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Nonzero mean PDF solution of nonlinear oscillators under external Gaussian white noise

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Abstract This paper is concerned with the nonzero mean stationary probability density function (PDF) solution for nonlinear oscillators under external Gaussian white noise. The PDF solution is governed by the well-known Fokker–Planck–Kolmogorov (FPK) equation and this equation is numerically solved by the exponential-polynomial closure (EPC) method. Different types of oscillators are further investigated in the case of nonzero mean response. Either weak or strong nonlinearity is considered to show the effectiveness of the EPC method. When the polynomial order equals 2, the results of the EPC method are identical with those given by equivalent linearization (EQL) method. These results obtained with the EQL method differ significantly from exact solution or simulated results. When the polynomial order is 4 or 6, the PDFs obtained with the EPC method present a good agreement with the exact solution or simulated results, especially in the tail regions. The numerical analysis also shows that the nonzero mean PDF of the response is nonsymmetrically distributed about its mean unlike the case of the zero mean PDF reported in the references.

Keywords Nonlinear oscillator · Nonzero mean · Probability density function · Gaussian white noise · Stochastic processes

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1 Introduction

The study of nonlinear stochastic dynamics has received more and more attention in the last few decades because it is widely applicable in the fields of science and engineering. In this research field, the excitation acting on systems is treated as white noise. Furthermore, most excitations are assumed to have zero mean, or the mean is preliminarily subtracted out so that the computation procedure can be significantly simplified. However, some physical excitations have nonzero mean in nature, e.g., wind gusts or extreme waves [1]. In addition, the response of the systems with some types of nonlinearity may have nonzero mean even under zero mean excitation [2].

Subjected to Gaussian white noise, the probability density function (PDF) solution of the response of nonlinear oscillators is governed by the well-known Fokker–Planck–Kolmogorov (FPK) equation [3]. Only a few exact stationary solutions were obtained under highly restricted conditions [4–6]. Therefore, some methods were developed for the approximate solutions to the FPK equation. Maximum entropy principle was employed to determine the approximate PDF solution of nonlinear oscillators, with which complicated nonlinear algebraic equations need to be formulated and solved [7]. Energy dissipation balancing method is another approximate approach [8]. In this method, the original nonlinear oscillator is approximately replaced by another oscillator which belongs to the class of generalized stationary potential and has obtainable exact solution [9]. This replacement criterion is based on the assumption that the average energy dissipation remains unchanged. Stochastic averaging method is also proposed for the PDF solution of response amplitude of nonlinear oscillators [10, 11]. It is suitable for the lightly damped oscillators excited by weak broad-band excitation. Perturbation method is also adopted for analyzing the weakly nonlinear oscillators [12]. Recently, the review on numerical solution methods was made, including weighted residuals method, finite element method, path integration (cell mapping) method, finite difference method, and variational method (eigenfunction) expansion [13]. Much computational effort is needed with these methods and negative values of PDF solutions may be obtained in the tail regions. Monte Carlo simulation is a versatile approach for the digital analysis of the PDF solutions of nonlinear stochastic oscillators [14]. Besides the PDF solutions, equivalent linearization (EQL) method is widely adopted for the evaluation on second moments [15]. When the system is excited by external Gaussian white noise, the EQL method is also equivalent to Gaussian closure method [16]. These two methods are inadequate when high nonlinearity or parametric excitation exists in nonlinear oscillators.

