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ORIGINAL PAPER

Optimal control of a class of fractional heat diffusion systems

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Abstract In this paper, a solution procedure for a class of optimal control problems involving distributed parameter systems described by a generalized, fractional-order heat equation is presented. The first step in the proposed procedure is to represent the original fractional distributed parameter model as an equivalent system of fractional-order ordinary differential equations. In the second step, the necessity for solving fractional Euler–Lagrange equations is avoided completely by suitable transformation of the obtained model to a classical, although infinite-dimensional, state-space form. It is shown, however, that relatively small number of state variables are sufficient for accurate computations. The main feature of the proposed approach is that results of the classical optimal control theory can be used directly. In particular, the well-known "linear-quadratic" (LQR) and "Bang-Bang" regulators can be designed. The proposed procedure is illustrated by a numerical example.

Keywords Fractional calculus · Optimal control · Distributed parameter systems · Fractional heat conduction · LQR · Bang-Bang control

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1 Introduction

Fractional calculus (FC) is a generalization of classical calculus that allows differentiation and integration to an arbitrary real (or even complex) order. In recent years, FC is emerging as a useful tool in modeling of a variety of physical phenomena. Fractional-order models are appearing in variational formulation of dissipative systems [1], in elasticity theory [2, 3], in particle physics, in the analysis of diffusive and electrical phenomena [4], but also in biomedical engineering [5] and in many other areas. In fact, it may be stated that the FC is presently being used in almost every field of science and engineering. The main advantage of fractional-order models in comparison to classical, integer-order ones is in the accommodation of hereditary and historical effects. These effects are common to many physical processes, yet they are simply neglected by the classical theory. In addition, fractional-order models can be seen as a superset of integer-order ones. They exhibit richer behavior in both time and complex domains. Fundamental solutions of fractional-order systems are expressed by means of a variety of special functions and their transfer functions are typically non-rational mappings of the Laplace variable s.

It is well known that fractional operators appear in analysis of several types of distributed parameter problems. These include long lines, heat transfer and diffusive processes in general. In fact, half-derivatives are reported to emerge during investigation of heat conduction both theoretically and empirically [4]. In recent years several generalizations of classical heat equation have been proposed that incorporate integro-differential operators of fractional order. A well-known generalization was proposed in [6, 7] by Mainardy and addressed by numerous other authors, including Agrawal [8]. A similar equation was studied by Metzler and his co-workers [9, 10]. Recently, a diffusion-wave equation with two fractional terms was addressed by Atanacković, Pilipović and Zorica [11]. However, although control of classical heat transfer is a well-studied area [12–14], there is little work addressing control of fractional heat processes reported in literature, one of the exceptions being [15]. In particular, to the authors' best knowledge there is no reported research addressing optimal control of generalized, fractional-order heat diffusion processes.

Even in the lumped parameters setting, where the models are given by *ordinary* differential equations, the optimal control of fractional-order systems has been solved only recently. The main contributions were given by Agrawal [16, 17] and recently by Atanacković and co-workers [18]. The exact solution of such problems entails solving the Euler–Lagrange equations with both left and right fractional derivatives included [19, 20]. This is a difficult task. A number of numerical procedures for solving the fractional optimal control problems have been proposed. A direct numerical procedure was proposed by Agrawal and Baleanu in [21], and a central difference scheme was addressed by Baleanu and co-workers in [22]. A novel procedure for numerical computation of the response of fractional-order systems was proposed by Agrawal in [23]. Recently Atanacković and Stanković proposed a method to represent a fractional derivative by means of a superposition of an infinite number of auxiliary variables each of which is the solution of an initial-value problem [24]. They used this method to solve a variety of ordinary fractional differential equations (FODE). Jeličić and N. Petrovački [25] used this approach in solving fractional optimal control problems of lumped parameter systems. The main advantage of this procedure is the use of the well-known classical optimal control theory in solving fractional control problems.

This paper addresses optimal control of a class of distributed parameter systems described by a generalized, fractional-order heat equation. The proposed solution consists of four distinct steps. In the first step, the original fractional-order partial differential equation (FPDE) is decomposed into an infinite set of ordinary differential equations of fractional order (FODE). The method used for decomposition is a generalization of the procedure developed by D. Petrovački for integer-order heat equation [26, 27]. By means of the above decomposition, the original distributed parameter system can be seen as a set of mutually coupled lumped parameter systems. The second step of the proposed procedure is to avoid the necessity of solving fractional Euler–Lagrange equation by representation of each fractional-order lumped parameter system as an infinite set of classical first-order systems by means of the decomposition proposed by Atanacković and Stanković and later used by Jeličić and N. Petrovački. After the second step, *the original FPDE is reduced without approximations to an infinite order classical state-space model*. In the third step, the obtained model is truncated and only a finite number of state variables are retained. Such a model can be controlled by classical methods [28, 29], including the well-known LQR approach [30], which are utilized in the fourth step. The presented procedure is illustrated by a numerical example. It is shown that the number of state variables relevant for computation of optimal control strategy is relatively small. Therefore, it is possible to work with approximate models of relatively small order.

2 Preliminaries on fractional calculus

Fractional calculus (FC) is a remarkably old topic. Its origins can be traced back to the end of 17th century, to the famous correspondence between Marquise de L'Hospital and G.W. Leibnitz in 1695. However, it was not until recent decades that FC was found to be a valuable tool in many applied disciplines. The first text devoted solely to fractional calculus is the book by Oldham and Spanier [31] published in 1974. Since then, numerous texts emerged. The primary references used within this paper are the book by Podlubny [32] and the recent one by Kilbas et al. [33].

Several definitions of fractional operators appear in literature. In the current paper the so-called Riemann–Liouville approach is adopted. The left Riemann–Liouville fractional integral of order $\alpha \geq 0$ (with respect to t) is defined as

$$ {}_0I_t^\alpha y = \frac{1}{\Gamma(\alpha)} \int_0^t y(\tau)(t-\tau)^{\alpha-1}\, d\tau, \tag{1} $$

where $y(t)$ is a scalar or a vector signal and $\Gamma(\alpha)$ is the Euler's gamma function

$$\Gamma(\alpha) = \int_0^\infty \nu^{\alpha-1} e^{-\nu}\, d\nu. \tag{2}$$

For integer values of integration order α the left Riemann–Liouville fractional integral is equivalent to the classical n-fold integral. In fact, in such a case, (1) reduces to the well-known Cauchy formula

$$\begin{aligned} {}_0I_t^n y &= \int_0^t \int_0^{t_1} \cdots \int_0^{t_{n-1}} y(t_n)\, dt_n \cdots dt_2\, dt_1 \\ &= \frac{1}{(n-1)!} \int_0^t y(\tau)(t-\tau)^{n-1}\, d\tau. \end{aligned} \tag{3}$$

The left Riemann–Liouville fractional derivative of order $\alpha \geq 0$ is defined as

$${}_0D_t^\alpha y = \frac{d^n}{dt^n} {}_0I_t^{n-\alpha} y, \tag{4}$$

where n is the smallest integer larger than α ($n-1 < \alpha \leq n$) and d/dt denotes the classical derivative. It can be shown that for broad class of functions and for integer values of α the fractional derivative coincides with the classical one. Within this paper, $\alpha \in (0,1)$ is of the primary interest. For such values of α the definition (4) becomes

$${}_0D_t^\alpha y = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t \frac{y(\tau)}{(t-\tau)^\alpha}\, d\tau. \tag{5}$$

In variational problems involving fractional-order operators, including optimal control problems, right fractional operators appear as well. In particular, necessary optimality conditions (Euler–Lagrange equations) of fractional optimal control problems contain both left and right fractional derivatives. Such equations are hard to solve. The need to solve these equations is avoided in the present work by suitable decomposition of fractional derivatives. For more information regarding right fractional operators and fractional Euler–Lagrange equations, the reader is referred to [16–18, 25].

