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Optimal nonlinear observers for chaotic synchronization with message embedded

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Abstract This paper presents an optimal nonlinear observer for synchronizing the transmitter-receiver pair with guaranteed optimal performance. In the proposed scheme, a generalized nonlinear state-space observer via uniform matrix transformations is constructed to estimate the transmitter state and the information signal, simultaneously. A nonlinear optimal design approach is used to synchronize chaotic systems. Solving the Hamilton–Jacobi–Bellman (H–J–B) equations we can obtain a linear optimal feedback scheme for piecewise-linear chaotic systems. Moreover, a robust scheme derived from the H_∞ optimization theory improves the synchronization performance of general nonlinear chaotic systems by suppressing the influence of their high order residual terms. Finally, two numerical simulation examples are illustrated by the chaotic Chua's circuit system and the Lorenz chaotic system to demonstrate the effectiveness of our scheme.

Keywords Chaos synchronization · Secure communication · Quadratic optimal control · H_∞ optimization

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1 Introduction

Synchronization of chaotic systems and its application to secure communication has attracted increasing attention over the last decade. Numerous researches have been proposed for chaos-based secure transmission of private information signals [1–3]. Maintaining synchronization between the transmitter and receiver is essential in recovering the message since chaotic signals are sensitive to variations in initial conditions and parameters. Various methods to synchronize chaotic signals of the drive and response systems have been investigated, such as adaptive synchronization method [4, 5], sliding–mode control method [6, 7] and impulsive synchronization method [8, 9]. Among various chaos–based communication schemes, two basic configurations can be identified. The first one is chaotic masking [10, 11], which consists of the addition of the message to the chaotic carrier. The overall signal is then transmitted to the receiver. Another approach is chaotic modulation [12, 13], which is based on the full-state model of the transmitter and receiver systems. The message is injected into the states of chaotic generators. The generated chaotic signal thus internally includes the information of the transmitted message. In [14], a homogeneous synchronization configuration that uses parameter modulation has been proposed to extend the capabilities for secure communication. This scheme enables the message signal to be merged as a driving signal. Ricardo et al. [15] present a robust asymptotic feedback scheme for the synchro-

nization of non-identical chaotic systems and its usage in secure communication. The response system whose dynamical model is not similar to the drive system can reconstruct the message signal. Although being demonstrated successful in practical implementation, these methods can not be extended easily to nonlinear systems in particular when the information signals to be constructed are injected nonlinearly into the model.

Observer-based approach in secure communication becomes one of the attractive techniques investigated in the current researches [16–18]. The observer is constructed from the transmitter model to yield the dynamical reconstruction of the transmitter states, in which only the input and output information of the transmitter system are used to construct part or all of the state information of the transmitter system. Chen et al. [19] presented a sliding mode observer-based response system to synchronize the drive system. The hidden message can be recovered via the concept of equivalent control. Jiang et al. [20] proposed a state-observer-based approach for synchronization of complex dynamical networks. Some synchronization criteria are established based on the Lyapunov stability theory and the linear matrix inequality technique. In [21], a nonlinear observer-based scheme was investigated for both chaos synchronization and secure communication. Regular matrix transformations were used to establish conditions for asymptotic convergence of synchronization error. Even though these observer-based approaches have been shown to be successful in many instances, no global synchronization can be claimed. Furthermore, the algorithms consume too long time in achieving synchronization and are only suitable for low-frequency message signals.

Convergence and optimality are the most important requirements for the secure communication of chaotic systems. Most of the existed works only described chaotic encryption and decryption algorithms, but none of them considered the problem of the computational effort required to break the system. The linear quadratic regulator (LQR) method can cause a process to satisfy physical constraints and at the same time minimize some performance measures. Grzybowski et al. [22] addressed the synchronization problem of the unified chaotic system via optimal linear feedback control and the application in secure communication. In [23], the design of attacking chaotic encryption algorithms has been developed through optimization processes. The chaotic encryption systems were broken in very short computer times. Though the favorable performance was guaranteed, such approach did not provide insight into what conditions must be satisfied to ensure convergence. Pertew et al. [24] proposed an observer design method for Lipschitz nonlinear systems via the H_∞ optimization theory. Generalized sufficient conditions are derived to ensure asymptotic convergence of state estimates. However, the design method requires an undesirable high optimal feedback gain.

In this paper, a novel observer-based scheme via a quadratic optimal approach is proposed to address the synchronization problem of chaotic systems and its usage in secure communication. There are two main objectives to be considered here. The first one is to construct a generalized nonlinear state-space observer for the chaos-based secure communication through uniform matrix transformations. Information signals may be nonlinearly injected into the chaotic transmitter system and also are added to output signals so that the level of security of transmitted chaotic signals is enhanced. The proposed observer may simultaneously estimate the transmitter state and the information signal. The latter objective is related with the quadratic optimal synchronization between the transmitter and receiver. The nonlinear optimal design method is formulated to synchronize nonlinear chaotic systems with continuously differentiable nonlinearities. From the theoretical results presented in this paper, the systematic design methodology for observer-based secure communication of chaotic systems is established to achieve the optimality and robustness in performance. Finally, two numerical simulation examples are illustrated by the chaotic Chua's circuit system and the Lorenz chaotic system to demonstrate the effectiveness of the proposed approach. The remainder of this paper is organized as follows. Section 2 introduces the observer-based secure communication problem of chaotic systems. Section 3 introduces the design of optimal synchronization for piecewise-linear chaotic systems. Section 4 presents robust synchronization strategy for general chaotic systems via H_∞ optimization approach. Section 5 provides simulation results, while Sect. 6 offers concluding remarks.

