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Analysis of a fractional order Van der Pol-like oscillator via describing function method

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Abstract In this paper, the behavior of a fractional order Van der Pol-like oscillator is investigated using a describing function method. A parametric function for the boundary between oscillatory and nonoscillatory regions of this system is extracted. The analytical results are evaluated by numerical simulations which demonstrate sufficient reliability of the proposed analyzing method.

Keywords Fractional Van der Pol oscillator · Describing function method · Oscillatory parametric region · Harmonic balance

1 Introduction

It has been claimed that Nature works with fractional time derivatives [1]. Relying on this claim, many real world processes have so far been modeled using fractional order dynamics. Moreover, calculus of the fractional order has recently been employed to design and implement the new generation of controllers. These applications are only two samples of the enormous applications of fractional calculus in practice. The growing interest in the application of fractional order differential equations necessitates more attention to and careful study on phenomena that might be observed in the systems represented by these equations. To this end, in recent years fractional order systems have been studied from different aspects of their applications and characteristics, including stability analysis [2, 3], system identification [4, 5], system approximation [6], control [7, 8], synchronization [9, 10], dynamical behavior analysis [11–18], and so on. The subject of this paper is related to dynamical behavior analysis of a fractional order system.

In recent years, the study of oscillatory behaviors in fractional order dynamical systems has been a subject of increasing attention [11–18]. These studies provide powerful tools to deepen our knowledge about complex behaviors of fractional order dynamical systems [15]. Also, these studies are useful in the design and implementation of fractional order oscillators [19–21]. Nowadays, it has been known that some fractional order systems can generate regular or irregular (chaotic) oscillations. For example, the fractional order Brusselator system [12], fractional order Van der Pol system [13–15], fractional order Bonhoeffer–Van der Pol system [16], fractional order Chen system [17, 18], and fractional order Lu system [17, 18] are some of these systems which have been studied in the literature in the sense of their abilities in generating regular or irregular (chaotic) oscillations. Thus far, different variants of the Van der Pol oscillator have been introduced, and their properties have been studied in the literature [13–15, 22, 23].

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The present work reports an investigation on the oscillatory behavior of a variant of the fractional order Van der Pol system. We base our analysis on the describing function method and determine whether or not an undamped oscillation can be generated by this system. In fact, in this paper, using a describing function-based analysis, we find the parametric range in which the considered Van der Pol model can be used as a nonlinear fractional order oscillator. It is worth mentioning that the describing function method has recently attracted the interest of researchers in the study of nonlinear fractional order systems [24–27].

The paper is organized as follows. Section 2 summarizes some basic concepts which are useful in the rest of the paper. This section also includes an introduction to the system under study and an explanation of the implemented analysis method, i.e., the describing function method. Use of the describing function method in the system oscillatory behavior analysis is discussed in Sect. 3. The numerical simulation results are presented in Sect. 4. The conclusions in Sect. 5 close the paper.

2 Preliminaries

This section presents the required basic definitions in fractional calculus, the introduction to the system under study, and the employed numerical method for the simulations. Also, a brief description of the method used in the system dynamical behavior analysis is presented.

2.1 Some basic definitions

Here some definitions of fractional order integration and fractional order differentiation are provided. The fractional integral, which is in fact an extension of the Cauchy formula for evaluating the integration, is defined by

$$I^{\beta} f(t) = \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} f(s)\,ds, \tag{1}$$

where $\beta \in R^{+}$ is the order of integration [28]. In the rest of the paper, two commonly used definitions for fractional order derivatives are employed. These definitions are known by their founders Grunwald–Letnikov and Caputo and are discussed in the sequel.

The Grunwald–Letnikov derivative with fractional order q is given by

$$_{\mathrm{GL}}D_t^q f(t) = \lim_{n\to\infty} \left[\frac{t}{n}\right]^{-q} \times \sum_{i=0}^{n} (-1)^i \binom{q}{i} f\left(t - i\left[\frac{t}{n}\right]\right), \tag{2}$$

where $q > 0$ is the order of derivation and

$$\binom{q}{i} \triangleq \frac{\Gamma(q+1)}{i!\,\Gamma(q+1-i)}$$

(where $\Gamma(.)$ is the gamma function). The Caputo fractional derivative with fractional order q is defined as

$$_{C}D_t^q f(t) = \begin{cases} I^{m-q} \frac{d^m}{dt^m} f(t), & m-1 < q < m, \\ \frac{d^m}{dt^m} f(t), & q = m, \end{cases} \tag{3}$$

where $m = \lceil q \rceil$.