As described above, although much effort has been made on the PDF solutions of nonlinear oscillators, the available methods are still restricted to some limitations, respectively. In particular, the nonzero mean PDF solutions as well as the tail behavior of the PDF solutions are scarcely mentioned in existing references. In this paper, the nonzero mean PDF solution is tentatively obtained with the exponential-polynomial closure (EPC) method by solving the well-known Fokker–Planck–Kolmogorov (FPK) equation [17, 18]. In order to evaluate the effectiveness of the proposed method, different types of oscillators is further investigated in the case of nonzero mean response. Either weak or strong nonlinearity is considered to show the effectiveness of the EPC method. When the polynomial order equals 2, the results of the EPC method are identical with those given by equivalent linearization (EQL) method. These EQL results differ significantly from exact solution or simulated results. When the polynomial order is 4 or 6, the PDFs obtained with the EPC method present good agreement with the exact solution or simulated results, especially in the tail regions. The numerical analysis also shows that the nonzero mean PDF of the response is nonsymmetrically distributed about its mean unlike the case of the zero mean PDF reported in previous references.

2 EPC solution procedure

A nonlinear oscillator excited by external excitation can be represented as

$$\ddot{X} + h\big(X, \dot{X}\big) = W(t), \tag{1}$$

where X and $\dot{X}$ are the responses of the oscillator (e.g., displacement, velocity) and $W(t)$ is external excitation. Under random excitation, X and $\dot{X}$ can be modeled as stochastic processes; $h(\cdot)$ is a function of X and $\dot{X}$, and the functional form is assumed to be deterministic; $W(t)$ is zero-mean Gaussian white noise which autocorrelation is

$$E\big[W(t)W(t+\tau)\big] = 2\pi K\delta(\tau), \tag{2}$$

where $E[\cdot]$ denotes the expectation of $(\cdot)$; $\delta(t)$ is Dirac delta function; K is a constant.

Setting $X = x_1$ and $\dot{X} = x_2$, (1) can be expressed as

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = -h(x_1, x_2) + W(t). \end{cases} \tag{3}$$

The response vector $\{x_1, x_2\}^T$ is Markovian and the PDFs of responses are governed by the following FPK equation

$$\frac{\partial p}{\partial t} = -x_2\frac{\partial p}{\partial x_1} + \frac{\partial}{\partial x_2}\big\{h(x_1, x_2)p\big\} + \frac{1}{2}\cdot 2\pi K\frac{\partial^2 p}{\partial x_2^2}. \tag{4}$$

In this paper, the stationary PDF solution is considered and the term on the left-hand side of (4) equals zero. Therefore, (4) is reduced to

$$-x_2\frac{\partial p}{\partial x_1}+\frac{\partial}{\partial x_2}\{h(x_1,x_2)p\}+\pi K\frac{\partial^2 p}{\partial x_2^2}=0. \quad (5)$$

In general, the following requirements are fulfilled by the PDF $p(x_1,x_2)$ of the stationary response of the nonlinear oscillator

$$\begin{cases} p(x_1,x_2)\geq 0, & x_1,x_2\in\Re^2,\\ \lim\limits_{x_i\to\pm\infty} p(x_1,x_2)=0, & i=1,2,\\ \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} p(x_1,x_2)\,dx_1\,dx_2=1, \end{cases} \quad (6)$$

where $\Re$ denotes real space. The exact PDF solution of (5) is usually not obtainable. To fulfill the conditions (6), an approximate PDF is used and expressed as an exponential-polynomial function of state variables. The approximate PDF solution $\tilde{p}(x_1,x_2;\mathbf{a})$ to (5) is assumed to be

$$\tilde{p}(x_1,x_2;\mathbf{a})=ce^{Q_n(x_1,x_2;\mathbf{a})}, \quad (7)$$

where c is a normalization constant and $\mathbf{a}$ is an unknown parameter vector containing N_p entries. The polynomial $Q_n(x_1,x_2;\mathbf{a})$ is expressed as

$$Q_n(x_1,x_2;\mathbf{a})=\sum_{i=1}^{n}\sum_{j=0}^{i}a_{ij}x_1^{i-j}x_2^{j}, \quad (8)$$

which is a n degree polynomial in x_1 and x_2. To fulfill the requirements (6), it is also required that

$$Q_n(x_1,x_2;\mathbf{a})=-\infty, \quad x_1,x_2\notin\Omega, \quad (9)$$

where $\Omega=[m_1-c_1\sigma_1,m_1+d_1\sigma_1]\times[m_2-c_2\sigma_2,m_2+d_2\sigma_2]\in\Re^2$ in which m_i, σ_i denote mean values and standard deviations of x_i $(i=1,2)$, respectively. $c_i>0$ and $d_i>0$ are constants; $m_i-c_i\sigma_i$ and $m_i+d_i\sigma_i$ locate in the tails of the PDFs of x_i. This means that the approximate PDF is assumed to vanish beyond these boundaries.

