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A chaotic system with Hölder continuity

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Abstract This paper presents a new chaotic system with infinitely many equilibria. The new system contains two system parameters and a nonlinear term which does not satisfy Lipschitz continuity but does satisfy $\frac{1}{2}$-Hölder continuity condition. The complicated dynamics are studied through theoretical analysis and numerical simulation. Synchronization for two identical systems by a piecewise linear feedback controller is investigated based on Lyapunov stability criteria.

Keywords Chaotic system · Hölder continuity · Synchronization · Piecewise linear feedback

1 Introduction

As an interesting nonlinear phenomenon, chaos has applications in many areas such as signal generator design, secure communication, biology, economics, many other engineering systems, and so on, and has been developed and studied thoroughly during the recent two decades [1]. The first chaotic attractor was discovered by Lorenz [2] in 1963. Based on the Lorenz system, Chen and Ueta [3] developed a new chaotic attractor called Chen's attractor from the anti-control method in 1999, which belongs to another canonical family of chaotic systems. Lü and Chen [4] proposed a new chaotic system, named Lü system, in 2002, which serves as a transition system between the Lorenz system and the Chen system. Liu et al. [5] presented a new three-dimensional chaotic system with five parameters. Qi et al. [6] reported a new chaotic system with five equilibria. We have recently [7–9] designed a class of new chaotic systems where the nonlinear terms take the form of trigonometric functions, and the chaotic attractors are similar to the multiscroll chaotic attractor from piecewise linear function series proposed in [10]. The generation of chaos has been studied with increasing interest.

Recently, chaos synchronization of the chaotic systems has also become an active research topic [11]. Generally, there are several effective strategies to achieve the control and synchronization of continuous time chaotic systems, such as OGY method [12], parametric resonance method [13], linear and nonlinear feedback method [11], adaptive feedback method [14], delay feedback method [15], sliding mode control method [16], and so on. However, these methods achieved chaos synchronization either by applying the local linearization scheme which could not produce the global analysis, or by using the nonlinear feedback

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controller which was difficult to operate in practice. In fact, it is significant to design a practical linear or piecewise linear feedback controller to globally synchronize chaotic systems.

Different from the existing chaotic systems that have been reported, e.g., the systems in [2–10], where the nonlinear terms satisfy Lipschitz continuity condition, uniform or local, and the systems have a finite number of equilibria, this paper develops a new chaotic system with two parameters and one nonlinear term which does not satisfy Lipschitz continuity but does satisfy $\frac{1}{2}$-Hölder continuity condition, and the new system has infinitely many equilibria. The complicated dynamics are analyzed by theoretical analysis and numerical simulation. Furthermore, the chaos synchronization via a piecewise linear state feedback controller for the new system is studied. It is shown that the proposed controller can synchronize the new system globally. This paper is organized as follows. Section 2 presents the system model and analyzes the complicated dynamics properties. Section 3 proposed the chaos synchronization of the new system. Finally, Sect. 4 draws the conclusions.

2 System description and dynamic properties

In this section, system model is presented and Hölder continuity is analyzed. Furthermore, the complicated dynamics are studied by theoretical analysis and numerical simulation.

2.1 System model

Consider the nonlinear system of the form

$$\begin{cases} \dot{x} = y, \\ \dot{y} = z, \\ \dot{z} = -a(x+y+z) + bf(x), \end{cases} \tag{1}$$

where the parameters $a, b \in R^+$, and the nonlinear term $f(x)$ takes the form

$$f(x) = \begin{cases} |x| \sin\frac{1}{x}, & x \neq 0, \\ 0, & x = 0. \end{cases} \tag{2}$$

It is easy to see from (2) that the nonlinear function $f(\cdot)$ is an odd function with the range $(-1, 1)$. Figure 1 shows the graph of $f(x)$.

Let the system parameters be

$$a = 0.5, \qquad b = 5. \tag{3}$$

This nonlinear system exhibits a double-scroll chaotic attractor shown in Fig. 2.

It should be pointed out that, with different parameters a and b, system (1) can evolve to other complex dynamics such as single-scroll chaotic attractor and periodic orbit. In addition, with parameters (3), the Lyapunov exponents of system (1) can be calculated as $(0.1345, 0, -0.6436)$. It is shown that there are only one positive Lyapunov exponent and one negative Lyapunov exponent, which indicates that system (1) is chaotic. Moreover, the bifurcation diagram of the state variable x in system (1) with the parameter $a \in [0.45, 1.05]$ is shown in Fig. 3, where the cross-section is selected as $\Sigma = \{(x, y, z)^T \in R^3 \mid y = 0\}$. The figure displays the abundant dynamic properties versus parameter a.