In this paper the Caputo definition is used to represent the studied fractional order system. However, to simulate the fractional order differential equations of the system, a numerical method based on the Grunwald–Letnikov definition is implemented.

2.2 System description

The classic Van der Pol system is described by the following differential equation:

$$y^{(2)} + \mu\left(y^2 - 1\right) y^{(1)} + y = 0, \tag{4}$$

where y is the position coordinate, which is a function of time t, and $\mu > 0$ is a scalar parameter indicating the strength of the nonlinear damping. Analysis of an extended version of the Van der Pol oscillator which contains a fractional power of y or its derivative has been done in [22, 23]. Also, numerical and theoretical analyses of a modified version of Van der Pol oscillator containing derivatives of fractional order have been respectively given in [13–15]. The approach employed in [15] to analyze this system is based on stability analysis of incommensurate fractional order systems. In this paper, another extension of the fractional Van der Pol system is studied. The analysis approach presented in the next section is based on the describing function method.

Now, we describe the model considered in this paper. A fractional order version of Van der Pol system can be represented as

$$ {}_{\mathrm{C}}D_t^{1+q} y + \mu\big(y^2 - 1\big) y^{(1)} + y = 0, \tag{5} $$

where $0 < q < 1$. In this paper the term $\frac{\varepsilon\mu}{3} y^3$ has been added to the fractional Van der Pol oscillator equation in (5). The addition of this term increases the complexity of the system dynamics in the sense of nonlinearity. That is, we are concerned with the following system as the extended Van der Pol oscillator in the rest of the paper:

$$ {}_{\mathrm{C}}D_t^{1+q} y + \mu\big(y^2 - 1\big) y^{(1)} + y + \frac{\varepsilon\mu}{3} y^3 = 0. \tag{6} $$

For $\varepsilon = 0$ the fractional Van der Pol system defined in (5) is achieved.

2.3 Numerical method used in the simulations

There are two distinct methods employed in numerical simulations of fractional differential equations: frequency domain and time domain methods [29]. Due to the existing limitations of the frequency domain methods in simulating nonlinear dynamics in some critical regions (such as the existence of oscillations) [21, 29, 30], the numerical simulations in this paper were performed by a time domain method. For numerical simulation of the system in (6), the Grunwald–Letnikov fractional derivative-based equivalent of the system is found. Define $y_1(t) = y(t)$, and $y_2(t) = \dot{y}(t)$. By a similar proof procedure presented in [31], it can be proved that the system given in (6) is equivalent to the one given by the following set of equations:

$$ \begin{cases} \dot{y}_1(t) = y_2(t), \\ \dot{y}_2(t) = -{}_{\mathrm{GL}}D_t^{1-q}\big[\mu(y_1^2 - 1)y_2 + y_1 + \frac{\varepsilon\mu}{3} y_1^2\big]. \end{cases} \tag{7} $$

Since the Grunwald–Letnikov fractional derivative can be discretized in the time domain by using the following formula [28]:

$$ {}_{\mathrm{GL}}D_t^{q} f(t) = \lim_{h\to 0+} h^{-q} \sum_{k=0}^{[t/h]} (-1)^k \binom{q}{k} f(t - kh), \tag{8} $$

simulation of system (7) is easily done in the time domain. Numerical simulations of this paper are performed based on the discretization of system (7) as an equivalent model for the differential equation in (6). It is worth mentioning that another way to find the numerical solution of system (7) is by rewriting the second equation of (7) in the Caputo sense and using the methods proposed for solving fractional order differential equations which are constructed based on the Caputo definition (for example, using the method introduced based on the predictor-corrector approach in [35]). In fact, it can be easily shown that the second equation of (7) is equivalent to ${}_{\mathrm{C}}D_t^{q} y_2(t) = -\mu(y_1^2 - 1)y_2 - y_1 - \frac{\varepsilon\mu}{3} y_1^2$ (for more details, see [35]).

2.4 Describing function method

To analyze any system using the describing function method, it is necessary to convert the system into the basic feedback structure of Fig. 1 where L is the linear time-invariant dynamic part of the system and N represents the nonlinear time-invariant static and memoryless part of the system.