Generally the FPK equation (5) cannot be satisfied exactly with $\tilde{p}(x_1,x_2;\mathbf{a})$ because $\tilde{p}(x_1,x_2;\mathbf{a})$ is only an approximation of $p(x_1,x_2)$ and the number of unknown parameters N_p is always limited in practice. Substituting $\tilde{p}(x_1,x_2;\mathbf{a})$ for $p(x_1,x_2)$ leads to the following residual error:

$$\begin{aligned}\Delta(x_1,x_2;\mathbf{a})\\ &=-x_2\frac{\partial\tilde{p}}{\partial x_1}+\frac{\partial}{\partial x_2}\{h(x_1,x_2)\tilde{p}\}+\pi K\frac{\partial^2\tilde{p}}{\partial x_2^2}\\ &=\left\{-x_2\frac{\partial Q_n}{\partial x_1}+\frac{\partial h(x_1,x_2)}{\partial x_2}+h(x_1,x_2)\frac{\partial Q_n}{\partial x_2}\right.\\ &\quad\left.+\pi K\left[\left(\frac{\partial Q_n}{\partial x_2}\right)^2+\frac{\partial^2 Q_n}{\partial x_2^2}\right]\right\}\tilde{p}(x_1,x_2;\mathbf{a})\\ &=F(x_1,x_2;\mathbf{a})\tilde{p}(x_1,x_2;\mathbf{a}). \end{aligned} \quad (10)$$

Because $\tilde{p}(x_1,x_2;\mathbf{a})\neq 0$, the only possibility for $\tilde{p}(x_1,x_2;\mathbf{a})$ to satisfy (5) is $F(x_1,x_2;\mathbf{a})=0$. However, $F(x_1,x_2;\mathbf{a})\neq 0$ in general because $\tilde{p}(x_1,x_2;\mathbf{a})$ is only an approximation of $p(x_1,x_2)$. In this case, another set of mutually independent functions $H_s(x_1,x_2)$ that span space $\Re^{N_p}$ are introduced to make the projection of $F(x_1,x_2;\mathbf{a})$ on $\Re^{N_p}$ vanish, which leads to

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}F(x_1,x_2;\mathbf{a})H_s(x_1,x_2)\,dx_1\,dx_2=0. \quad (11)$$

This means that the reduced FPK equation (5) is fulfilled with $\tilde{p}(x_1,x_2;\mathbf{a})$ in the weak sense of integration if $F(x_1,x_2;\mathbf{a})H_s(x_1,x_2)$ is integrable in $\Re^{N_p}$. Selecting $H_s(x_1,x_2)$ as

$$H_s(x_1,x_2)=x_1^{k-l}x_2^{l}f_1(x_1)f_2(x_2), \quad (12)$$

where $k=1,2,\dots,n$; $l=0,1,2,\dots,k$ and $s=\frac{1}{2}(k+2)(k-1)+l+1$; N_p nonlinear algebraic equations in terms of N_p unknown parameters can be formulated. The algebraic equations can be solved with any available method to determine the parameters. Numerical experience shows that a convenient and effective choice for $f_1(x_1)$ and $f_2(x_2)$ is the PDFs obtained with the EQL method or Gaussian closure method as follows:

$$f_1(x_1)=\frac{1}{\sqrt{2\pi}\sigma_1}\exp\left\{-\frac{(x_1-m_1)^2}{2\sigma_1^2}\right\}, \quad (13)$$

$$f_2(x_2)=\frac{1}{\sqrt{2\pi}\sigma_2}\exp\left\{-\frac{(x_2-m_2)^2}{2\sigma_2^2}\right\}. \quad (14)$$