Note that $f(\cdot)$ is an odd function. It is easy to see that system (1) has a natural symmetry which persists for all values of the system parameters under the coordinate transform $(x, y, z) \to (-x, -y, -z)$.

Furthermore, for system (1), one can get

$$\nabla V = \frac{\partial \dot{x}}{\partial x} + \frac{\partial \dot{y}}{\partial y} + \frac{\partial \dot{z}}{\partial z} = -a.$$

Obviously, the dynamical system (1) is dissipative while $a > 0$, and an exponential contraction of the system is $\exp(-at)$.

In addition, it should be pointed out that the nonlinear function $f(\cdot)$ of (2) does not satisfy the Lipschitz continuity condition. And we give the following result.

Proposition 1 *Consider the nonlinear function $f(\cdot)$ of (2). For any $\alpha > \frac{1}{2}$, the ratio $|f(x_1) - f(x_2)|/|x_1 - x_2|^\alpha$ is unbounded.*

Proof Let $x_{1k} = (k\pi)^{-1}$, $x_{2k} = (k\pi + \frac{\pi}{2})^{-1}$ $(k = 1, 2, \ldots)$. Then it can be calculated that

$$\frac{|f(x_{1k}) - f(x_{2k})|}{|x_{1k} - x_{2k}|^\alpha} = 2^\alpha \pi^{\alpha-1} k^{2\alpha-1} \left(1 + \frac{1}{2k}\right)^{\alpha-1}, \tag{4}$$