2 Problem formulation

Figure 1 shows the system diagram with message embedded coupling channel of the chaotic transmitter-

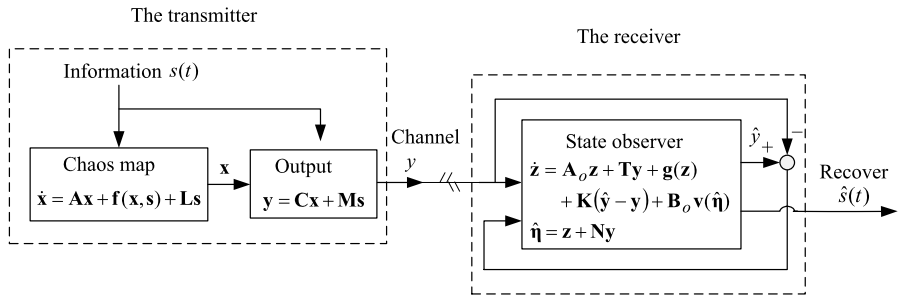

Fig. 1 The chaos-based secure communication system

receiver. It is observed that the information signal is nonlinearly injected into the chaotic transmitter system and also is added to the output signal. The chaotic transmitter system with multiple outputs and multiple information signals is

$$\dot{\mathbf{x}} = \mathbf{Ax} + \mathbf{f}(\mathbf{x},\mathbf{s}) + \mathbf{Ls} \tag{1}$$

$$\mathbf{y} = \mathbf{Cx} + \mathbf{Ms} \tag{2}$$

where $\mathbf{x} \in R^n$ is the system state, $\mathbf{y} \in R^p$ is the system output, $\mathbf{s} \in R^m$ is the information signal, $\mathbf{f}(\mathbf{x},\mathbf{s})$ is a real nonlinear vector field on R^n, and $\mathbf{A}, \mathbf{L}, \mathbf{C}$ and $\mathbf{M}$ are real matrices of appropriate dimensions. The output $\mathbf{y}$ is the signal transmitted to the receiver for secure communication applications. The chaotic receiver is then used to recover the information signal at the receiving end of the communication. The main purpose of this paper lies in building an observer from the output signal $\mathbf{y}$ to estimate both $\mathbf{x}$ and $\mathbf{s}$. For this purpose, the following assumptions are made.

Assumption 1 $(\mathbf{A}, \mathbf{C})$ *is a detectable pair.*

Assumption 2 *The matrices* $\mathbf{C}$ *and* $\mathbf{M}$ *are of full rank. For single output systems, this condition turns to be* $\mathbf{C}, \mathbf{M} \neq 0$.

The following notations are introduced for ease of presentation:

$$\boldsymbol{\eta} = \begin{bmatrix}\mathbf{x}^T & \mathbf{s}^T\end{bmatrix}^T \in R^{n+m} \tag{3}$$

$$\mathbf{F} = [\mathbf{A} \quad \mathbf{L}] \in R^{n\times(n+m)} \tag{4}$$

$$\mathbf{G} = [\mathbf{C} \quad \mathbf{M}] \in R^{p\times(n+m)} \tag{5}$$

A generalized nonlinear state-space observer corresponding to (1) can be constructed in the following form:

$$\dot{\mathbf{z}} = \mathbf{A}_o\mathbf{z} + \mathbf{Ty} + \mathbf{g}(\mathbf{z}) + \mathbf{K}(\hat{\mathbf{y}} - \mathbf{y}) + \mathbf{B}_o\mathbf{v}(t) \tag{6}$$

$$\hat{\boldsymbol{\eta}} = \mathbf{z} + \mathbf{Ny} \tag{7}$$

where $\hat{\boldsymbol{\eta}}$ and $\hat{\mathbf{y}}$ are the estimation vectors of $\boldsymbol{\eta}$ and $\mathbf{y}$, respectively; $\mathbf{K}$ is a feedback matrix; $\mathbf{v}(t) \in R^p$ is a linear feedback control vector; $\mathbf{A}_o, \mathbf{T}, \mathbf{B}_o$ and $\mathbf{N}$ are all constant matrices; and $\mathbf{g}(\mathbf{z})$ is a real nonlinear vector field on $R^{(n+m)}$. Define a synchronization error as $\mathbf{e} = \hat{\boldsymbol{\eta}} - \boldsymbol{\eta}$. If the synchronization error $\mathbf{e}(t)$ tends to zero as t tends to infinity, then chaotic transmitter system (1) is synchronized and also the information signal is recovered. Substituting (2) into (7), the synchronization error $\mathbf{e}(t)$ is rewritten as

$$\mathbf{e} = \mathbf{z} + (\mathbf{NG} - \mathbf{I}_{n+m})\boldsymbol{\eta} \tag{8}$$

Since the matrix $\mathbf{G}$ is of full rank from the Assumption 2, the following matrices can be build as [21]

$$[\mathbf{D} \quad \mathbf{N}] = \left(\begin{bmatrix}\mathbf{H}\\ \mathbf{G}\end{bmatrix}^T \begin{bmatrix}\mathbf{H}\\ \mathbf{G}\end{bmatrix}\right)^{-1}\begin{bmatrix}\mathbf{H}\\ \mathbf{G}\end{bmatrix} \tag{9}$$

where $\mathbf{D} \in R^{(n+m)\times n}$ and $\mathbf{H} = [\mathbf{I}_n \ \ \mathbf{0}] \in R^{n\times(n+m)}$. This deduces that

$$\mathbf{DH} + \mathbf{NG} = \mathbf{I}_{n+m} \tag{10}$$

Substituting (10) into (8) yields

$$\mathbf{e} = \mathbf{z} - \mathbf{DH}\boldsymbol{\eta} \tag{11}$$

It follows from (1) and (6) that the error dynamic can be described by

$$\begin{aligned}\dot{\mathbf{e}} &= \dot{\mathbf{z}} - \mathbf{DH}\dot{\boldsymbol{\eta}} = \dot{\mathbf{z}} - \mathbf{D}\dot{\mathbf{x}}\\ &= \mathbf{A}_o\mathbf{z} + \mathbf{TG}\boldsymbol{\eta} + \mathbf{g}(\mathbf{z}) + \mathbf{KGe} + \mathbf{B}_o\mathbf{v}(t)\\ &\quad - \mathbf{D}\big(\mathbf{F}\boldsymbol{\eta} + \mathbf{f}(\boldsymbol{\eta})\big)\\ &= (\mathbf{A}_o + \mathbf{KG})\mathbf{e} + [\mathbf{A}_o\mathbf{DH} + \mathbf{TG} - \mathbf{DF}]\boldsymbol{\eta} + \mathbf{g}(\mathbf{z})\\ &\quad - \mathbf{Df}(\boldsymbol{\eta}) + \mathbf{B}_o\mathbf{v}(t)\end{aligned} \tag{12}$$

If $\mathbf{A}_o$, $\mathbf{T}$ and $\mathbf{g}(\mathbf{z})$ are chosen to satisfy

$$\mathbf{A}_o\mathbf{DH} = (\mathbf{DF} - \mathbf{TG}) \tag{13}$$

$$\mathbf{g}(\mathbf{z}) = \mathbf{Df}(\hat{\boldsymbol{\eta}}) \tag{14}$$

then the error dynamic becomes

$$\dot{\mathbf{e}} = (\mathbf{A}_o + \mathbf{KG})\mathbf{e} + \mathbf{D}\big(\mathbf{f}(\hat{\boldsymbol{\eta}}) - \mathbf{f}(\boldsymbol{\eta})\big) + \mathbf{B}_o\mathbf{v}(t) \tag{15}$$