Because of the particular choice for $f_1(x_1)$ and $f_2(x_2)$, the integration in (11) can be easily evaluated in closed forms by the following equation due to the formulation of Gaussian PDF:

$$\int_{-\infty}^{+\infty} x^n \cdot \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\} dx = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} (2k-1)!!\mu^{n-2k}\sigma^{2k}, \tag{15}$$

where x is a variable; μ is mean value; σ is standard deviation; the binomial coefficient is given below

$$\binom{n}{k} = \frac{n!}{(n-k)!k!}, \tag{16}$$

where n and k are nonnegative integers; $\lfloor n \rfloor$ is the floor function.

3 Analysis on nonlinear oscillators

The zero mean response of different nonlinear oscillators has been widely investigated with the EPC method [17, 18]. In this paper, The PDF of the nonzero mean response is examined using different types of oscillators under external Gaussian white noise. Either exact available solution or Monte Carlo simulations are also presented to evaluate the effectiveness of the EPC method in this case.

3.1 Example 1

A Duffing oscillator with nonzero mean response is considered first in the following:

$$\ddot{X} + 2\zeta\omega_0\dot{X} + \omega_0^2(X + \varepsilon X^3) + m = W(t). \tag{17}$$

In this case, $h(X, \dot{X}) = 2\zeta\omega_0\dot{X} + \omega_0^2(X + \varepsilon X^3) + m$; $W(t)$ is Gaussian white noise with zero mean; and m is a constant. The autocorrelation of $W(t)$ is defined by (2). The exact stationary PDF solution is obtainable to be

$$p = C\exp\left\{-\frac{\zeta\omega_0}{\pi K}\left(\dot{x}^2 + \omega_0^2 x^2 + \frac{1}{2}\varepsilon\omega_0^2 x^4 + 2mx\right)\right\}, \tag{18}$$

where $x = X$ and $\dot{x} = \dot{X}$. Equation (18) presents a nonzero mean PDF solution of displacement in this case. The values of the system parameters are: $\zeta = 0.05$, $\omega_0 = 1.0$, $2\pi K = 1.0$, $\varepsilon = 0.1$, and $m = 1$.

The PDFs of displacement are obtained with the EPC method ($n = 2$ and $n = 4$) are shown in Fig. 1(a) and the exact PDF solution of displacement is also presented. The numerical results show that the EPC method ($n = 2$) is equivalent to the EQL method. This observation is also found in the following two cases. It is seen that the result given by the EQL method differs significantly from the exact solution. This difference is more pronounced in the tail region of the PDF solution as shown in Fig. 1(b) about the logarithmic PDFs of displacement. When n increases to 4, the result obtained with the EPC method coincides with the exact solution, particularly in the tail region of the PDF solutions.

As expected, the PDF solution of displacement has nonzero mean and the PDF distribution is not symmetric with respect to its mean. Comparatively, the PDF solutions of velocity given by each method is represented in Figs. 1(c) and (d). The results obtained with both EPC ($n = 2$) and EPC ($n = 4$) coincide with the exact PDF solution of velocity. The PDF solution of velocity has zero mean and the PDF distribution is symmetric about its zero mean. According to (18), the PDF solution of velocity is Gaussian PDF. The symmetry and zero-mean property of the PDF of velocity are not influenced by the addition of the constant m.