which is unbounded with k increasing. □

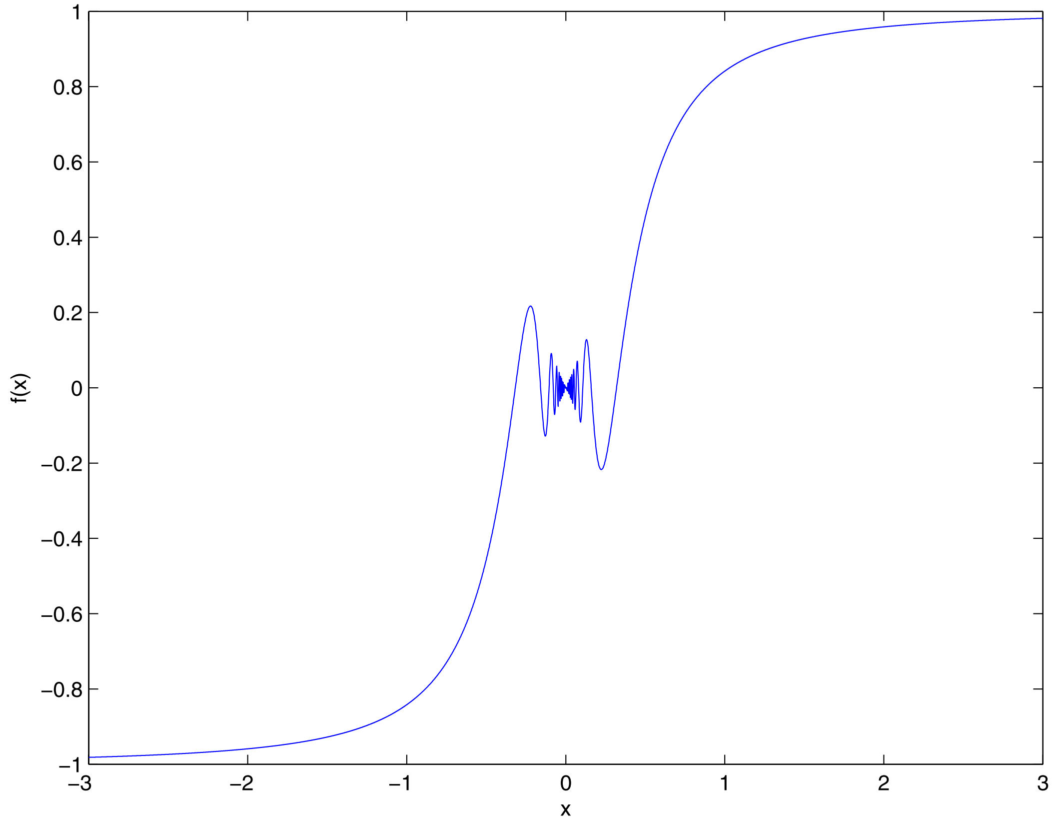

Fig. 1 Graph of the function $f(\cdot)$

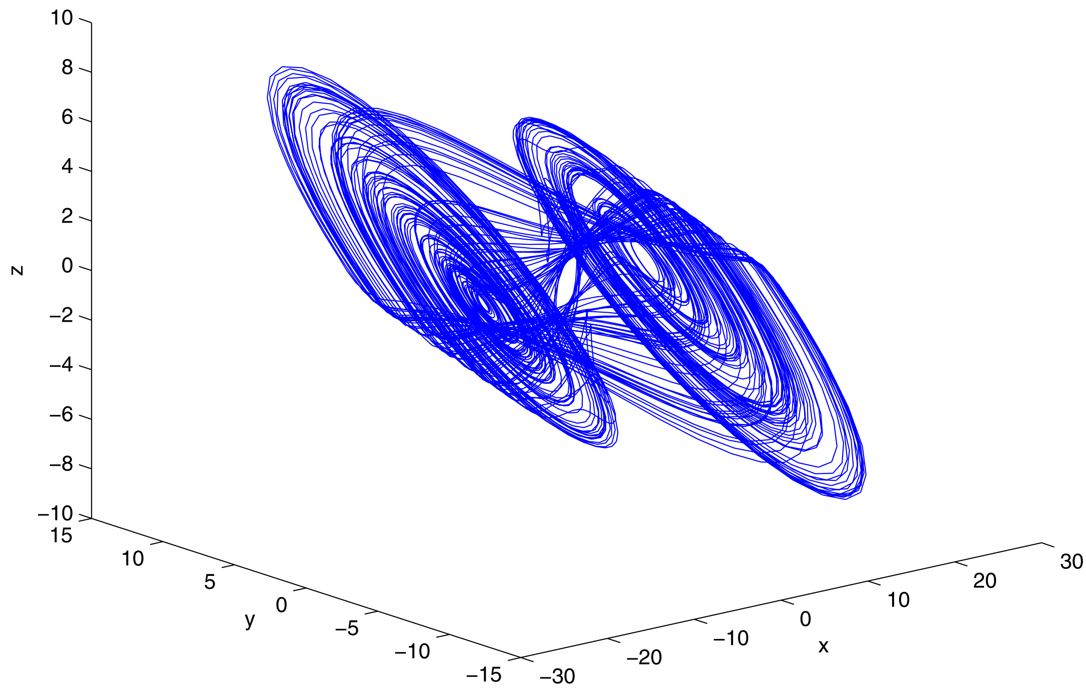

Fig. 2 Phase portraits of system (1) with $a = 0.5$, $b = 5$

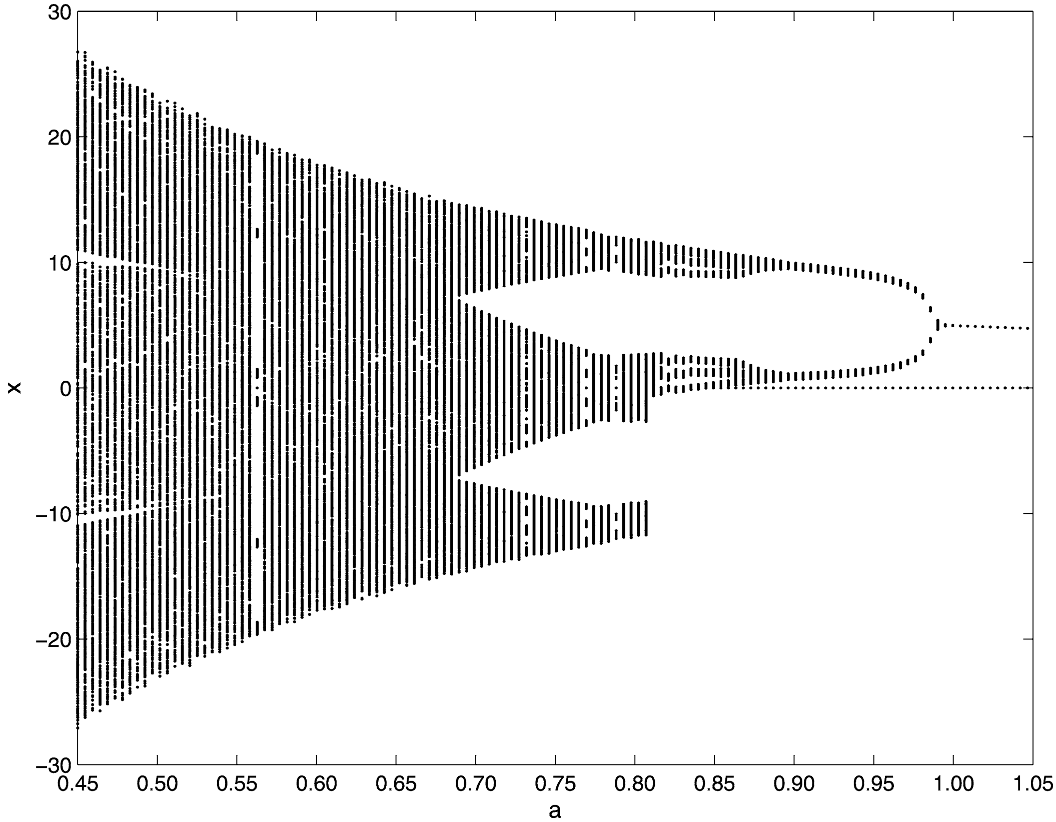

Fig. 3 Bifurcation diagram of x versus a

It is obvious from Proposition 1 that $f(\cdot)$ does not satisfy the Lipschitz continuity condition when $\alpha = 1$. Furthermore, since $\lim_{k\to\infty} x_{1k} = \lim_{k\to\infty} x_{2k} = 0$, it is shown from (4) that the derivative $f'(0)$ does not exist. For $x \neq 0$, the derivative of $f(\cdot)$ is

$$f'(x) = \begin{cases} \sin\frac{1}{x} - \frac{1}{x}\cos\frac{1}{x}, & x > 0, \\ -\sin\frac{1}{x} + \frac{1}{x}\cos\frac{1}{x}, & x < 0. \end{cases} \tag{5}$$

It is worthwhile to mention that the nonlinear function $f(\cdot)$ satisfies the $\frac{1}{2}$-Hölder continuity condition. Noticing the boundedness of a chaotic attractor, without loss of generality, one dimension of the state space of a chaotic system can be confined to a bounded closed interval $[-d, d]$ with $d \in R^+$. And we give the following result.

Proposition 2 *Consider the nonlinear function $f(\cdot)$ of (2). There exists a positive constant h such that*

$$\bigl|f(x_1) - f(x_2)\bigr| \le h|x_1 - x_2|^{1/2},$$