In order to make notations more compact, partial fractional derivatives with respect to time will be denoted by $\partial^\alpha/\partial t^\alpha$ and by $\partial/\partial t$ in a special case of the first derivative. In a similar fashion, partial and total derivatives with respect to the space coordinate x will be denoted by $\partial/\partial x$ and d/dx, respectively.

3 Problem formulation

Consider a process governed by the partial fractional differential equation

$$\begin{aligned} &\frac{\partial Q(x,t)}{\partial t} + \sum_{m=0}^{M} \gamma_m \frac{\partial^{\alpha_m} Q(x,t)}{\partial t^{\alpha_m}} \\ &\quad = \mathbf{D} \frac{\partial^2 Q(x,t)}{\partial x^2} + g(x,t), \end{aligned} \tag{6}$$

where the time variable is denoted by $t \geq 0$ and the space variable by $x \in [0, L]$. Assuming that (6) describes one-dimensional heat diffusion process within a bounded domain, a solid body, for example $Q(x,t)$, can be interpreted as the temperature distribution within the body, $g(x,t)$ is the heat generated inside the body and $\mathbf{D} > 0$ is the thermal diffusivity of the body. The fractional dynamics is defined by a number of additional parameters, $\alpha_m \in (0,1)$ and $\gamma_m \in \mathbb{R}$ with $m \in \{0, \ldots, M\}$. For $\gamma_m = 0$, (6) reduces to the classical heat equation. The process is also subject to a set of inhomogeneous boundary (BC) and initial (IC) conditions:

$$Q(0,t) = c_0(t), \qquad Q(L,t) = c_1(t), \tag{7}$$

$$Q(x,0) = Q_0(x), \tag{8}$$

where $c_0(t)$ and $c_1(t)$ are temperatures at the boundaries of the considered body, and $Q_0(x)$ is the initial temperature distribution. The Dirichlet-type boundary conditions (7) are used without loss of generality. The presented procedure is applicable regardless of the actual form of the BC, and the results obtained in a more general case are presented later (see Sect. 4.3).

Equation (6) can be seen as a generalization of the fractional diffusion-wave equation addressed in [18], since it allows multiple terms of fractional order to appear on the left side. However, in order to keep the initial conditions in the form (8), the highest derivative is constrained to be of the first order.

A common assumption [26, 27] is that the process can be affected through its boundaries, by manipulation of the $c_0(t)$ and $c_1(t)$ functions appearing in the boundary conditions (7). Another way to manipulate the process is through $g(x,t)$. Let us denote by $u(t)$ the selected control variable. By means of the procedure introduced in the sequel of this paper, numerous optimal control problems involving generalized heat equation can be treated. The following ones are addressed here:

- *Optimal control of the temperature within the target cross section, with respect to the quadratic optimality criterium:* Find the optimal control strategy $u(t)$ that steers the temperature in the given cross section $x = x_t \in [0, L]$ of the body to zero. The equations of the process are (6) and the optimality criterium is

$$J_1 = \frac{1}{2}\Psi Q^2(x_t, T) + \frac{1}{2}\int_0^T \left[Q^2(x_t, t) + \beta u^2(t)\right] dt, \tag{9}$$

  where T is the final instant of time under consideration and $\beta > 0$ and $\Psi > 0$ are design parameters.
- "*Bang-Bang*" *control of the temperature within the target cross section*: Find the optimal control strategy $u(t)$ that steers the temperature in the given cross section $x = x_t \in [0, L]$ of the body to zero. The equations of the process are the same as before, i.e. (6); however, the optimality criterium is

$$J_2 = \frac{1}{2}\Psi Q^2(x_t, T) + \int_0^T \left[\frac{1}{2}Q^2(x_t, t) + \beta u(t)\right] dt, \tag{10}$$

  with T, Ψ and β the same as before, but with the additional constraint on the control variable, $u_{\min} \le u(t) \le u_{\max}$ for all $t \in [0, T]$. Notice that the optimality criterium is linear with respect to the control variable.

4 The solution procedure

4.1 Step 1: Reduction to a set of lumped parameter systems

As in the case of the classical heat equation [26], the solution will be sought in the form

$$Q(x, t) = Q_{\mathrm{BC}}(x, t) + \sum_{k=0}^{\infty} a_k(t)\varphi_k(x), \tag{11}$$

where $Q_{\mathrm{BC}}(x, t)$ is a function satisfying the inhomogeneous boundary conditions, $\varphi_k(x)$ are the eigenfunctions and $a_k(t)$ are the coefficients of the eigenfunction expansion of $Q(x, t) - Q_{\mathrm{BC}}(x, t)$. In the particular case of Dirichlet boundary conditions (7), a possible choice of $Q_{\mathrm{BC}}(x, t)$ is

$$Q_{\mathrm{BC}}(x, t) = \frac{L - x}{L}c_0(t) + \frac{x}{L}c_1(t). \tag{12}$$

The eigenfunctions are solutions to the homogeneous boundary-value problem

$$\frac{d^2\varphi_k(x)}{dx^2} + \lambda_k^2\varphi_k(x) = 0, \tag{13}$$

$$\varphi_k(0) = 0, \qquad \varphi_k(L) = 0. \tag{14}$$

For other types of boundary conditions (7), the differential equation (13) remains the same, but the boundary conditions (14) change. However, irrelevant of the form of the boundary conditions, the boundary-value problem (13), (14) is solvable only for a countable set of eigenvalues λ_k. For the Dirichlet boundary conditions (7) these values are

$$\lambda_k = \frac{k\pi}{L} \tag{15}$$

and eigenfunctions themselves are

$$\varphi_k(x) = \sin(\lambda_k x). \tag{16}$$

An important property of the eigenfunctions is their mutual orthogonality:

$$\int_0^L \varphi_k(x)\varphi_n(x)\, dx = \begin{cases} 0, & k \ne n, \\ \Phi_n, & k = n. \end{cases} \tag{17}$$

In the case of the Dirichlet BC, it is easy to show that $\Phi_n = L/2$.