The problem considered in this paper is to design a generalized nonlinear observer in the form of (6) and (7) with the following properties:

(1) It achieves global asymptotic synchronization of the chaotic transmitter system (1) and (2).
(2) The linear feedback control $\mathbf{v}(t)$ is optimal with respect to the performance index

$$J(\boldsymbol{\eta}, \mathbf{e}, \mathbf{v}) = \int_{t_0}^{\infty} l(\boldsymbol{\eta}, \mathbf{e}, \mathbf{v})\, \mathrm{d}t \tag{16}$$

where $l(\boldsymbol{\eta}, \mathbf{e}, \mathbf{v}) > 0$ for all $\mathbf{e} \neq \mathbf{0}$. In general, the optimal control law for linear dynamic systems can be obtained by solving the Hamilton–Jacobi–Bellman (H–J–B) equation based on the dynamic programming approach. However, solving the corresponding H–J–B equation by a numerical integration method becomes intractable since the design problem considered in this paper is a nonlinear optimization problem. This paper introduces theoretical criteria to determine $\mathbf{v}(t)$ for two typical classes of chaotic systems.

3 Optimal synchronization for piecewise-linear chaotic systems

For piecewise-linear chaotic systems, such as the Chua's circuit and hyper-Rössler system, the nonlinearity can be described by [25]

$$\mathbf{f}(\hat{\boldsymbol{\eta}}) - \mathbf{f}(\boldsymbol{\eta}) = \mathbf{f}(\boldsymbol{\eta} + \mathbf{e}) - \mathbf{f}(\boldsymbol{\eta}) = \mathbf{E}(\boldsymbol{\eta})\,\mathbf{e} \tag{17}$$

where $\mathbf{E}(\boldsymbol{\eta})$ is a matrix dependent on $\boldsymbol{\eta}$, but is uniformly bounded. Substituting (17) into (15) leads to

$$\dot{\mathbf{e}} = \big[\mathbf{A}_o + \mathbf{KG} + \mathbf{DE}(\boldsymbol{\eta})\big]\mathbf{e} + \mathbf{B}_o\mathbf{v}(t) \tag{18}$$

We have the following theorem.

Theorem 1 *Consider the chaotic transmitter system* (1) *and* (2) *and give the quadratic performance index* $J(\boldsymbol{\eta}, \mathbf{e}, \mathbf{v})$ *in* (16). *If* $\mathbf{f}(\hat{\boldsymbol{\eta}}) - \mathbf{f}(\boldsymbol{\eta})$ *is a piecewise-linear function as described by* (17) *and the following conditions are satisfied*:

(1) *The gain matrix* $\mathbf{K}$ *is chosen such that* $\mathbf{A}_o + \mathbf{KG}$ *is a Hurwitz matrix*;
(2) *There exists positive definite matrices* $\mathbf{Q} = \mathbf{Q}^T$ *and* $\mathbf{R} = \mathbf{R}^T$ *such that the index function*

$$l(\boldsymbol{\eta}, \mathbf{e}, \mathbf{v}) = \mathbf{e}^T\big[\mathbf{Q} - \mathbf{E}^T(\boldsymbol{\eta})\mathbf{D}^T\mathbf{P} - \mathbf{PDE}(\boldsymbol{\eta})\big]\mathbf{e} + \mathbf{v}^T\mathbf{Rv} > 0 \quad \textit{for all } \mathbf{e} \neq \mathbf{0} \tag{19}$$

where the matrix $\mathbf{P} = \mathbf{P}^T > 0$ *satisfying*

$$(\mathbf{A}_o + \mathbf{KG})^T\mathbf{P} + \mathbf{P}(\mathbf{A}_o + \mathbf{KG}) - \mathbf{PB}_o\mathbf{R}^{-1}\mathbf{B}_o^T\mathbf{P} + \mathbf{Q} = \mathbf{0} \tag{20}$$

$$\mathbf{B}_o^T\mathbf{P} = \mathbf{G} \tag{21}$$

then the linear feedback control $\mathbf{v}(t)$ *that minimizes* (16) *subject to* (18) *is*

$$\mathbf{v}(t) = -\mathbf{R}^{-1}\mathbf{B}_o^T\mathbf{Pe} = -\mathbf{R}^{-1}(\hat{\mathbf{y}} - \mathbf{y}) \tag{22}$$

and the observer (6) *and* (7) *can globally exponentially synchronize the transmitter* (1) *and* (2).

Proof According to the dynamic programming rules, a necessary and sufficient condition for $\mathbf{v}(t)$ to minimize (16) subject to (18) is that there exists a value function $V(\mathbf{e}, t)$ satisfying the H–J–B equation [26]:

$$\min_{\mathbf{v}}\left[\frac{\mathrm{d}V(\mathbf{e}, t)}{\mathrm{d}t} + l(\boldsymbol{\eta}, \mathbf{e}, \mathbf{v})\right] = 0 \tag{23}$$

Choose a value function as

$$V(\mathbf{e}, t) = \mathbf{e}^T\mathbf{Pe} \tag{24}$$

Differentiating $V(\mathbf{e}, t)$ with respect to time leads to

$$\begin{aligned}\dot{V}(\mathbf{e}, t) &= \dot{\mathbf{e}}^T\mathbf{Pe} + \mathbf{e}^T\mathbf{P}\dot{\mathbf{e}} \\ &= \mathbf{e}^T\big[\big(\mathbf{A}_o + \mathbf{KG} + \mathbf{DE}(\boldsymbol{\eta})\big)^T\mathbf{P} \\ &\quad + \mathbf{P}\big(\mathbf{A}_o + \mathbf{KG} + \mathbf{DE}(\boldsymbol{\eta})\big)\big]\mathbf{e} \\ &\quad + 2\mathbf{e}^T\mathbf{PB}_o\mathbf{v}(t)\end{aligned} \tag{25}$$