$$\forall x_1, x_2 \in [-d, d]. \tag{6}$$

Proof The inequality (6) comes naturally with $x_1 = x_2$. In the following, assume $x_1 < x_2$, without loss of generality. We first consider the case $d = 1$. Then it is shown from (5) that

$$\bigl|f'(x)\bigr| = \left|\sin\frac{1}{x} - \frac{1}{x}\cos\frac{1}{x}\right| \le 1 + \bigl|x^{-1}\bigr|. \tag{7}$$

Consider $0 \le x_1 < x_2 \le 1$. Note that the codomain of $f(\cdot)$ on the interval $[(x_2^{-1} + 2\pi)^{-1}, x_2]$ covers the codomain of $f(\cdot)$ on $[0, x_2]$. Denote

$$x_m = \max_{0\le x\le x_2} \bigl\{x \mid f(x) = f(x_1)\bigr\}, \tag{8}$$

it is obvious that $x_m \ge (x_2^{-1} + 2\pi)^{-1}$, that is,

$$x_m^{-1} \le x_2^{-1} + 2\pi. \tag{9}$$

Then, by using the Cauchy–Schwarz inequality, we have

$$\bigl|f(x_1) - f(x_2)\bigr|$$
$$= \bigl|f(x_m) - f(x_2)\bigr|$$

$$\begin{aligned}
&\le \int_{x_m}^{x_2} \left|f'(x)\right| \mathrm{d}x \\
&\le |x_2 - x_m|^{1/2} \left(\int_{x_m}^{x_2} \left|f'(x)\right|^2 \mathrm{d}x \right)^{1/2} \\
&\le |x_2 - x_1|^{1/2} \left(\int_{x_m}^{x_2} \left(1 + x^{-1}\right)^2 \mathrm{d}x \right)^{1/2} \\
&\le |x_2 - x_1|^{1/2} \big(x_2 - x_m + 2\ln(x_2/x_m) \\
&\quad + x_m^{-1} - x_2^{-1}\big)^{1/2} \\
&\le \big(1 + 2\ln(1+2\pi) + 2\pi\big)^{1/2} |x_1 - x_2|^{1/2}. \qquad (10)
\end{aligned}$$