The coefficients of the eigenfunction expansion, $a_k(t)$, can be found by multiplying both sides of the generalized heat equation (6) by $\varphi_n(x)$, and then integrating with respect to x from 0 to L. Due to (11) and (12), it is straightforward that

$$\frac{\partial^\alpha Q(x, t)}{\partial t^\alpha} = \frac{L - x}{L}\,{}_0D_t^\alpha c_0(t) + \frac{x}{L}\,{}_0D_t^\alpha c_1(t) + \sum_{k=0}^{\infty} {}_0D_t^\alpha a_k(t)\varphi_k(x) \tag{18}$$

for $\alpha \in (0, 1]$. Consequently, by exploiting the orthogonality of the eigenfunctions, it can be verified that

$$\int_0^L \frac{\partial^\alpha Q(x,t)}{\partial t^\alpha}\varphi_n(x)\,dx = \frac{P_n}{L}{}_0D_t^\alpha c_0(t) + \frac{K_n}{L}{}_0D_t^\alpha c_1(t) + {}_0D_t^\alpha a_n(t)\Phi_n, \tag{19}$$

where $P_n = \int_0^L (L-x)\varphi_n(x)\,dx$ and $K_n = \int_0^L x\varphi_n(x)\,dx$. Similarly,

$$\frac{\partial^2 Q(x,t)}{\partial x^2} = \sum_{k=0}^{\infty} a_k(t)\frac{d^2\varphi_k(x)}{dx^2} \tag{20}$$

and consequently, due to (13) and (17),

$$\int_0^L \frac{\partial^2 Q(x,t)}{\partial x^2}\varphi_n(x)\,dx = -\lambda_n^2\Phi_n a_n(t). \tag{21}$$

Denote by $g_n(t)$ the coefficients of the eigenfunction expansion of $g(x,t)$:

$$g_n(t) = \int_0^L g(x,t)\varphi_n(x)\,dx. \tag{22}$$

Expressions (19), (21) and (22), combined with (6) and (11), lead to the following set of ordinary differential equations of fractional order:

$$\frac{d}{dt}a_n(t) + \sum_{m=1}^{M}\gamma_m {}_0D_t^{\alpha_m} a_n + \lambda_n^2 \mathbf{D} a_n(t) = \frac{1}{\Phi_n}g_n(t) - \frac{P_n}{\Phi_n L}v_0(t) - \frac{K_n}{\Phi_n L}v_1(t) \tag{23}$$

where the auxiliary control variables are defined as

$$v_0(t) = \frac{d}{dt}c_0(t) + \sum_{m=1}^{M}\gamma_n\, {}_0D_t^{\alpha_m} c_0, \tag{24}$$

$$v_1(t) = \frac{d}{dt}c_1(t) + \sum_{m=1}^{M}\gamma_n\, {}_0D_t^{\alpha_m} c_1. \tag{25}$$

The respective initial conditions accompanying differential equations (23), (24) and (25) are

$$a_n(0) = \frac{1}{\Phi_n}q_n - \frac{P_n}{\Phi_n L}c_0(0) - \frac{K_n}{\Phi_n L}c_1(0), \tag{26}$$

$$c_0(0) = Q_0(0), \tag{27}$$

$$c_1(0) = Q_0(L), \tag{28}$$

with q_n being the coefficients of the eigenvalue expansion of the initial temperature distribution $Q_0(x)$,

$$q_n = \int_0^L Q_0(x)\varphi_n(x)\,dx. \tag{29}$$

By means of the procedure introduced in this subsection, the initial distributed parameter model described by a single partial differential equation of fractional order (6) is rewritten using an infinite number of ordinary differential equations of fractional order (23), (24) and (25). However, the two representations are completely equivalent.

4.2 Step 2: Reduction to a classical, infinite order system

In order to avoid difficulties encountered when solving optimal control problems involving fractional-order models, each of the fractional differential equations (23), (24) and (25) will be transformed into a system of infinitely many ordinary differential equations of the first order. To do so, let us first introduce a decomposition method recently proposed in [24] and utilized in [25]. The Atanacković–Stanković decomposition (ASD) states that fractional derivative of order $\alpha \in (0,1)$ of an arbitrary signal $y(t)$ can be computed as

$${}_0D_t^\alpha y = \frac{y(t)}{t^\alpha}\mathcal{A}(\alpha) + \sum_{i=2}^{\infty}\mathcal{B}(\alpha,i)\frac{\tilde{V}_i(t)}{t^{i-1+\alpha}} \tag{30}$$

where

$$\mathcal{A}(\alpha) = \frac{1}{\Gamma(1-\alpha)} - \frac{1}{\Gamma(1-\alpha)\Gamma(\alpha-2)} \times \sum_{p=2}^{\infty}\frac{\Gamma(p-1+\alpha)}{(p-1)!}, \tag{31}$$

$$\mathcal{B}(\alpha,i) = \frac{1}{\Gamma(1-\alpha)\Gamma(\alpha-2)}\frac{\Gamma(i-1+\alpha)}{(i-1)!}, \tag{32}$$

and weighted moments $\tilde{V}_i(t)$ are defined by

$$\tilde{V}_i(t) = -(i-1)\int_0^t \tau^{i-2}y(\tau)\,d\tau. \tag{33}$$

By applying the above decomposition to $a_n(t)$, $c_0(t)$ and $c_1(t)$, it is readily obtained that

$$\frac{d}{dt}a_n(t) = -a_n(t)\sum_{m=1}^{M}\gamma_m\left[\frac{\mathcal{A}(\alpha_m)}{t^{\alpha_m}} + \lambda_n^2\mathbf{D}\right] - \sum_{i=2}^{\infty}\tilde{V}_n, i(t)\sum_{m=1}^{M}\gamma_m\frac{\mathcal{B}(\alpha_m, i)}{t^{i-1+\alpha_m}} + \frac{1}{\Phi_n}g_n(t) - \frac{P_n}{\Phi_n L}v_0(t) - \frac{K_n}{\Phi_n L}v_1(t), \tag{34}$$

$$\frac{d}{dt}c_0(t) = -c_0(t)\sum_{m=1}^{M}\gamma_m\frac{\mathcal{A}(\alpha_m)}{t^{\alpha_m}} - \sum_{i=2}^{\infty}\tilde{U}_{0,i}\sum_{m=1}^{M}\gamma_m\frac{\mathcal{B}(\alpha_m, i)}{t^{i-1+\alpha_m}} + v_0(t), \tag{35}$$

$$\frac{d}{dt}c_1(t) = -c_1(t)\sum_{m=1}^{M}\gamma_m\frac{\mathcal{A}(\alpha_m)}{t^{\alpha_m}} - \sum_{i=2}^{\infty}\tilde{U}_{1,i}\sum_{m=1}^{M}\gamma_m\frac{\mathcal{B}(\alpha_m, i)}{t^{i-1+\alpha_m}} + v_1(t), \tag{36}$$

with $\tilde{V}_{n,i}$, $\tilde{U}_{0,i}$ and $\tilde{U}_{1,i}$ being the weighted moments of the $a_n(t)$, $c_0(t)$ and $c_1(t)$, respectively. As a direct consequence of (33), these variables are solutions to the initial value problems

$$\frac{d}{dt}\tilde{V}_{n,i}(t) = -(i-1)t^{i-2}a_n(t), \quad \tilde{V}_{n,i}(0) = 0, \tag{37}$$

$$\frac{d}{dt}\tilde{U}_{0,i}(t) = -(i-1)t^{i-2}c_0(t), \quad \tilde{U}_{0,i}(0) = 0, \tag{38}$$

$$\frac{d}{dt}\tilde{U}_{1,i}(t) = -(i-1)t^{i-2}c_1(t), \quad \tilde{U}_{0,i}(0) = 0. \tag{39}$$