Substituting (25) and (22) into the H–J–B equation (23) yields

$$\mathbf{e}^T\big[\big(\mathbf{A}_o+\mathbf{KG}+\mathbf{DE}(\boldsymbol{\eta})\big)^T\mathbf{P}+\mathbf{P}\big(\mathbf{A}_o+\mathbf{KG}+\mathbf{DE}(\boldsymbol{\eta})\big) - 2\mathbf{P}\mathbf{B}_o\mathbf{R}^{-1}\mathbf{B}_o^T\mathbf{P}\big]\mathbf{e}+l(\boldsymbol{\eta},\mathbf{e},\mathbf{v})=\mathbf{0} \tag{26}$$

$$\mathbf{e}^T\big[(\mathbf{A}_o+\mathbf{KG})^T\mathbf{P}+\mathbf{P}(\mathbf{A}_o+\mathbf{KG}) - \mathbf{P}\mathbf{B}_o\mathbf{R}^{-1}\mathbf{B}_o^T\mathbf{P}+\mathbf{Q}\big]\mathbf{e} - \mathbf{e}^T\big(\mathbf{Q}-\mathbf{E}^T(\boldsymbol{\eta})\mathbf{D}^T\mathbf{P}-\mathbf{P}\mathbf{D}\mathbf{E}(\boldsymbol{\eta})\big)\mathbf{e} - \mathbf{v}^T\mathbf{R}\mathbf{v}+l(\boldsymbol{\eta},\mathbf{e},\mathbf{v})=\mathbf{0} \tag{27}$$

If $l(\boldsymbol{\eta},\mathbf{e},\mathbf{v})$ is chosen as (19), it can be concluded that

$$\mathbf{e}^T\big[(\mathbf{A}_o+\mathbf{KG})^T\mathbf{P}+\mathbf{P}(\mathbf{A}_o+\mathbf{KG})-\mathbf{P}\mathbf{B}_o\mathbf{R}^{-1}\mathbf{B}_o^T\mathbf{P}+\mathbf{Q}\big]\mathbf{e}=\mathbf{0} \tag{28}$$

This derives the Riccati-like equation (20). Hence, if a matrix $\mathbf{P}$ can be found that satisfies (20) $\forall t\in(t_0,\infty)$, then the value function $V(\mathbf{e},t)$ in (24) satisfies the H–J–B equation (23). In this case, the optimal control $\mathbf{v}(t)$ is obtained as (22). Moreover, since the quadratic function $V(\mathbf{e},t)$ in (26) is positive, it can be chosen as a Lyapunov function candidate. From the solution of the H–J–B equation (23) and using (19), we have

$$\begin{aligned}\frac{\mathrm{d}V(\mathbf{e},t)}{\mathrm{d}t}&=-l(\boldsymbol{\eta},\mathbf{e},\mathbf{v})\\&=-\big[\mathbf{e}^T\big(\mathbf{Q}-\mathbf{E}^T(\boldsymbol{\eta})\mathbf{D}^T\mathbf{P}-\mathbf{P}\mathbf{D}\mathbf{E}(\boldsymbol{\eta})\big)\mathbf{e}+\mathbf{v}^T\mathbf{R}\mathbf{v}\big]\\&<0\end{aligned} \tag{29}$$

Hence, the globally exponentially stability of the closed-loop system can be ensured according to the Lyapunov stability theorem [27]. In turn, this implies that the observer (6) and (7) can globally exponentially synchronize the transmitter (1) and (2), i.e., $\hat{\boldsymbol{\eta}}(t)\to\boldsymbol{\eta}(t)$ as $t\to\infty$, and thus this completes the theorem proof. □

Remark 1 The approach proposed in Theorem 1 is applicable to chaotic systems with nonlinearities satisfying condition (17), such as the Chua's circuit, hyper-Rössler system, and so on. Additionally, some other nonlinearities [28] can also be described by (17).

Remark 2 Solving the Riccati-like equation (20) can naturally obtain the optimal control $\mathbf{v}(t)$ in (22) that achieves the optimal performance requirement. However, once the matrix $\mathbf{A}_o$ in (6) is not a Hurwitz matrix, the optimal performance may be destroyed. Thus, introducing a linear feedback gain matrix $\mathbf{K}$ into the linear quadratic optimal design is aimed to guarantee the stability of the closed system.

4 Robust Synchronization for general chaotic systems

For general chaotic systems with continuously differentiable nonlinearities, using Taylor series expansion of $\mathbf{f}(\hat{\boldsymbol{\eta}})$ at $\boldsymbol{\eta}$ yields

$$\begin{aligned}&\mathbf{f}(\hat{\boldsymbol{\eta}})-\mathbf{f}(\boldsymbol{\eta})\\&=\left[\mathbf{f}(\boldsymbol{\eta})+\left.\frac{\partial\mathbf{f}(\hat{\boldsymbol{\eta}})}{\partial\hat{\boldsymbol{\eta}}}\right|_{\hat{\boldsymbol{\eta}}=\boldsymbol{\eta}}(\hat{\boldsymbol{\eta}}-\boldsymbol{\eta})+\mathbf{h}(\mathbf{e})\right]-\mathbf{f}(\boldsymbol{\eta})\\&=\mathbf{F}'(\boldsymbol{\eta})\mathbf{e}+\mathbf{h}(\mathbf{e})\end{aligned} \tag{30}$$

were $\mathbf{F}'(\boldsymbol{\eta})=\partial\mathbf{f}(\hat{\boldsymbol{\eta}})/\partial\hat{\boldsymbol{\eta}}|_{\hat{\boldsymbol{\eta}}=\boldsymbol{\eta}}$, and $\mathbf{h}(\mathbf{e})$ is the high order residual term. Substituting (30) into (15), the error dynamics becomes

$$\dot{\mathbf{e}}=\big[\mathbf{A}_o+\mathbf{KG}+\mathbf{DF}'(\boldsymbol{\eta})\big]\mathbf{e}+\mathbf{D}\mathbf{h}(\mathbf{e})+\mathbf{B}_o\mathbf{v}(t) \tag{31}$$

This leads to the following lemma and theorem.