Noting that $f(\cdot)$ is an odd function, for the case $-1 \le x_1 < x_2 \le 0$, one can also get the result of (10).

On the other hand, with $-1 \le x_1 < 0 < x_2 \le 1$, it is obvious that

$$\begin{aligned}
\left|f(x_1) - f(x_2)\right| &\le \left|f(x_1)\right| + \left|f(x_2)\right| \\
&\le |x_1| + |x_2| \\
&= |x_1 - x_2| \\
&\le \sqrt{2}|x_1 - x_2|^{1/2}. \qquad (11)
\end{aligned}$$

Then, combining (10) with (11), we have

$$\begin{aligned}
&\left|f(x_1) - f(x_2)\right| \le h_1 |x_1 - x_2|^{1/2}, \\
&\forall x_1, x_2 \in [-1, 1], \qquad (12)
\end{aligned}$$

where $h_1 = (1 + 2\pi + 2\ln(1+2\pi))^{1/2}$.

Next, consider the case $d > 1$. It is not hard to verify from (5) that the derivative of $f(\cdot)$ is a monotonically decreasing function when $x \ge 1/\pi$. Denote $c = 1/\pi$. Then, one can get from (5) that

$$\max_{\{x|x \ge c\}} f'(x) = f'(c) = \pi. \qquad (13)$$

Noticing that $f(\cdot)$ is an odd function and $|f(x_1) - f(x_2)| \le |x_1 - x_2|$ with $x_1 x_2 \le 0$, therefore, $\forall |x_1|$, $|x_2| \in [c, d]$, one can obtain

$$\begin{aligned}
\left|f(x_1) - f(x_2)\right| &\le \max_{\{x|x \in [c,d]\}} f'(x)|x_1 - x_2| \\
&\le \pi\sqrt{d}|x_1 - x_2|^{1/2}. \qquad (14)
\end{aligned}$$

In addition, it can be calculated that $f(x) > -1/4$ with $x \in [0, 1]$. Then, $\forall |x_1| \in [0, c]$, $|x_2| \in [1, d]$, it is shown

$$\begin{aligned}
&\left|f(x_1) - f(x_2)\right| \\
&\quad \le \frac{5}{4}/(1-c)^{1/2}|x_1 - x_2|^{1/2} \\
&\quad \le \frac{5}{4}\big(\pi/(\pi - 1)\big)^{1/2}|x_1 - x_2|^{1/2}. \qquad (15)
\end{aligned}$$

Then, considering (12), (14) and (15), one can get the result of (6) where the parameter h takes the value

$$h = \max\left\{\big(1 + 2\pi + 2\ln(1+2\pi)\big)^{1/2}, \pi\sqrt{d}\right\}. \qquad (16)$$

And the proof is completed. □

Remark 1 Lipschitz continuity condition for the nonlinear term in chaotic systems, uniform or local, is crucial to achieve the chaos synchronization. It is shown from Propositions 1 and 2 that the nonlinear term $f(x)$ of (2) does not satisfy Lipschitz continuity but does satisfy $\frac{1}{2}$-Hölder continuity condition, and is topologically not equivalent to the quadratic or cross product terms of the chaotic systems reported in [2–6], the trigonometric functions reported in [7–9] and the piecewise linear function series presented in [10], which all satisfy the Lipschitz continuity condition. Then system (1) is a novel one in essence, and the synchronization for this special system is proposed based on the Lyapunov stability criteria in the next section.