By the procedure introduced in this and the previous subsection, the original fractional-order distributed parameter model (6) has been reduced, with no approximations, to the classical state-space model (34), (35), (36), (37), (38), (39). These equations play central role in our further study of optimal control strategies. The obtained model is, as expected, infinite-dimensional.

4.3 A note regarding a more general class of boundary conditions

As mentioned earlier, most of the discussion presented in the current paper relates to distributed parameter models accompanied by boundary conditions of Dirichlet type. The case of the more general boundary conditions is briefly discussed next.

A list of commonly used boundary conditions is given in [26]. All of these can be written as

$$b_{00}Q(0,t) + b_{01}\frac{\partial Q(0,t)}{\partial t} = c_0(t), \tag{40}$$

$$b_{10}Q(L,t) + b_{11}\frac{\partial Q(L,t)}{\partial t} = c_1(t) \tag{41}$$

with constants b_{00}, b_{01}, b_{10} and b_{11} satisfying

$$b_{00}L^2 - 2b_{01}L \neq 0, \tag{42}$$

$$b_{10}L^2 + 2b_{11}L \neq 0. \tag{43}$$

In such a case, one may choose

$$Q_{\mathrm{BC}}(x,t) = k_0c_0(t)(L-x)^2 + k_1c_1(t)x^2, \tag{44}$$

with $k_0 = (b_{00}L^2 - 2b_{01}L)^{-1}$ and $k_1 = (b_{10}L^2 + 2b_{11}L)^{-1}$.

Eigenfunctions and eigenvalues are chosen to satisfy (13), together with boundary conditions

$$b_{00}\varphi(0) + b_{01} + \frac{d\varphi(0)}{dx} = 0, \tag{45}$$

$$b_{10}\varphi(L) + b_{11} + \frac{d\varphi(L)}{dx} = 0. \tag{46}$$

The orthogonality of the eigenfunctions (17) still holds. By reapplying the procedure described in the previous two sections, one obtains

$$\frac{d}{dt}a_n(t) + \sum_{m=1}^{M}\gamma_m\, {}_0D_t^{\alpha_m}a_n + \lambda_n^2\mathbf{D}a_n(t) = \frac{1}{\Phi_n}g_n(t) - \frac{\tilde{P}_nk_0}{\Phi_n L}v_0(t) - \frac{\tilde{K}_nk_1}{\Phi_n L}v_1(t) + \frac{2k_0I_n}{\Phi_n}\mathbf{D}c_0(t) + \frac{2k_1I_n}{\Phi_n}\mathbf{D}c_1(t), \tag{47}$$

with auxiliary control variables defined as before in (35), (36), $\tilde{P}_n = \int_0^L(L-x)^2\varphi_n(x)dx$, $\tilde{K}_n = \int_0^L x^2\varphi_n(x)dx$ and $I_n = \int_0^L\varphi_n(x)dx$. The initial conditions associated with (47) are

$$a_n(0) = \frac{1}{\Phi_n}q_n - \frac{\tilde{P}_nk_0}{\Phi_n L}c_0(0) - \frac{\tilde{K}_nk_1}{\Phi_n L}c_1(0). \tag{48}$$

After application of the Atanacković–Stanković formula, (47) becomes

$$\frac{d}{dt}a_n(t) = -a_n(t)\sum_{m=1}^{M}\gamma_m\left[\frac{\mathcal{A}(\alpha_m)}{t^{\alpha_m}} + \lambda_n^2\mathbf{D}\right] - \sum_{i=2}^{\infty}\tilde{V}_n,i(t)\sum_{m=1}^{M}\gamma_m\frac{\mathcal{B}(\alpha_m,i)}{t^{i-1+\alpha_m}} + \frac{1}{\Phi_n}g_n(t) - \frac{\tilde{K}_n k_1}{\Phi_n L}v_1(t) - \frac{\tilde{K}_n k_1}{\Phi_n L}v_1(t) + \frac{2k_0 I_n}{\Phi_n}\mathbf{D}c_0(t) + \frac{2k_1 I_n}{\Phi_n}\mathbf{D}c_1(t), \tag{49}$$

with $\tilde{V}_{n,i}(t)$ satisfying (37), as before.

As in the case of the Dirichlet boundary conditions, the original fractional-order distributed parameter model has been reduced to classical state-space representation, although with infinite number of state variables. The structure of the obtained model is almost identical as in the case of the Dirichlet BCs. It is therefore clear that control strategies investigated in the sequel are applicable even in this more general case.

4.4 Step 3: Approximations

In order to make the problem computationally tractable, it is necessary to truncate both the Atanacković–Stanković expansion (30), (31) and the number of terms in (11). Consequently, the original distributed parameter system is approximated using only a finite number of expansion functions a_n and weighted moments $\tilde{V}_{n,i}$ and $\tilde{U}_{j,i}$.

Without loss of generality, it is assumed in the sequel that no heat is generated inside the body and that the process is influenced through the initial boundary $x = 0$, while the temperature at the other boundary $x = L$ is fixed at zero, $Q(L,t) = 0$. In other words, $g(x,t) = 0$ and $c_1(t) = 0$, and consequently $g_n(t) = 0$ and $v_1(t) = 0$. The control variable is $u(t) = v_0(t)$.

In the approximate setting, (11) is replaced by

$$Q(x,t) = Q_{\mathrm{BC}}(x,t) + \sum_{k=0}^{N_1}a_k(t)\varphi_k(x) = \frac{L-x}{L}c_0(t) + \sum_{k=0}^{N_1}a_k(t)\varphi_k(x). \tag{50}$$

The approximate Atanacković–Stanković formula is

$${}_0D_t^{\alpha}y \approx \frac{y(t)}{t^{\alpha}}\mathcal{A}_{N_2}(\alpha) + \sum_{i=2}^{N_2}\mathcal{B}(\alpha,i)\frac{\tilde{V}_i(t)}{t^{i-1+\alpha}}, \tag{51}$$

$$\mathcal{A}_{N_2}(\alpha) = \frac{1}{\Gamma(1-\alpha)} - \frac{1}{\Gamma(1-\alpha)\Gamma(\alpha-2)}\sum_{p=2}^{N_2}\frac{\Gamma(p-1+\alpha)}{(p-1)!}. \tag{52}$$

The formula is reported to give good approximations even for small values of N_2 [24, 25].