Usually the block L is described by its transfer function

$$ L(s) = \frac{b(s)}{a(s)}, \tag{9} $$

where s is the Laplace transform variable. The block N is represented by the nonlinear single-valued function $n(.)$. The system is unforced and $y(t)$ is the scalar output. The governing differential equation of this system is

$$ y(t) + L(p) n\big(y(t)\big) = 0, \tag{10} $$

where p is the differential operator. Assume the hypothetical solution of (10) as

$$ \tilde{y}(t) = A + B\sin(\omega t), \tag{11} $$

where $\omega > 0$ and $B > 0$. Also assume that the nonlinear function $n(y(t))$ can be approximated by

$$ n\big(\tilde{y}(t)\big) \approx N_0(A, B)A + N_1(A, B)B\sin(\omega t), \tag{12} $$

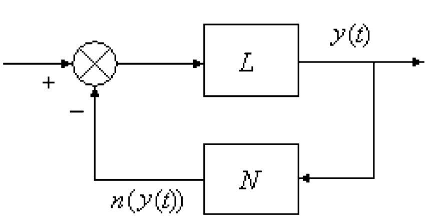

Fig. 1 Block diagram of basic feedback structure

where $N_0(A,B)$ and $N_1(A,B)$ are well-known describing function terms that are defined by

$$N_0(A,B)=\frac{1}{2\pi A}\int_{-\pi}^{\pi} n\bigl(A+B\sin(\omega t)\bigr)\,d\omega t, \tag{13}$$

$$N_1(A,B)=\frac{1}{\pi B}\int_{-\pi}^{\pi} n\bigl(A+B\sin(\omega t)\bigr)\sin(\omega t)\,d\omega t. \tag{14}$$

The hypothetical solution $\tilde{y}(t)$ is derived by the describing function method. The limit cycle existence conditions are

$$A\bigl(1+L(0)N_0(A,B)\bigr)=0, \tag{15}$$

$$1+L(j\omega)N_1(A,B)=0, \tag{16}$$

which are determined from (11), (13), and (14), imposing the harmonic balance principle [32] along with the system loop of Fig. 1. Equations (15) and (16) are solved for A, B and ω. Since the higher harmonic terms of $n(\tilde{y}(t))$ have been ignored in the derivation of (15) and (16), the derived results for the predicted limit cycles would be sufficiently accurate whenever the following inequality is held for any integer k greater than 1:

$$\bigl|L(j\omega)\bigr| \gg \bigl|L(kj\omega)\bigr|. \tag{17}$$

3 Analysis based on describing function method

To analyze the system of (6) with the describing function method, one should convert it into the basic feedback structure in Fig. 1. The linear and nonlinear parts of the system are determined as follows:

$$L(s)=\frac{s+\varepsilon}{s^{1+q}-\mu s+1}, \tag{18}$$

and

$$n\bigl(y(t)\bigr)=\mu\frac{y(t)^3}{3}. \tag{19}$$

Considering the nonlinear function of (19), and using (13) and (14), the describing functions are determined as follows:

$$N_0(A,B)=\mu\left(\frac{1}{3}A^2+\frac{1}{2}B^2\right), \tag{20}$$

$$N_1(A,B)=\mu\left(A^2+\frac{1}{4}B^2\right). \tag{21}$$

Using (16) and (21) and equating the imaginary part of (16) to zero, the following relation is obtained for the frequency of the oscillation:

$$f(\omega)=\omega^{1+q}+\left(\varepsilon\cot\frac{q\pi}{2}\right)\omega^q+\frac{-\varepsilon\mu-1}{\sin\frac{q\pi}{2}}=0. \tag{22}$$

The study is continued in two cases: $\varepsilon\geq 0$ and $\varepsilon<0$.

a. $\varepsilon\geq 0$: Since $0\leq q\leq 1$ and also μ is a positive value, for any value of the parameters, (22) is an increasing function with respect to ω. Thus, according to inequality $f(0)<0$, (22) definitely has a real positive root, which indicates that only a unique acceptable oscillating frequency exists. The same consequence may be obtained by differentiating (22) with respect to ω. Since $\partial f(\omega)/\partial\omega$ has no positive root, there would be no maximum or minimum point. Considering (15) and (20) and applying some mathematical simplifications, one gets

$$A=0. \tag{23}$$

Consequently,

$$1+\frac{1}{-\mu+\omega_0^q\cos\frac{q\pi}{2}}\left\{\frac{\mu}{4}B^2\right\}=0, \tag{24}$$

which means

$$B=\left(\frac{4(\mu-\omega_0^q\cos\frac{q\pi}{2})}{\mu}\right)^{1/2}. \tag{25}$$