2.2 Equilibria

The equilibria of system (1) can be obtained by solving the following algebraic equations simultaneously

$$y = 0, \qquad z = 0, \qquad -ax + bf(x) = 0. \qquad (17)$$

With the parameters $a \le b$, we solve the third equation of (17) and get the analytical solutions as

$$\begin{aligned}
&x = 0, \qquad \pm 1/\big(2k\pi + \arcsin(a/b)\big), \\
&\quad k = 0, 1, 2, \ldots. \qquad (18)
\end{aligned}$$

Thus, system (1) has infinitely many equilibria

$$\begin{aligned}
&S_0(0,0,0), \quad S_1(x_0,0,0), \quad S_2(-x_0,0,0), \quad \ldots, \\
&\quad S_{2k+1}(x_k,0,0), \quad S_{2k+2}(-x_k,0,0), \quad \ldots \qquad (19)
\end{aligned}$$

where $x_k = 1/(2k\pi + \arcsin(a/b)), k = 0, 1, 2, \ldots$. And it is obvious that $\lim_{k\to\infty} S_k = S_0$.

By linearizing system (1) at S, one obtains the Jacobian matrix as follows

$$J_s = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a + bf'(x_s) & -a & -a \end{bmatrix}, \tag{20}$$

where x_s corresponds to the first element of the equilibrium S.

In view of (19) and parameters (3), the equilibria of system (1) are computed as

$$S_0(0,0,0), \quad \ldots, \quad S_{2k+1}\big(1/(2k\pi + 0.1002), 0, 0\big), \\ S_{2k+2}\big(-1/(2k\pi + 0.1002), 0, 0\big), \quad \ldots$$

with $k = 0, 1, 2, \ldots$.

For the nonzero equilibria S_{2k+1}, S_{2k+2}, $k = 0, 1, 2, \ldots$, which correspond to $x_s = \pm 1/(2k\pi + 0.1002)$, respectively, the characteristic values of the Jacobian matrix J_s can be obtained from $|\lambda I - J_s| = 0$. For example, for the nonzero equilibria S_1 and S_2 with $k = 0$, which correspond to $x_s = \pm 9.9833$, respectively, three characteristic values are computed as follows

$$\lambda_1 = -0.7378, \qquad \lambda_2 = 0.1189 + j0.8132, \\ \lambda_3 = 0.1189 - j0.8132,$$

where λ_1 is a negative real number, λ_2 and λ_3 become a pair of complex conjugate characteristic values with positive real parts. The equilibria S_1 and S_2 are saddle-focus points which are unstable. For the other $S_{2k+1}, k \geq 1$, the characteristic values of J_s can also be computed, and these equilibria are also saddle-focus points. In addition, note that the derivative $f'(x_s)$ does not exist with $x_s = 0$, and the characteristic values of J_s for the zero equilibrium S_0 cannot be computed directly. However, by virtue of the fact that $\lim_{k\to\infty} S_k = S_0$, we conclude that the zero equilibrium S_0 is unstable.

3 Chaos synchronization of the new system

By applying the simple linear state feedback controllers, we have recently [9] studied the chaos synchronization for a class of new chaotic systems where the nonlinear terms are needed to satisfy the Lipschitz continuity condition. Unfortunately, noting from Proposition 1 that the nonlinear term $f(x)$ of (2) does not satisfy Lipschitz continuity condition, so it is very hard to find a linear feedback controller for achieving the chaos synchronization, perhaps thus a controller does not exist at all. On the other hand, it is shown from Proposition 2 that the nonlinear term $f(x)$ satisfies the $\frac{1}{2}$-Hölder continuity condition. Based on this consideration, we will design a piecewise linear feedback controller to achieve chaos synchronization between two identical chaotic systems in this section. Assume the drive system is

$$\begin{cases} \dot{x}_1 = y_1, \\ \dot{y}_1 = z_1, \\ \dot{z}_1 = -a(x_1 + y_1 + z_1) + bf(x_1), \end{cases} \tag{21}$$

and the response system is

$$\begin{cases} \dot{x}_2 = y_2 + u_1, \\ \dot{y}_2 = z_2 + u_2, \\ \dot{z}_2 = -a(x_2 + y_2 + z_2) + bf(x_2) + u_3, \end{cases} \tag{22}$$

where u_i, $i = 1, 2, 3$, are the active control laws to be designed.

Then the error system between the drive system (21) and the response system (22) is

$$\begin{cases} \dot{e}_1 = e_2 + u_1, \\ \dot{e}_2 = e_3 + u_2, \\ \dot{e}_3 = -a(e_1 + e_2 + e_3) \\ \qquad + b(f(x_2) - f(x_1)) + u_3, \end{cases} \tag{23}$$

where $e_1 = x_2 - x_1$, $e_2 = y_2 - y_1$ and $e_3 = z_2 - z_1$.