Lemma 1 *Let* $\mathbf{x}$ *and* $\mathbf{y}$ *be real vectors of appropriate dimensions and assuming that* $\mathbf{A}$ *and* $\mathbf{B}$ *are matrices with corresponding dimensions, by choosing a constant* $\varepsilon>0$, *the following inequality hold*:

$$2\mathbf{x}^T\mathbf{A}^T\mathbf{B}\mathbf{y}\le\varepsilon^{-2}\mathbf{x}^T\mathbf{A}^T\mathbf{B}\mathbf{B}^T\mathbf{A}\mathbf{x}+\varepsilon^2\mathbf{y}^T\mathbf{y} \tag{32}$$

Theorem 2 *Consider the chaotic transmitter system* (1) *and* (2) *and the observation dynamic system* (6) *and* (7) *where the optimal control* $\mathbf{v}(t)$ *is given by* (22). *If the gain matrix* $\mathbf{K}$ *is chosen ensuring that* $\mathbf{A}_o+\mathbf{KG}$ *is a Hurwitz matrix and there exists positive definite matrices* $\mathbf{Q}=\mathbf{Q}^T$ *and* $\mathbf{R}=\mathbf{R}^T$ *such that*

$$l(\boldsymbol{\eta},\mathbf{e},\mathbf{v})=\mathbf{e}^T\big[\mathbf{Q}-\mathbf{F}'^T(\boldsymbol{\eta})\mathbf{D}^T\mathbf{P}-\mathbf{P}\mathbf{D}\mathbf{F}'(\boldsymbol{\eta})\big]\mathbf{e}+\mathbf{e}^T\mathbf{R}\mathbf{e}>0,\quad \textit{for all } \mathbf{e}\neq\mathbf{0} \tag{33}$$

where the matrix $\mathbf{P}=\mathbf{P}^T>0$ *is the solution of the following Riccati-like equation*:

$$(\mathbf{A}_o+\mathbf{KG})^T\mathbf{P}+\mathbf{P}(\mathbf{A}_o+\mathbf{KG})-\mathbf{P}\mathbf{B}_o\mathbf{R}^{-1}\mathbf{B}_o^T\mathbf{P}+\rho^{-2}\mathbf{P}\mathbf{D}\mathbf{D}^T\mathbf{P}+\mathbf{Q}=\mathbf{0} \tag{34}$$

$$\mathbf{B}_o^T\mathbf{P}=\mathbf{G} \tag{35}$$

then the quadratic optimal synchronization performance

$$\int_{t_0}^{\infty} l(\boldsymbol{\eta}, \mathbf{e}, \mathbf{v})\,\mathrm{d}t \le \rho^2 \int_{t_0}^{\infty} \left\|\mathbf{h}(\mathbf{e})\right\|^2 \mathrm{d}t \tag{36}$$

is ensured for the attenuation level $\rho > 0$ *and the synchronization error* $\mathbf{e}(t)$ *asymptotically converge to zero for* $\mathbf{h}(\mathbf{e}) \in L_2$.

Proof Consider the Lyapunov function candidate

$$V(\mathbf{e}, t) = \mathbf{e}^T(t)\mathbf{P}\mathbf{e}(t) \tag{37}$$

The time derivative of $V(\mathbf{e}, t)$ along the error dynamic equation (31) is

$$\begin{aligned}\dot{V} &= \mathbf{e}^T\left[(\mathbf{A}_o + \mathbf{K}\mathbf{G})^T\mathbf{P} + \mathbf{P}(\mathbf{A}_o + \mathbf{K}\mathbf{G})\right]\mathbf{e} \\ &\quad + \mathbf{e}^T\left(\mathbf{F}'^T(\boldsymbol{\eta})\mathbf{D}^T\mathbf{P} + \mathbf{P}\mathbf{D}\mathbf{F}'(\boldsymbol{\eta})\right)\mathbf{e} \\ &\quad + 2\mathbf{e}^T\mathbf{P}\left[\mathbf{D}\mathbf{h}(\mathbf{e}) + \mathbf{B}_o\mathbf{v}(t)\right]\end{aligned} \tag{38}$$

Using Lemma 1, we have

$$2\mathbf{e}^T\mathbf{P}\mathbf{D}\mathbf{h}(\mathbf{e}) \le \rho^{-2}\mathbf{e}^T\mathbf{P}\mathbf{D}\mathbf{D}^T\mathbf{P}\mathbf{e} + \rho^2\left\|\mathbf{h}(\mathbf{e})\right\|^2 \tag{39}$$

Substituting (39) into (38) and based on the design of $\mathbf{v}(t)$ in (22), (38) can be represented as

$$\begin{aligned}\dot{V} &\le \mathbf{e}^T\left[(\mathbf{A}_o + \mathbf{K}\mathbf{G})^T\mathbf{P} + \mathbf{P}(\mathbf{A}_o + \mathbf{K}\mathbf{G})\right. \\ &\quad \left. - 2\mathbf{P}\mathbf{B}_o\mathbf{R}^{-1}\mathbf{B}_o^T\mathbf{P} + \rho^{-2}\mathbf{P}\mathbf{D}\mathbf{D}^T\mathbf{P}\right]\mathbf{e} \\ &\quad + \mathbf{e}^T\left(\mathbf{F}'^T(\boldsymbol{\eta})\mathbf{D}^T\mathbf{P} + \mathbf{P}\mathbf{D}\mathbf{F}'(\boldsymbol{\eta})\right)\mathbf{e} \\ &\quad + \rho^2\left\|\mathbf{h}(\mathbf{e})\right\|^2\end{aligned} \tag{40}$$

According to the Riccati-like equation in (34) leads to

$$\begin{aligned}\dot{V} &\le -\mathbf{e}^T\left[\mathbf{Q} + \mathbf{P}\mathbf{B}_o\mathbf{R}^{-1}\mathbf{B}_o^T\mathbf{P}\right]\mathbf{e} \\ &\quad + \mathbf{e}^T\left(\mathbf{F}'^T(\boldsymbol{\eta})\mathbf{D}^T\mathbf{P} + \mathbf{P}\mathbf{D}\mathbf{F}'(\boldsymbol{\eta})\right)\mathbf{e} + \rho^2\left\|\mathbf{h}(\mathbf{e})\right\|^2 \\ &= -\mathbf{e}^T\left(\mathbf{Q} - \mathbf{F}'^T(\boldsymbol{\eta})\mathbf{D}^T\mathbf{P} - \mathbf{P}\mathbf{D}\mathbf{F}'(\boldsymbol{\eta})\right)\mathbf{e} \\ &\quad - \mathbf{v}^T\mathbf{R}\mathbf{v} + \rho^2\left\|\mathbf{h}(\mathbf{e})\right\|^2 \\ &= -l(\boldsymbol{\eta}, \mathbf{e}, \mathbf{v}) + \rho^2\left\|\mathbf{h}(\mathbf{e})\right\|^2\end{aligned} \tag{41}$$