To have a real value for B, the condition in (26) has to be satisfied:

$$\mu-\omega_0^q\cos\frac{q\pi}{2}\geq 0. \tag{26}$$

This is equivalent to having

$$\omega_0\leq\left(\frac{\mu}{\cos\frac{q\pi}{2}}\right)^{1/q}. \tag{27}$$

Substituting (27) into (22) results in a relation between μ and q that indicates the condition in which the limit cycle can be generated by the system in (6):

$$\mu\geq\frac{(\sin\frac{q\pi}{2})^{\frac{1}{q+1}}}{\tan\frac{q\pi}{2}}. \tag{28}$$

It is worth noting that the condition (28) is in fact the instability condition for the equilibrium point of the system, i.e., $y^*=0$ (see the Appendix).

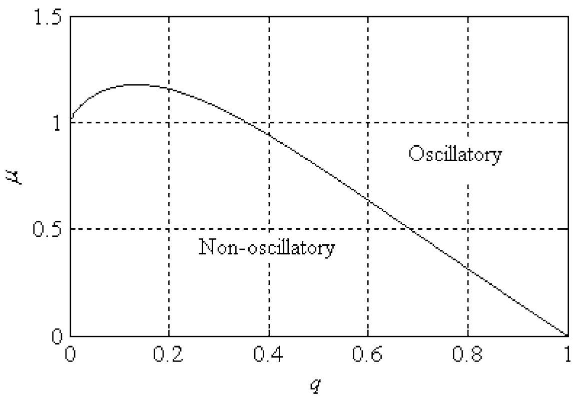

Fig. 2 Stability and instability regions for the system (6)

As a result, system (6) might have undamped oscillation for $\mu \geq g(q)$ where $g(q) = (\sin q\pi/2)^{1/(q+1)}/\tan q\pi/2$. The function $g(q)$ determines the boundary between oscillatory and nonoscillatory regions in the (μ, q) plane for the extended fractional Van der Pol oscillator (Fig. 2).

The stability of the predicted limit cycle may be checked via the Loeb criterion ([33]) or any other available method.

Based on the approximate Loeb criterion, the stability of the predicted limit cycle depends on the relative location of $L(j\omega)$ with respect to ω and the locus of $-1/N(A(B), B)$ with respect to B, where A is considered as a function of B. The predicted limit cycle is stable if the locus of $-1/N(A(B), B)$ enters a stable interval of the real axis by increasing B and enters an unstable interval by decreasing B (see [33, Fig. 6]). From (21) and (23),

$$N_1\big(A(B), B\big) = \frac{\mu}{4}B^2, \tag{29}$$

and consequently

$$\frac{dN_1(A(B), B)}{dB} = \frac{\mu}{2}B > 0.$$

Therefore, for an unstable transfer function of (18) (for proper selection of q and μ considering inequality (28) this system is unstable), $-1/N(A(B), B)$ enters in a stable interval of the real axis by increasing B. This means that the predicted limit cycle is stable according to the Loeb criterion.

b. $\varepsilon < 0$: In this case, there are three distinct equilibrium points as follows.

$$\begin{cases} y_1^* = 0, \\ y_{2,3}^* = \pm(\frac{-3}{\varepsilon\mu})^{1/2}. \end{cases} \tag{30}$$

The point y_1^* is the same equilibrium point of the previous case. Therefore, the stability of two other equilibrium points will be discussed. The characteristic equation of the system in equilibrium points $y_{2,3}^*$ is

$$s^{1+q} - \left(\frac{3}{\varepsilon} + \mu\right)s - 2 = 0. \tag{31}$$

The left-hand side of (31) is a negative value for $s = 0$ and a positive one for $s \to \infty$; thus, it has at least one real positive root which indicates that the equilibrium points $y_{2,3}^*$ are unstable.

Now, from (15), (20) and (21) and some simplifications, one finds three distinct solutions for A and thus for N_1 as given below.

$$\begin{cases} A_1 = 0, \\ N_1(A_1, B_1) = \frac{\mu}{4}B_1^2, \end{cases} \tag{32}$$

$$\begin{cases} A_{2,3} = (\frac{-3}{\varepsilon\mu} - \frac{3}{2}B_{2,3}^2)^{1/2}, \\ N_1(A_{2,3}(B_{2,3}), B_{2,3}) \\ \quad = \mu(\frac{-3}{\varepsilon\mu} - \frac{3}{2}B_{2,3}^2 + \frac{1}{4}B_{2,3}^2) = \mu(\frac{-3}{\varepsilon\mu} - \frac{5}{4}B_{2,3}^2). \end{cases} \tag{33}$$