After approximations, (34) reduces to

$$\frac{d}{dt}a_n(t) = -a_n(t)\sum_{m=1}^{M}\gamma_m\left[\frac{\mathcal{A}_{N_2}(\alpha_m)}{t^{\alpha_m}} + \lambda_n^2\mathbf{D}\right] - \sum_{i=2}^{N_2}\tilde{V}_n,i(t)\sum_{m=1}^{M}\gamma_m\frac{\mathcal{B}(\alpha_m,i)}{t^{i-1+\alpha_m}} - \frac{P_n}{\Phi_n L}v_0(t). \tag{53}$$

Denote by $\mathbf{a}_n$ the column-vector $[a_n \;\; \tilde{V}_{n,2} \;\; \cdots \;\; \tilde{V}_{n,N_2}]^T$ (T in the right superscript denotes matrix transposition). The vector $\mathbf{a}_n$ consists, therefore, of a_n and all the related weighted moments $\tilde{U}_{n,i}$. For a fixed $n \in [0, N_1]$, differential equations (53) and corresponding equations (37) can be compactly written in a matrix form

$$\frac{d}{dt}\mathbf{a}_n(t) = \mathbf{A}_n(t)\mathbf{a}_n(t) + \mathbf{B}_n(t)u(t), \tag{54}$$

with the initial condition

$$\mathbf{a}_n(0) = \left[a_n(0)\; 0 \dots 0\right]^T. \tag{55}$$

Similarly, let $\mathbf{c}_0 = [c_0 \;\; \tilde{U}_{0,2} \;\; \cdots \;\; \tilde{U}_{0,N_2}]^T$. After approximations, (35) becomes

$$\frac{d}{dt}c_0(t) = -c_0(t)\sum_{m=1}^{M}\gamma_m\frac{\mathcal{A}_{N_2}(\alpha_m)}{t^{\alpha_m}} - \sum_{i=2}^{N_1}\tilde{U}_{0,i}\sum_{m=1}^{M}\gamma_m\frac{\mathcal{B}(\alpha_m,i)}{t^{i-1+\alpha_m}} + v_0(t), \tag{56}$$

or, combined with (38), in a matrix form

$$\frac{d}{dt}\mathbf{c}_0(t) = \mathbf{A}_c(t)\mathbf{c}_0(t) + \mathbf{B}_c(t)u(t), \tag{57}$$

with the initial condition

$$\mathbf{c}_0(0) = \left[\, Q_0(0)\ 0 \ldots 0 \,\right]^T. \tag{58}$$

Values of the particular entries of matrices $\mathbf{A}_c$, $\mathbf{B}_c$, $\mathbf{A}_n$ and $\mathbf{B}_n$ can readily be obtained from (53), (37), (56) and (38). The complete finite-order approximate model is obtained by combining (54) for all $n \in [0, N_1]$ and (57). Denote the complete state-vector by $\mathbf{a} = [\mathbf{c}_0^T \ \mathbf{a}_0^T \ \cdots \ \mathbf{a}_{N_1}^T]^T$. The approximate model can be written as

$$\frac{d}{dt}\mathbf{a}(t) = \mathbf{A}(t)\mathbf{a}(t) + \mathbf{B}(t)u(t), \tag{59}$$

with

$$\mathbf{A}(t) = \begin{bmatrix} \mathbf{A}_c(t) & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_0(t) & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{A}_{N_1}(t) \end{bmatrix}, \quad \mathbf{B}(t) = \begin{bmatrix} \mathbf{B}_c(t) \\ \mathbf{B}_0(t) \\ \vdots \\ \mathbf{B}_{N_1}(t) \end{bmatrix}. \tag{60}$$

Equation (50) can also be written in a more compact, matrix form

$$Q(x,t) = \mathbf{C}(x)\mathbf{a}(t) \tag{61}$$

with the output matrix $\mathbf{C}(x)$ defined as

$$\mathbf{C}(x) = \left[\, \frac{L-x}{L}\ 0 \ldots 0\ \varphi_0(x)\ 0 \ldots \varphi_{N_1}(x)\ 0 \ldots 0 \,\right]. \tag{62}$$

The initial conditions are

$$\mathbf{a}(0) = \left[\, \mathbf{c}_0(0)\ \mathbf{a}_0(0) \ldots \mathbf{a}_{N_1}(0) \,\right]^T. \tag{63}$$

It is interesting to investigate the dimension of the obtained system. Each of the $\mathbf{A}_n$ and $\mathbf{A}_c$ matrices is a square $N_2 \times N_2$ matrix. There are $N_1 + 2$ such matrices in the block diagonal matrix $\mathbf{A}$ (60). Therefore, the dimension of the approximate system is $(N_1 + 2)N_2$. Note that in the case of the Dirichlet boundary conditions, $\varphi_0(x) = 0$ for all x and therefore a_0 does not influence the solution. In such a case, the dimensionality can be reduced by N_2. If v_1 would also be considered as control variable, the dimensionality of the system would have to be increased by N_2. In this case the $\mathbf{B}$ matrix would have an additional column and $\mathbf{C}$ vector—N_2 additional entries. By introducing several (say N_3) g_n as additional control variables, the dimension of the system would remain the same, but the $\mathbf{B}$ matrix would have N_3 additional columns.

4.5 Step 4: Optimal control of the approximate model

Both of the optimal control problems defined in Sect. 3, with the optimality criteria given by (9) and (10), can now be approximated by the following classical optimal control problem: *Find an admissible control law $u(t)$ that minimizes*

$$J = \psi\big(Q(x_t, T)\big) + \int_0^T L\big(Q(x_t, t), u(t)\big)\, dt \tag{64}$$

subject to (59), (63) *and*

$$Q(x_t, t) = \mathbf{C}(x_t)\mathbf{a}(t). \tag{65}$$

The solution to the problem will be sought by means of the Hamiltonian formalism [28, 30]. In the considered case, the Hamiltonian function is

$$\begin{aligned} \mathcal{H}\big(\mathbf{a}(t), \mathbf{p}(t), u(t)\big) &= L\big(\mathbf{C}(x_t)\mathbf{a}(t), u(t)\big) \\ &\quad + \mathbf{p}^T(t)\big[\mathbf{A}(t)\mathbf{a}(t) \\ &\quad + \mathbf{B}(t)u(t)\big], \end{aligned} \tag{66}$$

with $\mathbf{p}(t)$ being the generalized impulses. The optimal control strategy, $u^*(t)$, is found as the admissible control vector that minimizes the Hamiltonian function

$$\mathcal{H}\big(\mathbf{a}(t), \mathbf{p}(t), u^*(t)\big) = \min_{u(t) \text{ is admissible}} \mathcal{H}\big(\mathbf{a}(t), \mathbf{p}(t), u(t)\big). \tag{67}$$