The active control laws take the form of piecewise linear feedback as

$$\begin{cases} u_1 = \begin{cases} -k_1 e_1 - \frac{bh^2}{2}, & e_1 > 0, \\ 0, & e_1 = 0, \\ -k_1 e_1 + \frac{bh^2}{2}, & e_1 < 0, \end{cases} \\ u_2 = -k_2 e_2, \\ u_3 = a(e_1 + e_2 + e_3) - k_3 e_3. \end{cases} \tag{24}$$

Then the error system (23) can be described as

$$\begin{cases} \dot{e}_1 = e_2 - k_1 e_1 - \operatorname{sgn}(e_1)\frac{bh^2}{2}, \\ \dot{e}_2 = e_3 - k_2 e_2, \\ \dot{e}_3 = b(f(x_2) - f(x_1)) - k_3 e_3, \end{cases} \tag{25}$$

where $\operatorname{sgn}(\cdot)$ denotes the sign function.

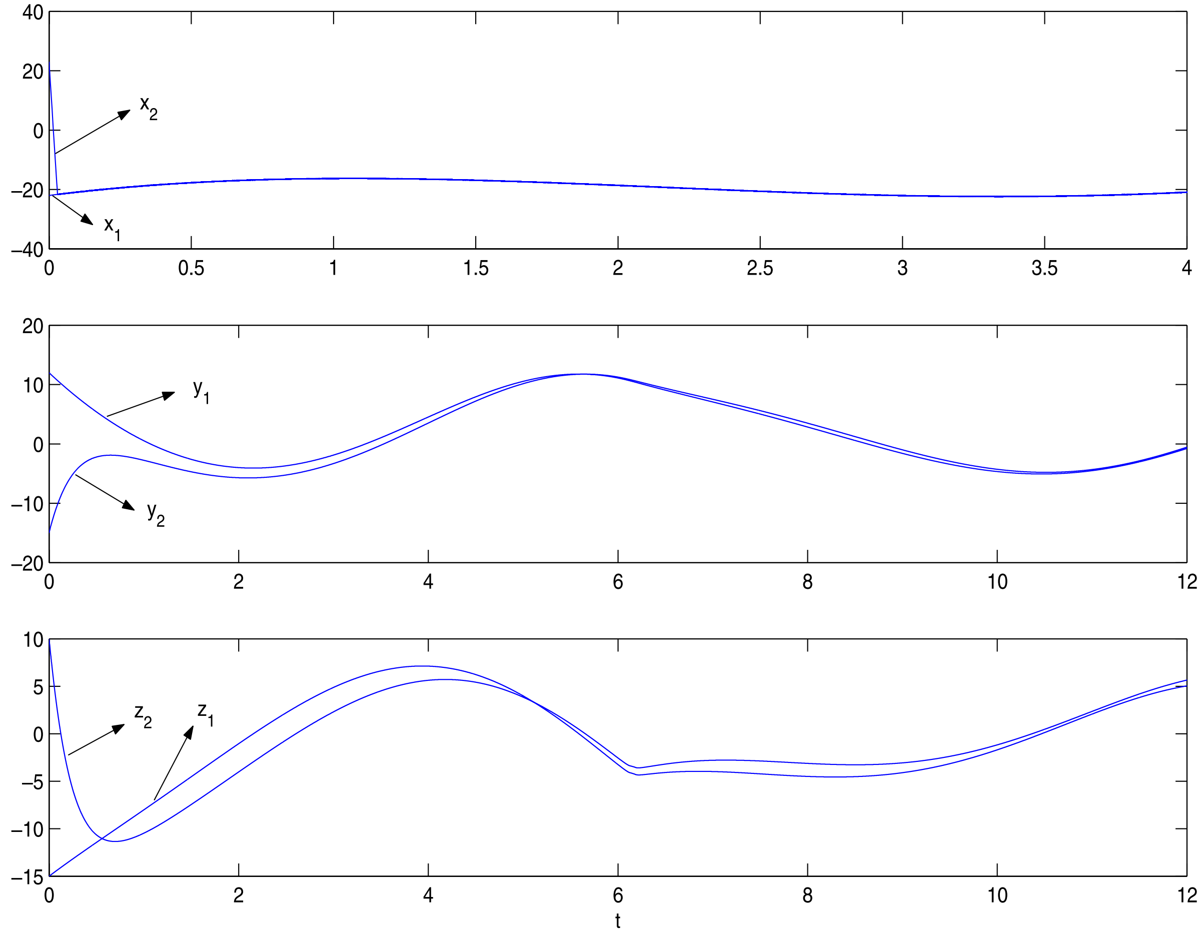

Fig. 4 Synchronization of systems (21) and (22)

Actually, by designing the suitable gains k_i, $i = 1, 2, 3$, the states of the error system (25) will be globally asymptotically stable at the origin. And we have the following result.