If the condition (33) is hold, then $V(t) > 0$ and $\dot{V}(t) < 0$ when $l(\boldsymbol{\eta}, \mathbf{e}, \mathbf{v}) > \rho^2\|\mathbf{h}(\mathbf{e})\|^2$; this implies $\mathbf{e}(t)$ is bounded for $t > 0$. Integrating the inequality (41) from $t = t_0$ to $t = T$ yields

$$\begin{aligned}V(T) - V(t_0) \le &-\int_{t_0}^{T} l(\boldsymbol{\eta}, \mathbf{e}, \mathbf{v})\,\mathrm{d}t \\ &+ \rho^2\int_0^T \left\|\mathbf{h}(\mathbf{e})\right\|^2 \mathrm{d}t\end{aligned} \tag{42}$$

Since $V(T) \ge 0$, and if the system starts with initial condition $\mathbf{e}(t_0) = 0$, this ensures the quadratic optimal synchronization performance (36). In other words, the error $\mathbf{e}(t)$ is attenuated to a prescribed level ρ. Moreover, $\dot{\mathbf{e}}(t) \in L_\infty$ is ensured from the boundedness of all terms on the right-hand side of (31). If $\mathbf{h}(\mathbf{e}) \in L_2[0, T), \forall T \in [0, \infty)$, then $\mathbf{e}(t) \in L_2[0, T)$. Hence, $\lim_{t\to\infty}\mathbf{e}(t) = 0$ is concluded as a result of $\dot{\mathbf{e}}(t) \in L_\infty$ and $\mathbf{e}(t) \in L_2$ via Barbalat's Lemma, and this completes the proof of the theorem. □

Remark 3 The approach proposed in Theorem 2 is applicable to general chaotic systems with nth-order differentiable nonlinearities, such as the Lorenz system, Duffing system, and so on.

Remark 4 Considering the dynamic of the high order residual term $\mathbf{h}(\mathbf{e})$, a quadratic optimal approach is introduced in this paper to attenuate its effect to a prescribed level. There is a trade-off between the solution of the Riccati-like equation in (34) and the attenuate capability because the Riccati-like equation restricts the choices of the matrix $\mathbf{P}$, which decides the optimal control $\mathbf{v}(t)$ in (22).

Remark 5 There are several merits of the quadratic optimal design approach. First, the stability design of the chaos observer is converted into a problem of quadratic optimal design issue. The proposed observer can handle a broader class of chaotic systems than the previous works [20, 21]. Second, the underlying concept of a robust approach is using variable structure control (VSC) to compensate the high order residual term $\mathbf{h}(\mathbf{e})$. It would induce a conservative upper-bounded estimation of the high order residual term $\mathbf{h}(\mathbf{e})$. In the proposed quadratic optimal design approach, the conservative bound estimation is avoided.

5 Simulation results

First example for Theorem 1 is the chaotic Chua's circuit. This oscillator is widely studied as a carrier for

secure communication schemes. The transmitter system is described by [29]

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 0- & \alpha \\ 0 & -\beta & \beta \\ \delta & \varsigma & -\varsigma \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ f(x_2) \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} s \qquad (43)$$

$$y = [1 \quad 0 \quad 0]\,\mathbf{x} + s$$

where $\alpha > 0, \beta > 0, \mathbf{x} = [x_1, x_2, x_3]^T$ is the state variable, and $f(x_2)$ is a piecewise-linear function defined by

$$f(x_2) = bx_2 + \frac{1}{2}(a-b)\big(|x_2+1| - |x_2-1|\big) \qquad (44)$$

with $a < -1 < b < 0$. If the parameters are chosen as $\alpha = 2.0, \beta = 1.9, \delta = 2.0, \varsigma = 1.5, a = -1.5$, and $b = -0.5$, then the system exhibits chaotic behavior. The information signal s is of the form $s = \sin(\pi t)$. The piecewise-linear function in (44) can be rewritten in the following form:

$$f(\hat{x}_2) - f(x_2) = k_x(\hat{x}_2 - x_2) \qquad (45)$$

where k_x is the slope of the linear segment, depending on both $\hat{x}_2$ and x_2. The slope varies within the interval $[a, b]$ for all $t \geq 0$, i.e., $a \leq k_x \leq b < 0$. Define $\boldsymbol{\eta} = [x_1, x_2, x_3, s]^T$. Therefore

$$\begin{aligned} \mathbf{f}(\hat{\boldsymbol{\eta}}) - \mathbf{f}(\boldsymbol{\eta}) &= \begin{bmatrix} 0 \\ f(\hat{\eta}_2) - f(\eta_2) \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & k_x & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} (\hat{\boldsymbol{\eta}} - \boldsymbol{\eta}) \\ &\equiv \mathbf{E}(\boldsymbol{\eta})\mathbf{e} \end{aligned} \qquad (46)$$

Using (9) and (13) and after calculation, matrices $\mathbf{D}, \mathbf{N}, \mathbf{A}_o$ and $\mathbf{T}$ of the state observer in (6) and (7) are

$$\mathbf{D} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix}, \qquad \mathbf{N} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\mathbf{A}_o = \begin{bmatrix} -2.0 & 0 & -2.0 & -1.0 \\ -2.0 & -1.0 & 1.0 & -1.0 \\ 3.0 & 1.5 & -1.5 & 1.0 \\ -1.0 & 0 & 2.0 & -2.0 \end{bmatrix} \qquad (47)$$

$$\mathbf{T} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix}$$

From Theorem 1, we choose $\mathbf{Q} = 10\mathbf{I}_4, \mathbf{R} = 1$, and the gain vector $\mathbf{K} = [-10, -10, 0, 0]^T$. The matrices $\mathbf{Q}$ and $\mathbf{R}$ are chosen so that $l(\boldsymbol{\eta}, \mathbf{e}, \mathbf{v})$ is a positive function for different initial conditions. The eigenvalues of the matrix $\mathbf{A}_o + \mathbf{KG}$ are $-13.0, -0.17$ and $-1.7 \pm j1.8$. Solving the algebraic Riccati equation (20) and (21), $\mathbf{P}$ and $\mathbf{B}_o$ are obtained as

$$\mathbf{P} = \begin{bmatrix} 25.5 & -32.0 & -18.1 & 28.0 \\ -32.0 & 41.6 & 24.4 & -36.2 \\ -18.1 & 24.4 & 18.5 & -19.0 \\ 28.0 & -36.2 & -19.0 & 37.9 \end{bmatrix}$$

$$\mathbf{B}_o = \begin{bmatrix} 1.45 \\ 1.57 \\ -0.37 \\ 0.27 \end{bmatrix} \qquad (48)$$

Figures 2–3 show the simulation results of the proposed scheme with the initial values $\mathbf{x}(0) = [1, 0, -1]^T$ and $\hat{\mathbf{x}}(0) = [1, 0, 0]$. From Fig. 2, the estimation of the transmitter state $\mathbf{x}$ is very quick and well. It means that the proposed observer system achieves the synchronization of the transmitter system. The transmitter signal y includes the information s and the chaotic signal of the transmitter. Figures 3(a) and 3(b) show that the recovered signal $\hat{s}(t)$ can succeed in recovering the emitted signal $s(t)$, and the transmitter signal y and the estimation signal $\hat{y}$ are almost identical. The evaluation of the performance index is indicated in Fig. 3(c).