3.2 Example 2

Consider the following system with nonlinear terms in both displacement and velocity:

$$\ddot{X} + 2\zeta\omega_0(\dot{X} + \varepsilon_1\dot{X}^3) + \omega_0^2(X + \varepsilon_2 X^3) + m = W(t). \tag{19}$$

The system parameters are set as $\zeta = 0.05$, $\omega_0 = 1.0$, $2\pi K = 1.0$, $\varepsilon_1 = 0.1$, $\varepsilon_2 = 1.0$, and $m = -1$. In this case, the exact solution is unknown and Monte Carlo simulation is conducted to verify the results obtained with the EPC method in this case. A sample size of 2×10^7 is adopted to get the simulated PDF values. The PDFs and logarithmic PDFs of displacement and velocity are shown and compared in Figs. 2(a)–(d), respectively. The results from the EPC method with $n = 6$ agree well with the simulated results for both displacement and velocity. The PDFs of displacement given by the EQL method (i.e., EPC $n = 2$) deviate much from the simulated results (MCS). The nonsymmetric distribution and nonzero mean of displacement

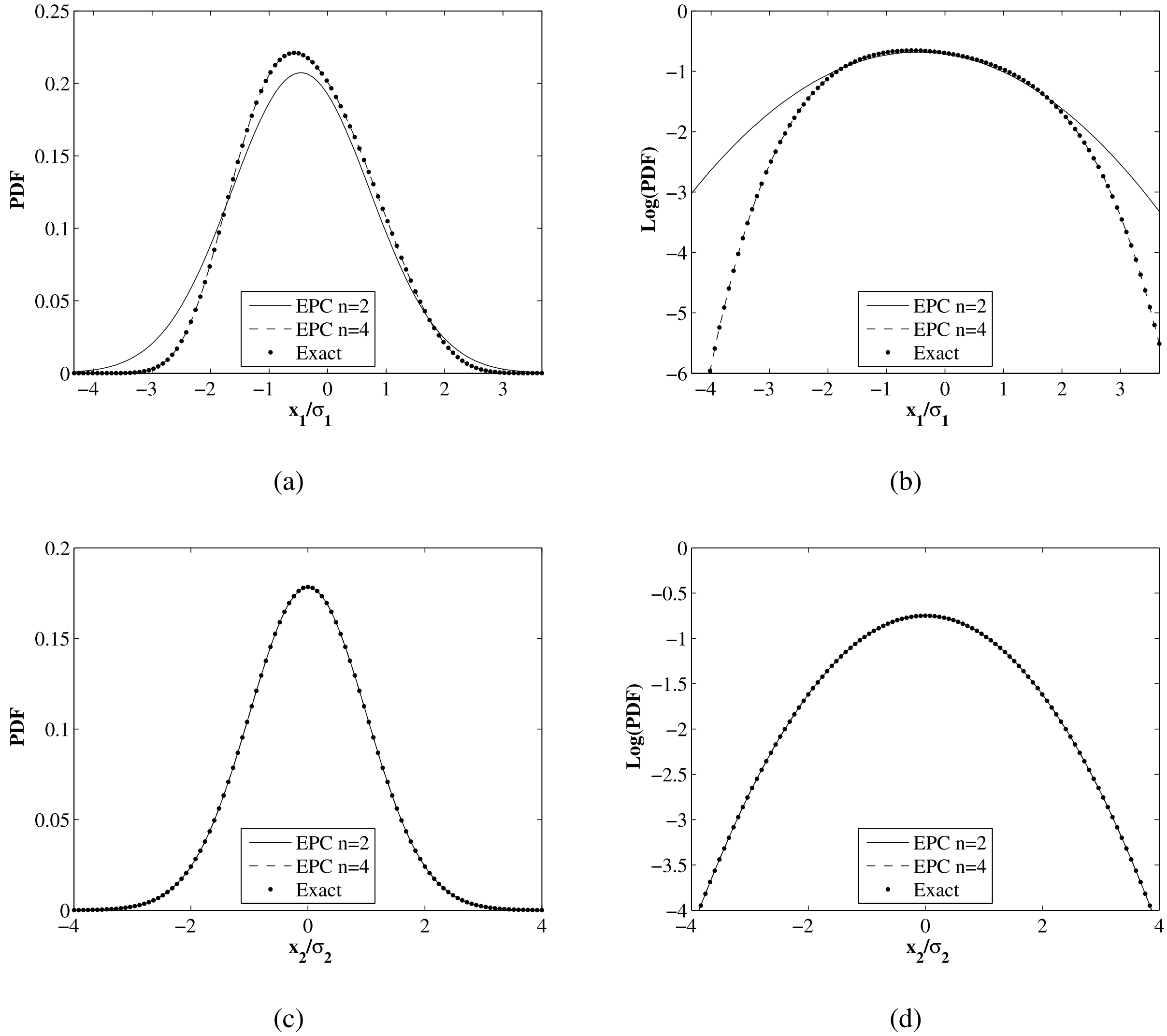

Fig. 1 Comparison of PDFs in Example 1: (**a**) PDFs of displacement; (**b**) Logarithmic PDFs of displacement; (**c**) PDFs of velocity; (**d**) Logarithmic PDFs of velocity

are also observed in this case. The PDF of velocity is not Gaussian but it is still symmetric about its zero mean. In this case, the solution procedure converges when σ_2 is replaced by $0.85\sigma_{x_2}$ in formulating the nonlinear algebraic equations and searching the solution.

3.3 Example 3

Consider the following system with complex nonlinearity

$$\ddot{X} + 2\zeta\omega_0\dot{X} + \varepsilon X^2\dot{X} + \omega_0^2 X + m = W(t). \tag{20}$$

The values of the system parameters are: $\zeta = 0.05$, $\omega_0 = 2.0$, $2\pi K = 1.0$, $\varepsilon = 0.2$, $m = -6$.

Monte Carlo simulation is also conducted in this case because no exact solution is obtainable. The PDFs and logarithmic PDFs of displacement and velocity obtained with the EPC method ($n = 2$ and $n = 4$) and the simulated solutions are shown in Figs. 3(a)–(b), respectively. In this case, the results from the EPC method with $n = 4$ agree well with the simulated results for both displacement and velocity. Although the PDFs of displacement and velocity is close to Gaussian as shown in Figs. 3(a) and (c), the non-