Solution (32) is the same as the one in the case **a**, and therefore a similar oscillation condition will be obtained. Therefore, the calculations are performed with the solution in (33). Substituting the second relation of (33) into (16) results in the following value for $B_{2,3}$:

$$B_{2,3} = \left(\frac{-4(\mu - \omega_0^q \cos\frac{q\pi}{2})}{5\mu} - \frac{12}{5\varepsilon\mu}\right)^{1/2}. \tag{34}$$

This parameter would have a real value whenever

$$\frac{-4(\mu - \omega_0^q \cos\frac{q\pi}{2})}{5\mu} - \frac{12}{5\varepsilon\mu} > 0 \tag{35}$$

is satisfied. Therefore,

$$\omega_0 > \left(\frac{\frac{3}{\varepsilon} + \mu}{\cos\frac{q\pi}{2}}\right)^{1/q}. \tag{36}$$

To investigate the stability of these limit cycles, the Loeb criterion is checked again. As declared in (33),

$$N_1\big(A_{2,3}(B_{2,3}), B_{2,3}\big) = \mu\left(\frac{-3}{\varepsilon\mu} - \frac{5}{4}B_{2,3}^2\right) \tag{37}$$

and thus

$$\frac{dN_1(A_{2,3}(B_{2,3}), B_{2,3})}{dB_{2,3}} = -\frac{5\mu}{2} B_{2,3} < 0.$$

Therefore, for an unstable transfer function of (18), $-1/N(A(B), B)$ enters an unstable interval of the real axis while B increases. This means that the predicted limit cycles are unstable according to the Loeb criterion, and consequently they will not be observed in the numerical simulations.

4 Numerical simulations

Equation (28) determines the oscillatory region with respect to q for any given μ. For example, for $\mu = 0.8$, the system of equation (6) may produce undamped oscillations if $0.49 < q < 1$ and for $\mu = 1$, the undamped oscillations exist if $0.35 < q < 1$. Figure 3 shows the simulation results for $\mu = 0.8$, $\varepsilon = 1$ and two values of $q = 0.50$ and $q = 0.48$. As predicted by performed analysis, the system with parameters $\mu = 0.8$, $\varepsilon = 1$ and $q = 0.50$ has undamped oscillations (Figs. 3(a) and (b)), while the system with parameters $\mu = 0.8$, $\varepsilon = 1$ and $q = 0.48$ has a damping oscillation toward the equilibrium point (Figs. 3(c) and (d)). Figure 4 shows the validity of the analysis in the previous section for $\mu = 1$, $\varepsilon = 0$ and two values of $q = 0.36$ and $q = 0.34$. The simulation result for $\mu = 1$, $\varepsilon = 0$ and $q = 0.36$, as sketched in Figs. 4(a) and (b), is expected to contain undamped oscillations and the simulation result for $\mu = 1$, $\varepsilon = 0$ and $q = 0.34$, as sketched in Figs. 4(c) and (d), is expected to have a damping characteristics. The results are consistent with the predicted properties. Note that the system of Fig. 4 is sketched for $\varepsilon = 0$ and thus it is, in fact, the well-known fractional Van der Pol oscillator.

As discussed, in the case of negative ε, the two other predicted limit cycles are unstable and will not be observed in the numerical simulations. Figure 5 is sketched for a negative value of ε.