Second example for Theorem 2 is the unified chaotic system described by

$$\dot{\mathbf{x}} = \begin{bmatrix} -(25\sigma + 10) & 25\sigma + 10 & 0 \\ (28 - 35\sigma) & 29\sigma - 1 & 0 \\ 0 & 0 & -\frac{\sigma+8}{3} \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ -x_1x_3 \\ x_1x_2 \end{bmatrix} + \begin{bmatrix} 10 \\ 20 \\ 0 \end{bmatrix} s \qquad (49)$$

$$y = [1 \quad 1 \quad 0]\,\mathbf{x} + s$$

The system is chaotic for any $\sigma \in [0, 1]$. We choose $\sigma = 0$ for illustration purposes. The nonlinear function

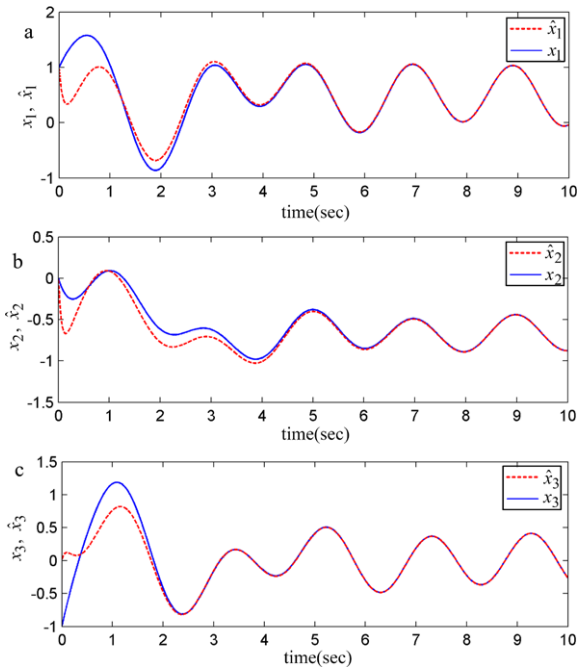

Fig. 2 Behavior of the communication of the Chua's circuit: observer state $\hat{\mathbf{x}}$ and transmitter state $\mathbf{x}$

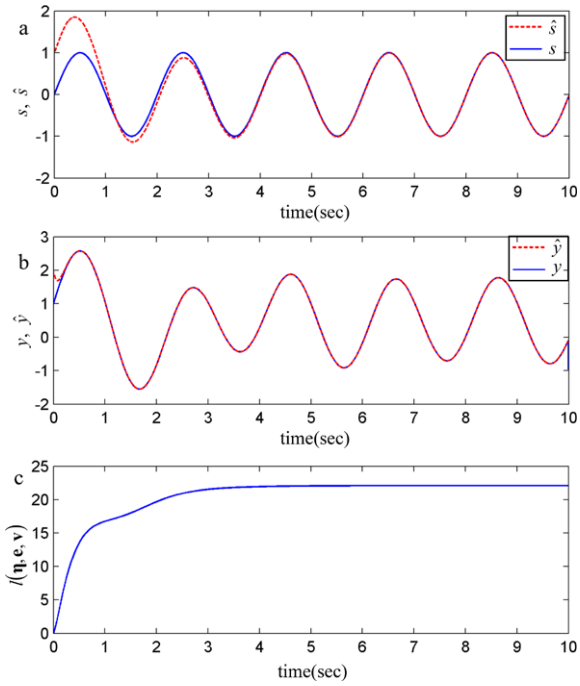

Fig. 3 Behavior of the communication of the Chua's circuit: (**a**) $\hat{s}$ and s; (**b**) $\hat{y}$ and y; (**c**) the evaluation of the performance index $\int_0^T l(\boldsymbol{\eta}, \mathbf{e}, \mathbf{v})\, \mathrm{d}t$

in (49) can be rewritten as

$$\begin{aligned}\mathbf{f}(\hat{\boldsymbol{\eta}}) - \mathbf{f}(\boldsymbol{\eta}) &= \begin{bmatrix} 0 \\ -(\hat{\eta}_1\hat{\eta}_3 - \eta_1\eta_3) \\ \hat{\eta}_1\hat{\eta}_2 - \eta_1\eta_2 \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ -(\eta_3 e_1 + \eta_1 e_3 + e_1 e_3) \\ \eta_2 e_1 + \eta_1 e_2 + e_1 e_2 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & 0 & 0 \\ -\eta_3 & 0 & -\eta_1 & 0 \\ \eta_2 & \eta_1 & 0 & 0 \end{bmatrix} \mathbf{e} + \begin{bmatrix} 0 \\ -e_1 e_3 \\ e_1 e_2 \end{bmatrix} \\ &\equiv \mathbf{F}'(\boldsymbol{\eta})\, \mathbf{e} + \mathbf{h}(\mathbf{e}) \end{aligned} \tag{50}$$

