (0.95, 0.9, 0.85)$ is achieved. The numerical simulations shown in Figs. 5 and 6 are coincident with this conclusion.

6 Conclusions

In this paper, we studied chaos and hybrid projective synchronizations of fractional-order Chen–Lee systems based on the stability of fractional-order systems. Previously, the existence of chaos in fractional-order Chen–Lee systems was only verified by numerical calculations of a positive Lyapunov exponent. In addition, the theoretical study of synchronization of chaotic fractional-order systems is still incomplete in the present literature. This reason motivated us to devote ourselves to accomplishing this work. The stability theorems of fractional-order systems guarantee that chaos is exhibited in commensurate and incommensurate fractional-order Chen–Lee systems, and that hybrid projective synchronization occurs if the necessary conditions are satisfied. Numerical simulations were coincident with the theoretical conclusions.

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