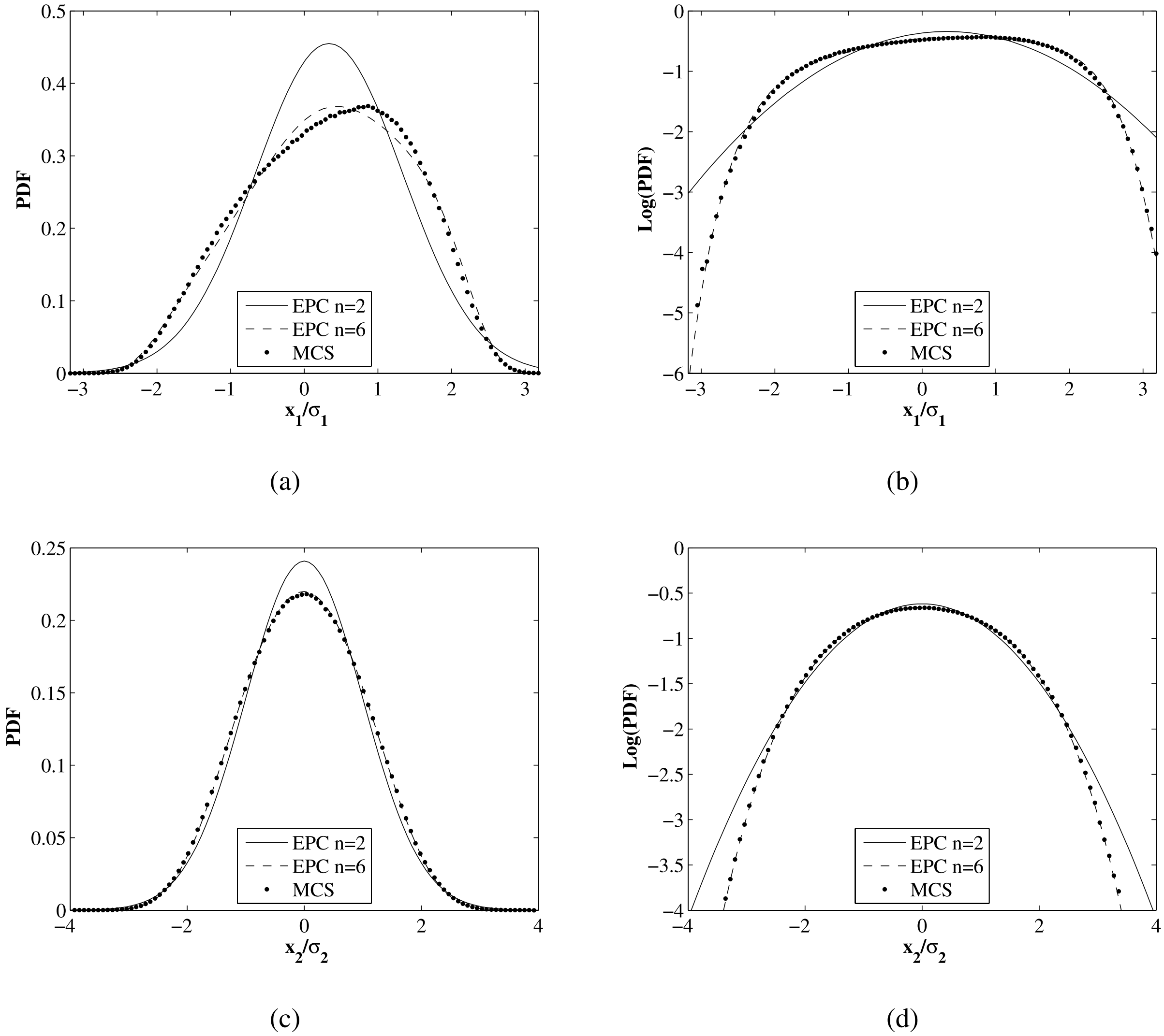

Fig. 2 Comparison of PDFs in Example 2: (**a**) PDFs of displacement; (**b**) Logarithmic PDFs of displacement; (**c**) PDFs of velocity; (**d**) Logarithmic PDFs of velocity

Gaussian behavior is obvious in the tail regions of the PDF solutions as shown in Figs. 3(b) and (d). The PDF solutions of both displacement and velocity have non-symmetric behavior for this type of nonlinear oscillator.

4 Conclusions

The EPC method is employed to solve the FPK equation in the case of the nonzero mean response of nonlinear oscillators under external Gaussian white noise. Different types of oscillators is investigated in the case of nonzero mean response to show the effectiveness of the EPC method. Both weak and strong nonlinearities are considered to show the effectiveness of the EPC method. When the polynomial order equals 2, the results of the EPC method are same as those given by equivalent linearization (EQL) method. The results obtained with the EQL method differ significantly from exact solution or simulated results. When the polynomial order equals 4 or 6, the PDFs obtained with the EPC method agree well with the exact solution or simulated results, especially in the tail re-