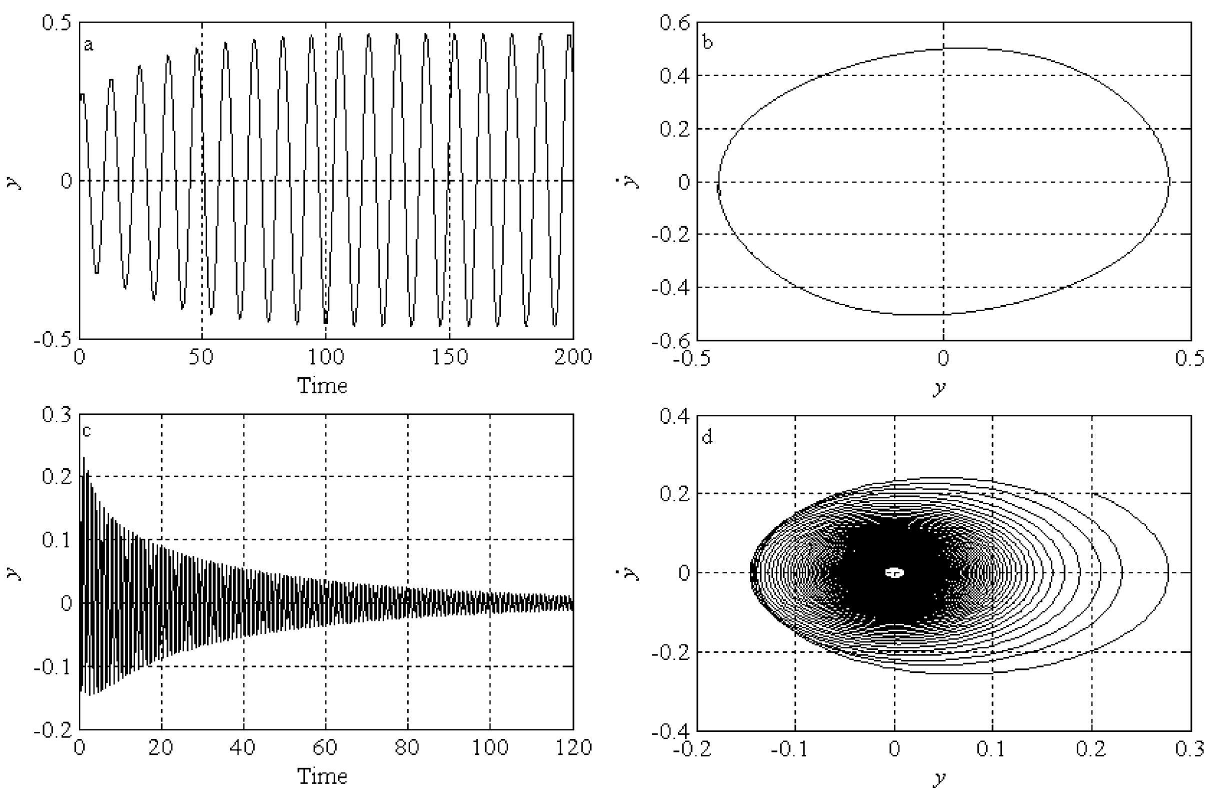

Fig. 3 Time response and state transition of system (6) for $\mu = 0.8$, $\varepsilon = 1$: (**a**), (**b**) $q = 0.50$ (undamped oscillations), (**c**), (**d**) $q = 0.48$ (damped oscillations)