The information signal is of the form $s = 10 \times [\mathrm{sign}(\sin(2\pi t)) + 1]$. From (9) and (13), matrices $\mathbf{D}, \mathbf{N}, \mathbf{A}_o$ and $\mathbf{T}$ of the state observer are obtained as

$$\mathbf{D} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -1 & 0 \end{bmatrix}, \qquad \mathbf{N} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\mathbf{A}_o = \begin{bmatrix} 5 & 25 & 0 & 25 \\ -28 & -57 & 0 & -36 \\ -9 & -9 & -8/3 & -9 \\ -24 & -15 & 0 & -36 \end{bmatrix} \tag{51}$$

$$\mathbf{T} = \begin{bmatrix} 10 \\ 20 \\ 0 \\ -30 \end{bmatrix}$$

For Theorem 2, we choose $\mathbf{Q} = 2\mathbf{I}_4$, $\mathbf{R} = 1$, $\rho = 0.6$, and the gain vector $\mathbf{K} = [-10, -10, 0, 0]^T$. The eigenvalues of the matrix $\mathbf{A}_o + \mathbf{KG}$ are -2.7, -20, -21 and -67. Solving the nonlinear algebraic Riccati equation (34) and (35), $\mathbf{P}$ and $\mathbf{B}_o$ are obtained as

$$\mathbf{P} = \begin{bmatrix} 0.092 & 0.013 & -0.067 & 0.006 \\ 0.013 & 0.022 & -0.067 & -0.012 \\ -0.067 & -0.067 & 0.511 & -0.067 \\ 0.006 & -0.012 & -0.067 & 0.049 \end{bmatrix} \tag{52}$$

$$\mathbf{B}_o = \begin{bmatrix} -11 \\ 621 \\ 126 \\ 347 \end{bmatrix}$$

For the initial states $\mathbf{x}(0) = [20, -20, 0]^T$ and $\hat{\mathbf{x}}(0) = [0, 0, 0]$, the observer state $\hat{\mathbf{x}}$ and the recovered sig-

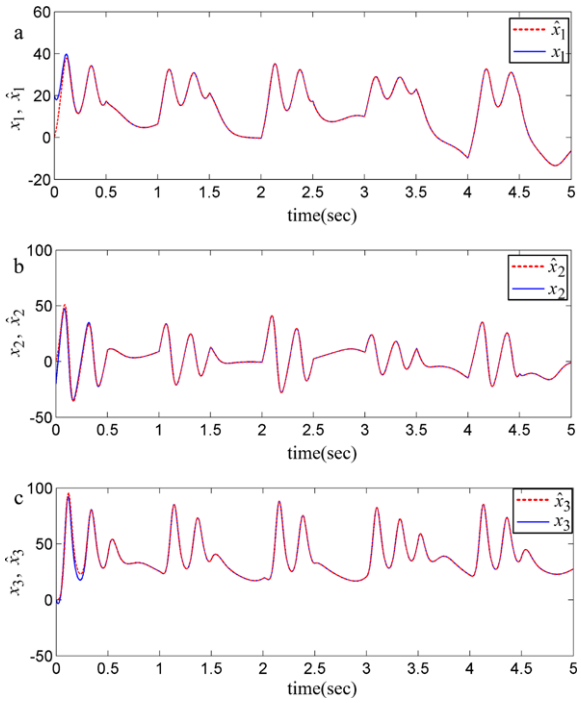

Fig. 4 Behavior of the communication of the Lorenz's system: observer state $\hat{\mathbf{x}}$ and transmitter state $\mathbf{x}$

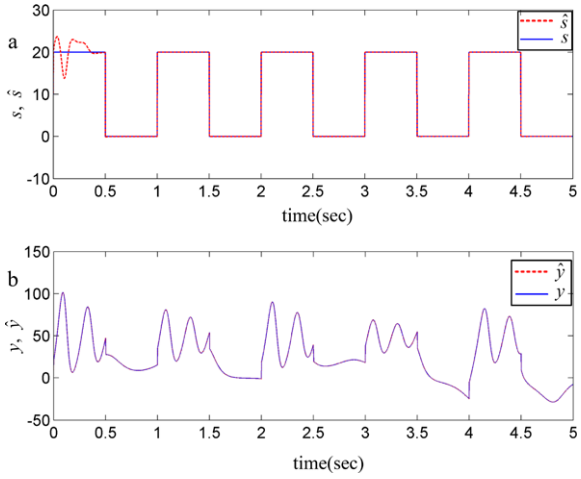

Fig. 5 Behavior of the communication of the Lorenz's system: (**a**) $\hat{s}$ and s; (**b**) $\hat{y}$ and y

nal $\hat{s}(t)$ versus time are shown in Figs. 4, 5, 6. Note also that the system (49) is globally synchronized and the message signal is recovered with good accuracy. Figure 6 shows the evaluations of the performance index $\int_0^T l(\boldsymbol{\eta}, \mathbf{e}, \mathbf{v})\, \mathrm{d}t$ with different attenuation levels $\rho = 0.6, 0.8$ and 1.0. It reveals that under lower attenuation level, i.e., $\rho = 1.0$, the H_∞ synchronization performance is often poor too.

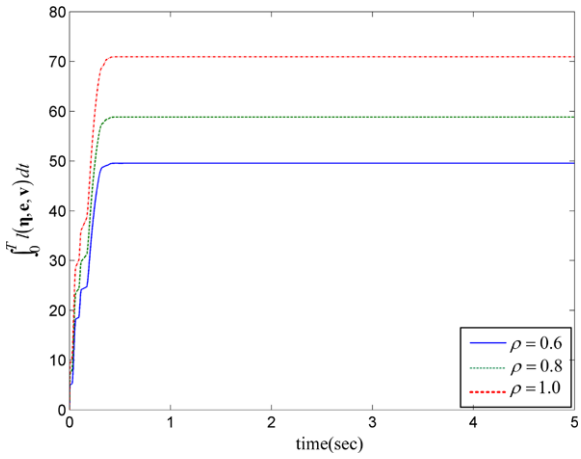

Fig. 6 Behavior of the communication of the Lorenz's system: families of $\int_0^T l(\boldsymbol{\eta}, \mathbf{e}, \mathbf{v})\, \mathrm{d}t$

6 Conclusion

This paper proposes an optimal observer-based scheme for synchronization of chaotic systems and presents its mathematical proof. It also successfully applies this scheme to chaos-based secure communication. One of the main features is that a generalized nonlinear state-space observer via uniform matrix transformations is constructed to estimate the state vector and the information signals, simultaneously. The nonlinear optimal design method is formulated to synchronize two typical classes of chaotic systems. For piecewise-linear chaotic systems, a linear optimal feedback scheme can be obtained by solving the H–J–B equation. For general nonlinear chaotic systems, the robust control and H_∞ optimization approach are integrated to overcome the effects of the high order residual term on the synchronization error. The proposed scheme takes advantage of the quadratic optimal and robust properties to make the transmitter and receiver synchronous. This paper also provides simulation results from the chaotic Chua's circuit system and the Lorenz chaotic system to illustrate the excellence of the proposed approach. The major contributions of this paper are that it presents a simple and systematic design method for chaos-based secure communication, and provides mathematical proof of the optimality and robustness of the synchronization system using the Lyapunov theorem.

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