The optimal trajectory, $\mathbf{a}^*(t)$, satisfies the canonical equation

$$\begin{aligned} \frac{d}{dt}\mathbf{a}(t) &= \frac{\partial}{\partial \mathbf{p}(t)}\mathcal{H}\big(\mathbf{a}(t), \mathbf{p}(t), u^*(t)\big)\Big|_{u(t)=u^*(t)} \\ &= \mathbf{A}(t)\mathbf{a}(t) + \mathbf{B}(t)u^*(t), \end{aligned} \tag{68}$$

while the generalized impulses are governed by

$$\frac{d}{dt}\mathbf{p}(t) = -\frac{\partial}{\partial \mathbf{a}(t)}\mathcal{H}\big(\mathbf{a}(t), \mathbf{p}(t), u^*(t)\big)\big|_{u(t)=u^*(t)} = -\frac{\partial}{\partial \mathbf{a}(t)}L\big(\mathbf{C}(x_t)\mathbf{a}(t), u(t)\big)\big|_{u(t)=u^*(t)} - \mathbf{A}^T(t)\mathbf{p}, \quad (69)$$

accompanied by the natural boundary condition

$$\mathbf{p}(T) = \frac{\partial}{\partial \mathbf{a}(T)}\psi\big(Q(x_t, T)\big). \quad (70)$$

- *Optimal control of the temperature within the target cross section, with respect to the quadratic optimality criterium.* In this particular case, the optimality criterium is defined by (9), the Hamiltonian (66) reduces to

$$\mathcal{H} = \frac{1}{2}\mathbf{a}^T(t)\mathbf{C}^T(x_t)\mathbf{C}(x_t)\mathbf{a}(t) + \frac{\beta}{2}u^2(t) + \mathbf{p}^T(t)\big[\mathbf{A}(t)\mathbf{a}(t) + \mathbf{B}(t)u(t)\big] \quad (71)$$

  and

$$\psi\big(Q(x_t, t)\big) = \frac{1}{2}\Psi\mathbf{a}^T(T)\mathbf{C}^T(x_t)\mathbf{C}(x_t)\mathbf{a}(T). \quad (72)$$

  No additional constraints have been imposed on the control action, therefore all control strategies are admissible. This is in fact the well-known LQR problem. The optimal control is given by

$$u^*(t) = -\frac{1}{\beta}\mathbf{B}^T(t)\mathbf{L}(t)\mathbf{a}(t), \quad (73)$$

  with $\mathbf{L}(t)$ being the symmetric matrix satisfying the Riccati equation

$$\frac{d}{dt}\mathbf{L}(t) = -\mathbf{L}(t)\mathbf{A}(t) - \frac{1}{\beta}\mathbf{A}^T(t)\mathbf{L}(t) + \mathbf{L}(t)\mathbf{B}(t)\mathbf{B}^T(t)\mathbf{L}(t) - \mathbf{C}^T(x_t)\mathbf{C}(x_t) \quad (74)$$

  and the terminal condition

$$\mathbf{L}(T) = \Psi\mathbf{C}^T(x_t)\mathbf{C}(x_t). \quad (75)$$

- "*Bang-Bang*" *control of the temperature within the target cross section.* The Hamiltonian function (66) is in this case

$$\mathcal{H} = \frac{1}{2}\mathbf{a}^T(t)\mathbf{C}^T(x_t)\mathbf{C}(x_t)\mathbf{a}(t)\beta u(t) + \mathbf{p}^T(t)\big[\mathbf{A}(t)\mathbf{a}(t) + \mathbf{B}(t)u(t)\big], \quad (76)$$

  which is linear in $u(t)$. However, there are constraints on the control variable, and (67) implies that the optimal control is given by

$$u^*(t) = \begin{cases} u_{\min}, & \frac{\partial}{\partial u(t)}\mathcal{H} > 0, \\ \text{any admissible } u, & \frac{\partial}{\partial u(t)}\mathcal{H} = 0, \\ u_{\max}, & \frac{\partial}{\partial u(t)}\mathcal{H} < 0. \end{cases} \quad (77)$$

  Since

$$\frac{\partial}{\partial u(t)}\mathcal{H} = \beta + \mathbf{p}^T(t)\mathbf{B}(t), \quad (78)$$

  the optimal control value is in each time instant determined by the values of the generalized impulses. The canonical equation (69) becomes

$$\frac{d}{dt}\mathbf{p}(t) = -\mathbf{C}^T(x_t)\mathbf{C}(x_t)\mathbf{a}(t) - \mathbf{A}^T(t)\mathbf{p} \quad (79)$$

  with the natural boundary condition (70) reduced to

$$\mathbf{p}(T) = \Psi\mathbf{C}^T(x_t)\mathbf{C}(x_t)\mathbf{a}(T). \quad (80)$$

  Contrary to previously discussed LQR problem, calculations of the "Bang-Bang" control strategy involve solving two-point boundary-value problem, which is far more complex.

5 A numerical example

Consider a process governed by

$$\frac{\partial Q(x,t)}{\partial t} + \frac{\partial^{0.5}Q(x,t)}{\partial t^{0.5}} = \frac{\partial^2 Q(x,t)}{\partial x^2}, \quad (81)$$

with $x \in [0, 1]$ $(L = 1)$. The number of fractional terms on the left-hand side is 1 $(M = 1)$, with $\gamma_1 = 1$ and $\alpha_1 = 0.5$. The thermal diffusivity of the body is assumed to be $\mathbf{D} = 1$. The objective is to control the temperature of a target cross section x_t. The process is subject to boundary conditions of the Dirichlet type

$$Q(0,t) = c_0(t), \qquad Q(1,t) = 0 \quad (82)$$

and initial conditions

$$Q_0(x) = \sum_{n=0}^{3} \frac{1}{n^2} \sin\left(\frac{n\pi}{L} x\right). \tag{83}$$

After *Step 1* of the procedure proposed in this paper, (81) is reduced to a system of fractional differential equations

$$\frac{d}{dt} a_n(t) + {}_0D_t^{0.5} a_n + \lambda_n^2 a_n(t) = -\frac{P_n}{\Phi_n L} u(t), \quad n \in \mathbb{N}, \tag{84}$$

$$\frac{d}{dt} c_0(t) + {}_0D_t^{0.5} c_0 = u(t), \tag{85}$$

with $\lambda_n = n\pi$, $\Phi_n = 1/2$ and

$$P_n = \int_0^1 (1-x)\sin(n\pi x)\, dx = \frac{1}{n\pi}. \tag{86}$$

Note that (84) and (86) imply that the impact of the control variable on the nth harmonic $a_n(t)$ decreases with n. Lower order harmonics are, in general, more influenced by the control variable than the higher order ones.