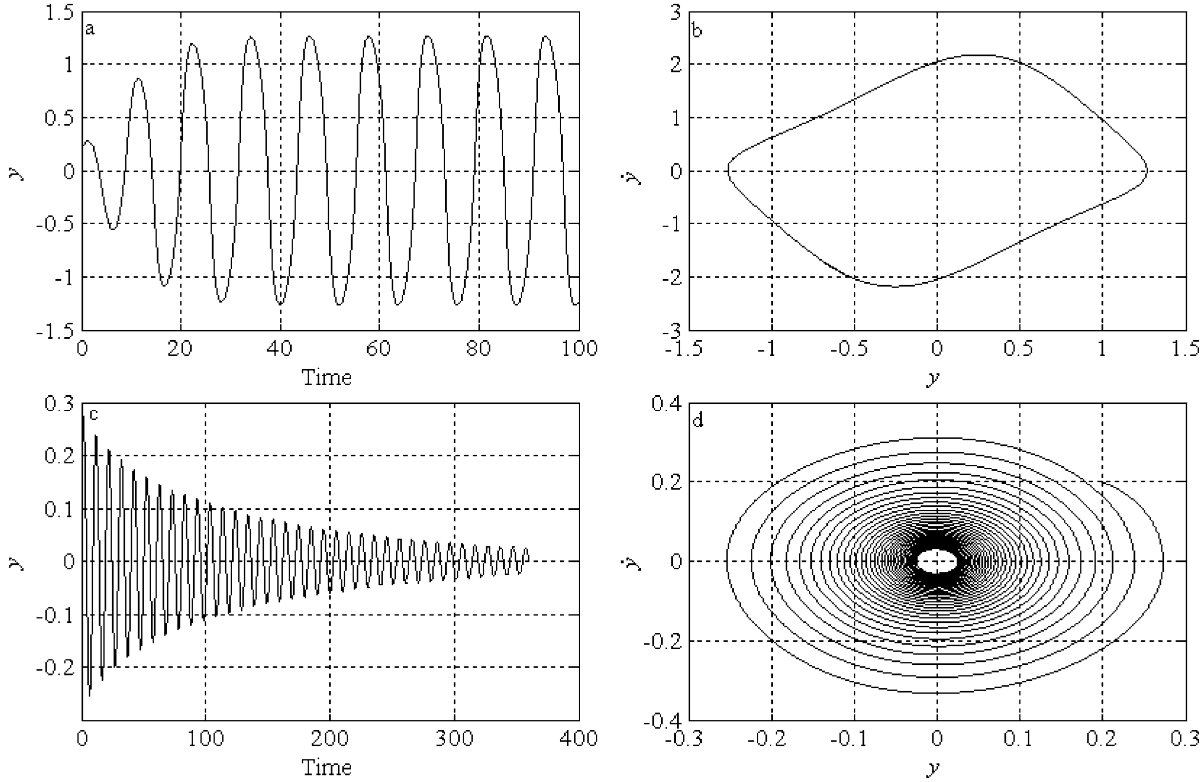

Fig. 4 Time response and state transition of system (6) for $\mu = 1$, $\varepsilon = 0$: (**a**), (**b**) $q = 0.36$ (undamped oscillations), (**c**), (**d**) $q = 0.34$ (damped oscillations)

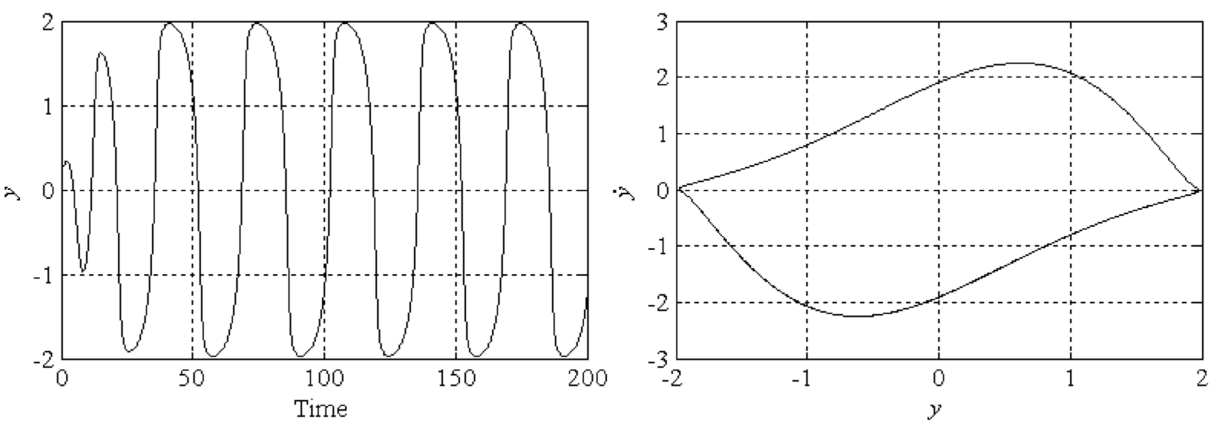

Fig. 5 Time response and state transition of system (6) for $\varepsilon = -0.7$, $\mu = 1$ and $q = 0.8$

5 Conclusion

In this paper, a describing function-based technique was proposed to determine the oscillatory parametric region of fractional order differential equations. This method had been previously introduced for predicting limit cycles in integer order systems. To illustrate the usefulness of the proposed technique in predicting the oscillatory parametric region of fractional order systems, this technique was applied to predict the oscillatory region of an extended version of the fractional order Van der Pol oscillator. The results are consistent with those of a previously introduced method in the literature based on stability theorems. The main advantage of the proposed method of this paper is that it is much simpler than the previously introduced methods. Based on this proposed method, the parametric range in which the fractional order Van der Pol model (6) can act as a fractional order oscillator was found.

Appendix

The method of this section is adopted from the method provided in [15]. The characteristic equation for the equilibrium point $y^* = 0$ is

$$w^{1+q} - \mu w + 1 = 0. \tag{A.1}$$

Supposing $q = v/u$, roots of (A.1) can be obtained from the following equation:

$$s^{u+v} - \mu s^u + 1 = 0. \tag{A.2}$$

The equilibrium point is asymptotically stable, if and only if the equation in (A.2) has no roots in the region $\{z \in \mathbb{C} \mid |\arg(z)| \leq \pi/2u\}$ [34]. Let us define $f(s) = s^{u+v} - \mu s^u + 1$. All coefficients of $f(s)$ are real numbers; thus, its roots are symmetrical with respect to the real axis. Thus, (A.2) has no roots in the region $\{z \in \mathbb{C} \mid |\arg(z)| \leq \pi/2u\}$ if and only if it has no roots in the region

$$R = \{z \in \mathbb{C} \mid 0 \leq \arg(z) \leq \pi/2u\}. \tag{A.3}$$

Therefore, the equilibrium point $y^* = 0$ is stable if and only if $f(R)$ excludes zero. Let β be the boundary of region R. This curve is composed of three curves as listed below.