Theorem 1 *The error system* (25) *is globally asymptotically stable if the gains satisfy*

$$k_1 > \frac{1}{2}, \quad k_2 > 1, \quad k_3 > \frac{1+b}{2}. \tag{26}$$

That is, the trajectories of the chaotic systems (21) *and* (22) *can be globally synchronized with different initial conditions* $(x_1(0), y_1(0), z_1(0))^T$ *and* $(x_2(0), y_2(0), z_2(0))^T$.

Proof Denote $\boldsymbol{e} = (e_1, e_2, e_3)^T$. For the error system (25), consider the Lyapunov function as

$$V(\boldsymbol{e}) = \frac{1}{2}\boldsymbol{e}^T\boldsymbol{e}. \tag{27}$$

It is shown from Proposition 2 that $|f(x_1) - f(x_2)|^2 \leq h^2|e_1|$. Then, the time derivative of the Lyapunov function (27) along the trajectory of the error system (25) is described as

$$\begin{aligned}\dot{V}(\boldsymbol{e}) &= \boldsymbol{e}^T\dot{\boldsymbol{e}}\\ &= e_1e_2 - k_1e_1^2 - \operatorname{sgn}(e_1)\frac{bh^2}{2}e_1 + e_2e_3 - k_2e_2^2\\ &\quad - k_3e_3^2 + b\big(f(x_2) - f(x_1)\big)e_3\\ &\leq \frac{1}{2}\big(e_1^2 + e_2^2\big) - k_1e_1^2 - \frac{bh^2}{2}|e_1|\\ &\quad + \frac{1}{2}\big(e_2^2 + e_3^2\big) - k_2e_2^2 - k_3e_3^2\\ &\quad + \frac{b}{2}\big(\big|f(x_2) - f(x_1)\big|^2 + e_3^2\big)\end{aligned}$$

$$\leq -\left(k_1 - \frac{1}{2}\right)e_1^2 - (k_2 - 1)e_2^2 - \left(k_3 - \frac{1}{2}\right)e_3^2 - \frac{bh^2}{2}|e_1| + \frac{b}{2}(h^2|e_1| + e_3^2) = -\left(k_1 - \frac{1}{2}\right)e_1^2 - (k_2 - 1)e_2^2 - \left(k_3 - \frac{1+b}{2}\right)e_3^2. \quad (28)$$

Noticing (26), we can get $\dot{V}(\boldsymbol{e}) < 0$. Then, according to the Lyapunov stability criteria, the error system (25) is globally asymptotically stable. And the proof is completed. □

Remark 2 It is shown from Theorem 1 that systems (21) and (22) can be synchronized globally by designing a suitable piecewise linear feedback controller (strictly, a piecewise affine feedback controller).

Consider parameters (3) and choose $k_1 = 1$, $k_2 = 2$, $k_3 = 4$ satisfying (26). The drive system (21) and the response system (22) with the initial states $(-22, 12, -15)^T$ and $(23, -15, 10)^T$, respectively, are synchronized via the piecewise linear feedback controller (24) where the parameter $h = 5\pi$ computed from (16) with $d = 25$; see Fig. 4.

Remark 3 It should be mentioned that, although only the chaos synchronization is considered in this paper, it is not hard to consider the chaos control for the new system (1), which can be achieved via a simple linear state feedback controller by noticing the fact that $|f(x)| \leq |x|, \forall x \in R$, and applying the chaos control method provided in [9].

4 Conclusion

This paper introduces a special chaotic system with the nonlinear term satisfying Hölder continuity condition. The complex dynamic properties are investigated by theoretical analysis and numerical simulation. Moreover, the chaos synchronization of the new system via a piecewise linear feedback controller is investigated based on Lyapunov stability criteria where the controller can globally synchronize the new chaotic system.

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