In open-loop simulations, the number of terms in Atanacković–Stanković decomposition needed to accurately calculate the responses may be up to 20. Solutions to FODE (84) with $n = 1$, $u(t) = 0$ and initial condition $a_1(0) = 1$ obtained with different N_2 are presented in Fig. 1. The results are compared to the solution obtained by series expansion in the Laplace domain proposed in [11]. The results suggest that $N_2 = 20$ or even more are needed. However, when calculating optimal control strategy, a significantly smaller number of terms are sufficient. In particular, when LQR is computed, only marginal differences are obtained when using ASD with 5, 7 and 10 terms. A detail with the maximal discrepancy of the temperature in the target cross section $x_t = 0.4$ controlled by LQR calculated for different N_2 is presented in Fig. 2. The results were obtained with $N_1 = 5$, $\Psi = 0$ and $\beta = 0.001$. The ordinate axis shows percentage of the peak value. The maximal discrepancy when using $N_2 = 7$ and $N_2 = 10$ is less than 1 percent. It therefore seems that $N_2 = 7$ provides a good trade-off between computational accuracy and load. This conclusion is in accordance with [24, 25].

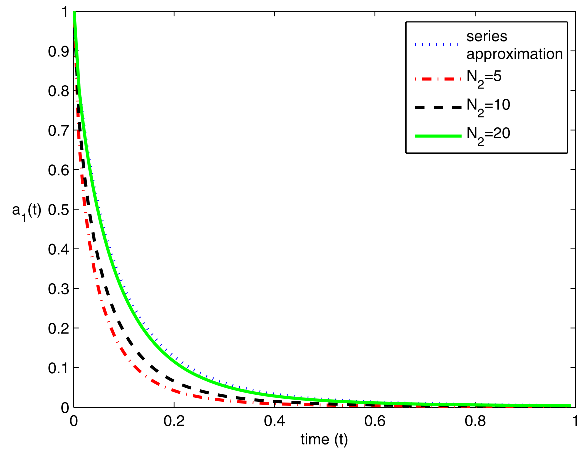

Fig. 1 Approximate responses of (84) for $n = 1$ to the perturbed initial condition $a_1(0) = 1$ obtained with different values of N_2. The results are compared to the solution obtained by the series expansion in the Laplace domain proposed in [11]

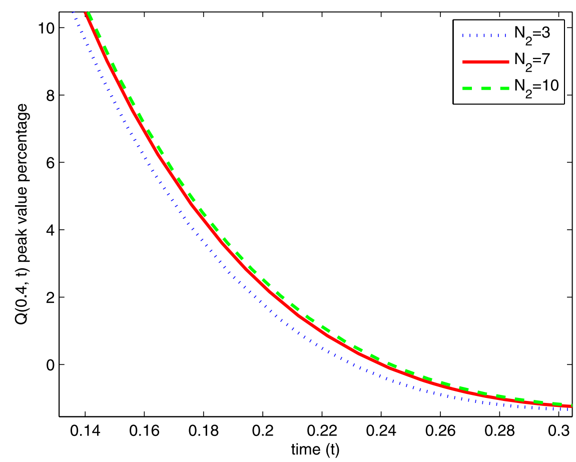

Fig. 2 A detail of the closed-loop response obtained with different values of N_2. LQR was used for regulation. The *ordinate axis* is normalized so that it shows percentage of the peak response instead of true values. The discrepancy in other parts of responses is even smaller

A detail of LQR controlled target cross section temperature calculated with different number of harmonics included (N_1) is presented in Fig. 3. Other simulation parameters are the same as before, with $N_2 = 10$. Percentages of the peak value are shown in the abscissa. Provided that higher-order spatial harmonics are not highly excited by the initial conditions, their influence on the overall solution is weak, since they are only mildly excited by the control variable. This is

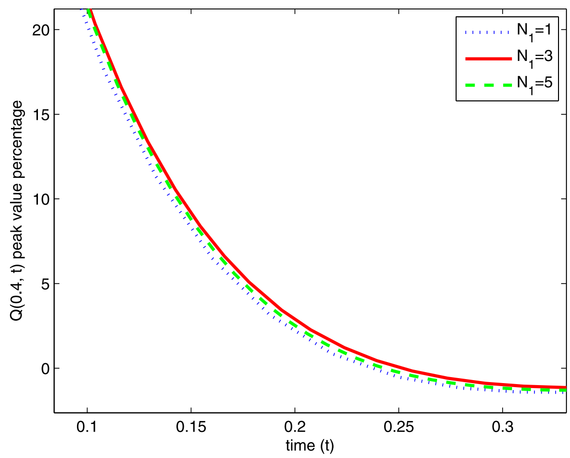

Fig. 3 A detail of the closed-loop response obtained with different values of N_1 with LQR regulation. The *ordinate axis* is normalized so that it shows percentage of the peak response instead of true values. The discrepancy of the simulation results is smaller in the parts of the responses not shown in the figure

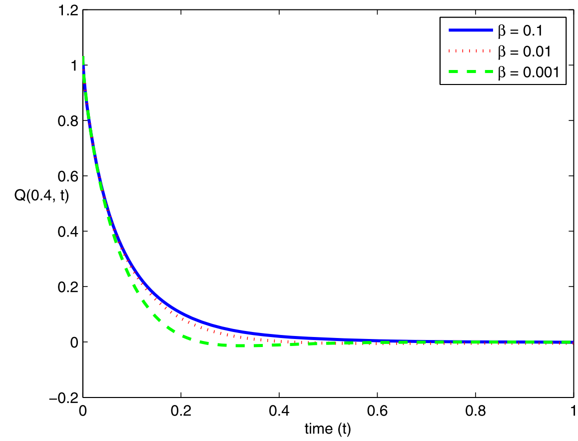

Fig. 5 The closed-loop responses of the temperature within the target cross section $x_t = 0.4$ obtained for different values of the β parameter in the criterium (9). In all cases, $\Psi = 0$, $N_1 = 5$ and $N_2 = 7$ were used

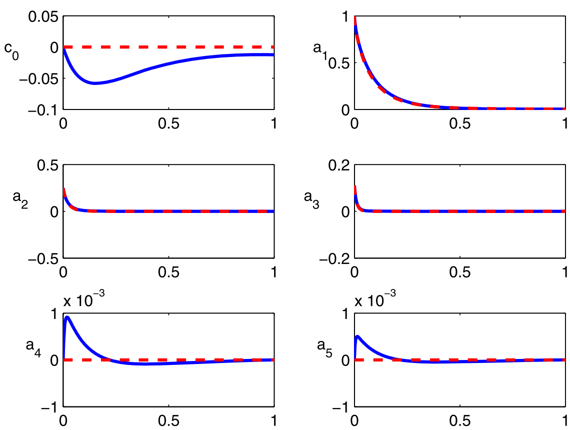

Fig. 4 Temperature at the initial cross section (c_0) and first few harmonics (a_i) of the LQR closed-loop (*solid line*) and open-loop (*dashed line*) responses. The time variable (t) is represented on the *abscissa axis*