$$\beta_1 = \{z \in \mathbb{C} \mid z = re^{j\theta},\ 0 \leq \pi/2u,\ r \to \infty\}, \tag{A.4}$$

$$\beta_2 = \{z \in \mathbb{C} \mid z = r,\ r \in \mathbb{R},\ r \geq 0\}, \tag{A.5}$$

$$\beta_3 = \{z \in \mathbb{C} \mid z = re^{j\pi/2u},\ r > 0\}. \tag{A.6}$$

$f(z)$ is an analytic function; therefore, examining $f(\beta_1)$, $f(\beta_2)$ and $f(\beta_2)$ is sufficient to determine $f(R)$. We have

$$f(\beta_1) = \{z \in \mathbb{C} \mid z = re^{j\theta},\ 0 \leq \theta \leq \pi/2 + \pi q/2,\ r \to \infty\}, \tag{A.7}$$

and

$$f(\beta_2) = \{z \in \mathbb{C} \mid z \in \mathbb{R},\ z \geq \lambda\}, \tag{A.8}$$

where $\lambda = \min\{f(z) \mid z \in \beta_2\}$. For $f(R)$ to exclude 0, λ should be a positive number. To find λ, we try to find the minimum point of $f(\beta_2)$. Equating $\partial f(z)/\partial z$ to zero, this point is obtained as follows:

$$z_0 = \left(\frac{\mu}{1+q}\right)^{1/v}. \tag{A.9}$$

Hence,

$$\lambda = f(z_0) = \left(\frac{\mu}{1+q}\right)^{\frac{u+v}{v}} - \mu\left(\frac{\mu}{1+q}\right)^{\frac{u}{v}} + 1. \tag{A.10}$$

Since $\lambda > 0$, with some manipulations, we get

$$\mu < \frac{1+q}{q^{q/(1+q)}}. \tag{A.11}$$

Now, let $z = re^{j\pi/2u} \in f(\beta_3)$. Then,

$$f\left(re^{j\pi/2u}\right) = \left(1 - r^{u+v}\sin(q\pi/2)\right) + j\left(r^{u+v}\cos(q\pi/2) - \mu r^u\right). \tag{A.12}$$

$f(\beta_3)$ intersects the real axis only once. In this point $\mathrm{Im}\{f(z)\} = 0 (z \in \beta_3)$, and therefore,

$$r^{u+v}\cos(q\pi/2) - \mu r^u = 0. \tag{A.13}$$

r is a positive value, thus (A.13) has a unique solution at

$$r_0 = \left(\frac{\mu}{\cos(q\pi/2)}\right)^{1/v}, \tag{A.14}$$

which shows that $f(\beta_3)$ intersects the real axis only once. The real value a is defined $f(r_0 e^{j\pi/2u})$. $f(\beta)$ will not encircle the zero point if and only if $\lambda > 0$ and $a > 0$. It has been shown that the inequality (A.11) is a result for $\lambda > 0$. With respect to (A.12) and (A.14) and assuming $a > 0$, one gets

$$a = f\left(r_0 e^{j\pi/2u}\right) = 1 - r_0^{u+v}\sin(q\pi/2) = 1 - \left(\frac{\mu}{\cos(q\pi/2)}\right)^{(u+v)/v}\sin(q\pi/2) > 0. \tag{A.15}$$

This requires that

$$\mu < \frac{(\sin(q\pi/2))^{1/(1+q)}}{\tan(q\pi/2)}. \tag{A.16}$$

The inequality (A.16) is stronger than (A.11) because

$$\frac{(\sin(q\pi/2))^{1/(1+q)}}{\tan(q\pi/2)} = \frac{\cos(q\pi/2)}{(\sin(q\pi/2))^{q/(1+q)}} \le \frac{1+q}{(\frac{2}{\pi}\frac{q\pi}{2})^{q/(1+q)}} = \frac{1+q}{q^{q/(1+q)}}. \tag{A.17}$$

The inequality in (A.16) is the reverse of inequality (28), which was proven to be the condition for the system in (6) to have undamped oscillation.

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