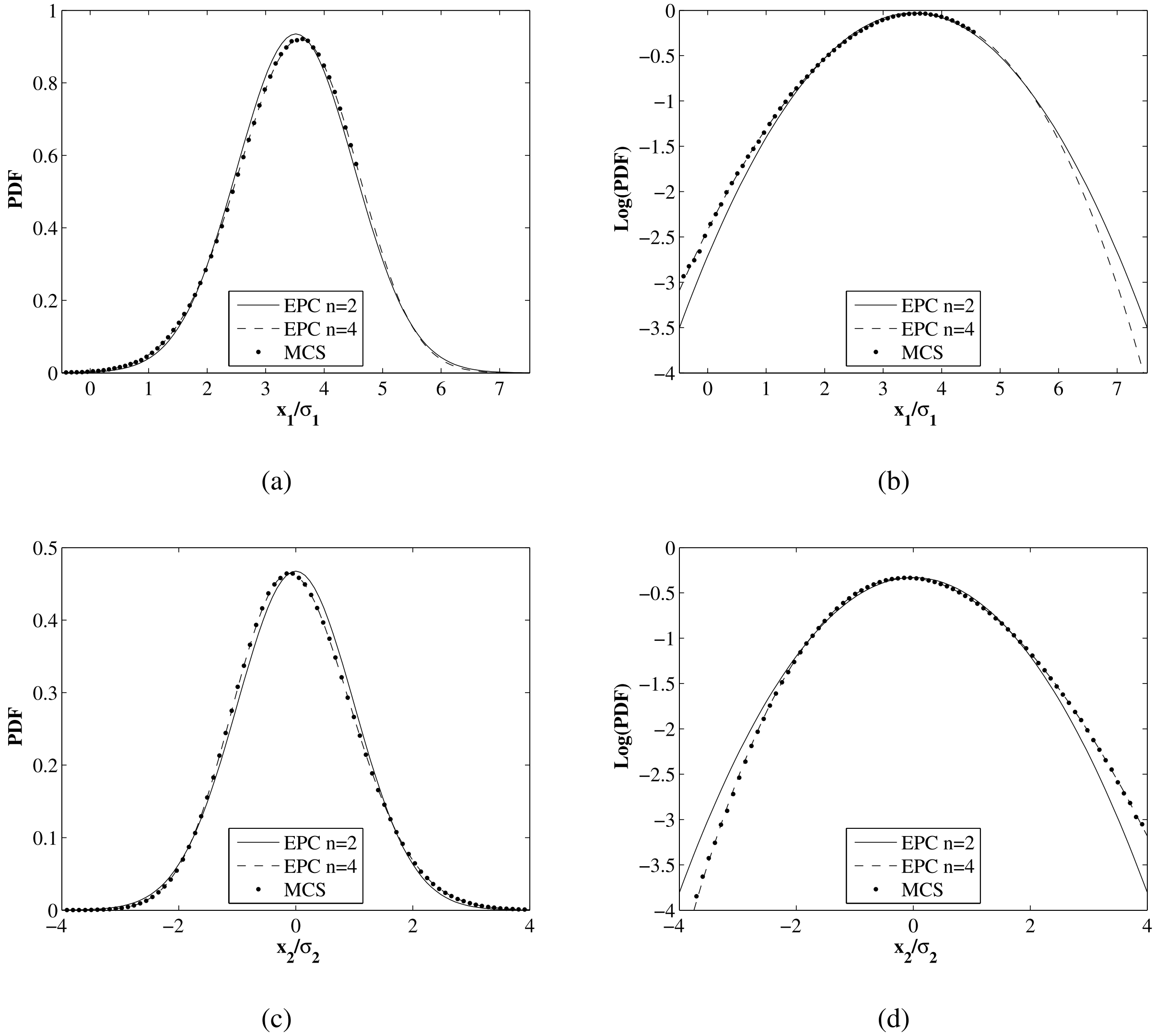

Fig. 3 Comparison of PDFs in Example 3: (**a**) PDFs of displacement; (**b**) Logarithmic PDFs of displacement; (**c**) PDFs of velocity; (**d**) Logarithmic PDFs of velocity

gions of the PDFs. For the oscillators with nonlinear terms in displacement, the PDF solutions of nonzero mean displacement are non-symmetric about its mean. Whereas, the PDF of velocity is still Gaussian and symmetric about its zero mean. In the cases that there is nonlinear term in velocity, the PDF of velocity is not Gaussian any more and the PDFs of both displacement and velocity may not be symmetrical.

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