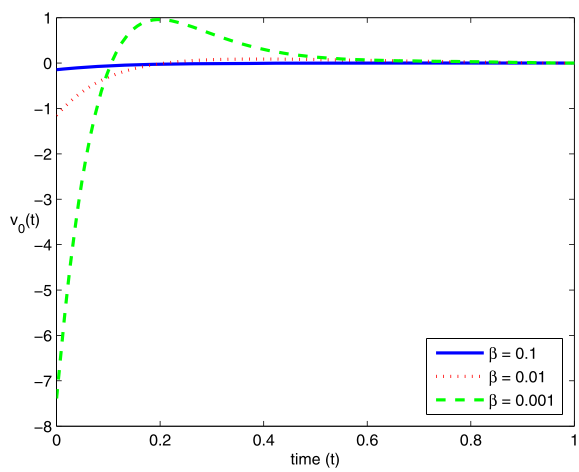

Fig. 6 The optimal control signal with respect to criterium (9) calculated with different values of the β parameter. In all cases, $\Psi = 0$, $N_1 = 5$ and $N_2 = 7$ were used

further illustrated in Fig. 4. It is seen that control variable mostly affects the first term of the target temperature, the one proportional to the boundary temperature. The first, second and third harmonics are excited by the initial condition, but with decreasing magnitude, which is a consequence of (83). They are also mildly influenced by the control variable. The fourth and fifth harmonics are influenced only by the control variable, yet their magnitude is comparatively small. This explains why often only a few harmonics are sufficient when calculating temperature distribution and optimal control.

Closed-loop responses of the target cross section $x_t = 0.4$ calculated for different values of the β parameter in criterium (9) are presented in Fig. 5. Respective optimal control signals are presented in Fig. 6. All results were obtained with $N_1 = 5$, $N_2 = 7$ and $\Psi = 0$. By decreasing the β parameter, the penalty (9) imposed on large control signals lowers. As a result, for small values of β the response is quicker, but the control signal is larger. In particular, both initial value

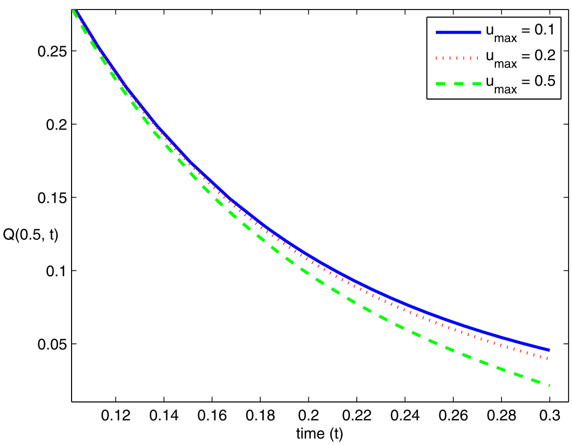

Fig. 7 A detail of the closed-loop response obtained with the "Bang-Bang" control strategy for different values of $u_{\max}$. The responses were obtained with $\beta = 0.001$, $\Psi = 1000$ and final time instant $T = 0.3$

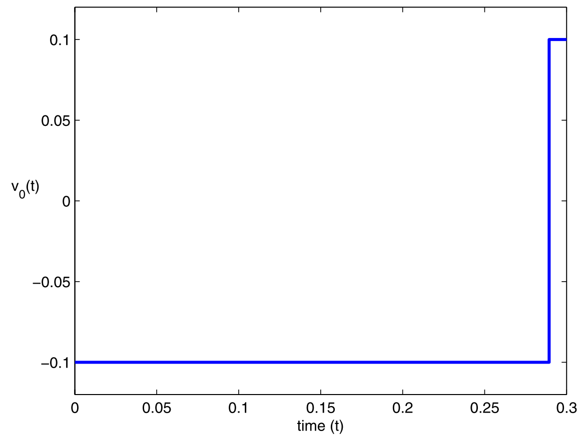

Fig. 8 Control signal obtained with the "Bang-Bang" control strategy for $u_{\max} = 0.1$. The parameters were $\beta = 0.001$, $\Psi = 10^6$ and final time instant $T = 0.3$

of the control signal and control overshoot strongly depend on β.

Temperature within the target cross section $x_t = 0.5$ when using the "Bang-Bang" control strategy is presented in Fig. 7. The control signal was subjected to be less than $u_{\max}$ in magnitude, $|u(t)| \leq u_{\max}$. Different curves in Fig. 7 correspond to different choices of $u_{\max}$. Clearly, by weakening the constraints on control variable, the response becomes quicker. The control variable for $u_{\max} = 0.1$ is shown in Fig. 8. Initial condition (83) was assumed. A single switch of the control signal is noticeable. Numerous numerical simulations conducted by the authors testify that even in the "Bang-Bang" case, $N_1 = 5$ and $N_2 = 7$ are sufficient for accurate calculations of optimal control in most cases.

6 Conclusions

A solution procedure for a class of optimal control problems involving distributed parameter systems described by generalized, fractional-order heat equation was presented and investigated in the present work. The procedure consists in reduction of distributed parameter fractional-order system to an integer order, lumped parameter system. More formally, the initial model given by fractional-order partial differential equation (FPDE) is transformed to a system of first-order ordinary differential equations (ODE). Optimal control strategies are then designed with respect to the transformed model.

It is quite important that the transformation from FPDE to ODE be exact, with no approximations involved. The obtained model is however infinite-dimensional and model reduction is needed prior to control design. Nevertheless, the proposed transformation is efficient, in the sense that the approximate models of relatively small degree can be obtained with no significant loss of accuracy.

The primary feature of the proposed solution procedure is that the necessity of solving fractional Euler–Lagrange equations is avoided completely. In fact, the solution is found by means of techniques well known in classical optimal control theory. As a consequence, the so-called "linear–quadratic" regulator, or LQR, can be constructed. LQR solution is computationally easy to obtain—it involves integrating two separate Cauchy-type problems instead of a single two-point boundary-value problem. On the other hand, the optimal control is given in state-feedback form, which is of considerable practical interest. Other optimal control problems can be solved as well. As an example, the "Bang-Bang" control was investigated in some detail.

Several directions for further research can be identified. First, a more general class of models can be considered, non-linear models and models involving distributed-order fractional operators in particular. Also, more difficult optimal control problems can be considered. It is interesting to note that even variational problems with variable upper time limit, such are minimum time problem and tracking problem, can be dealt with (at least approximately) by means of the proposed procedure. To the authors' best knowledge, no exact necessary conditions have been formulated for these problems when the model to be controlled is of fractional order.

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