Nonlinear Dyn (2010) 61: 217–228
DOI 10.1007/s11071-009-9643-4

ORIGINAL PAPER

# Approximate conservation laws of perturbed partial differential equations

**Yani Gan · Changzheng Qu**

Received: 28 March 2009 / Accepted: 14 December 2009 / Published online: 7 January 2010


**Abstract** This paper presents a general result on approximate conservation laws of perturbed partial differential equations. A method of constructing approximate conservation laws to systems of perturbed partial differential equations is given, which is based on approximate Noether symmetries of approximate and standard adjoint systems of the original system. The relationship between the Noether symmetry operators of approximate and standard adjoint system is established. As a result, the approach is applied to the perturbed wave equation and the perturbed KdV equation.

**Keywords** Conservation law · Lagrangian · Noether symmetry · Adjoint system

## 1 Introduction

Conservation laws play an important role in the study of nonlinear partial differential equations (PDEs). In particular, they are useful for integrability and linearization, constants of motion, analysis of solutions and numerical methods of nonlinear PDEs. So the construction of conservation laws is a key subject of much discussion.

The definition of conservation laws itself gives rise to a method for finding conservation laws of nonlinear PDEs, which is referred to as the direct method [1]. It has been used successfully to construct conservation laws for several well-known nonlinear PDEs. However, for some problems, certain assumptions about conserved vector's form are made to find conservation laws. Noether [2] introduced an algorithm to construct conservation laws for systems arising from a Lagrangian formulation. This approach reduces the construction of conservation laws to finding variational symmetries for which there exist several well-developed methods. Nevertheless, it can be applied only to equations with Euler–Lagrange structure. In order to wide the scope of Noether's theorem, Steudel [3] introduced the so-called characteristic method by writing the conservation law in a characteristic form. In this approach, one has to find the related characteristics, which are multipliers of differential equations, to determine a conservation law. Later, Olver [4] discovered that the characteristics can be obtained by computing the variational derivative of the determining equations. The characteristics obtained here are given explicitly in terms of independent and dependent variables. In [5] and [6], Anco and Bluman provided a direct construction formulae of local conservation laws for the PDEs expressed in a standard Cauchy–Koralevskaya form. In [7], Kara and Mahomed pre-

Y. Gan · C. Qu (✉)
Center for Nonlinear Studies, Northwest University,
Xi'an 710069, China
e-mail: czqu@nwu.edu.cn

Y. Gan · C. Qu
Department of Mathematics, Northwest University,
Xi'an 710069, China

sented a partial Noether approach, which is efficient for Euler–Lagrange-type equations, i.e., any system, considered together with its adjoint equations, has a Lagrangian. More recently, Ibragimov [8] provided a formulae to construct nonlocal conservation laws with an auxiliary variable by using the Lie symmetries, Lie–Bäcklund symmetries and nonlocal symmetries of the above system.

Another vital aspect is that many nonlinear PDEs in applications depend on a small parameter. The notion of approximate conservation law has been introduced [14]. Usually, the method of constructing approximate conservation laws for perturbed PDEs is to find conservation laws for unperturbed PDEs. A question is whether there are other symmetry-related methods to obtain conservation laws for perturbed PDEs.

To this end, Kara, Mahomed, Ünal, and Johnpillai [9, 10, 13] developed the theory of the approximate symmetry group method to construct approximate conservation laws. In the sequel, Johnpillai, Kara, and Mahomed [11] extended the ideas of partial Lagrangian and Noether-type symmetry operator to a given system of PDEs with a small parameter, and introduced the partial Noether method to obtain approximate conservation laws, which is used only for approximate Euler–Lagrange-type equations. In this paper, we extend the method related to the adjoint equation to system with one or more perturbed PDEs. Then the original system, considered together with its standard or approximate adjoint equations (I), has an approximate Lagrangian. While it, considered together with its standard or approximate adjoint equations (II), has a Lagrangian. Thus, we can construct approximate conservation laws for the initial system. Furthermore, we shall establish the relationship between the approximate Noether symmetries of standard adjoint system and approximate adjoint system. Finally, we use the approach to obtain the approximate conservation laws for the perturbed wave equation and the KdV equation.

## 2 Preliminaries and notations

Let $x=(x^1,x^2,\dots,x^n)$ be the independent variable with coordinate $x^i$ and $u=(u^1,u^2,\dots,u^m)$ be the dependent variable with coordinate $u^\alpha$. The derivatives of $u^\alpha$ with respect to $x^i$ are $u_i^\alpha=D_i(u^\alpha)$, $u_{ij}^\alpha=D_jD_i(u^\alpha),\dots,$ where

$$D_i=\frac{\partial}{\partial x^i}+u_i^\alpha\frac{\partial}{\partial u^\alpha}+u_{ij}^\alpha\frac{\partial}{\partial u_j^\alpha}+\cdots,\quad i=1,\dots,n$$

is a total differentiation operator with respect to $x^i$, the summation convention is adopted throughout the paper. Consider an $r$th-order system of perturbed PDEs of $n$ independent variables $x=(x^1,x^2,\dots,x^n)$ and $m$ dependent variables $u=(u^1,u^2,\dots,u^m)$ with a small parameter $\varepsilon$

$$E^\beta(x,u,u_{(1)},\dots,u_{(r)};\varepsilon)=0,\quad \beta=1,2,\dots,m, \tag{2.1}$$

where $u_{(1)},u_{(2)},\dots,u_{(r)}$ denote the collections of all first, second, …, $r$th-order partial derivatives.

**Definition 1** [12] The Euler–Lagrange operator, for each $\alpha$, is defined by

$$\frac{\delta}{\delta u^\alpha}=\frac{\partial}{\partial u^\alpha}+\sum_{s\ge1}(-1)^sD_{i_1}\cdots D_{i_s}\frac{\partial}{\partial u^\alpha_{i_1\cdots i_s}},\quad \alpha=1,2,\dots,m.$$

**Definition 2** [14] The $k$th-order approximate Lie–Bäcklund symmetry operator is given by

$$\chi=X_0+\varepsilon X_1+\cdots+\varepsilon^kX_k, \tag{2.2}$$

where

$$X_b=\xi_b^i\frac{\partial}{\partial x^i}+\eta_b^\alpha\frac{\partial}{\partial u^\alpha}+\zeta_{b,i}^\alpha\frac{\partial}{\partial u_i^\alpha}+\cdots,\quad b=0,1,\dots,k$$

is a Lie–Bäcklund symmetry operator, the coefficients are $\zeta_{b,i}^\alpha=D_i(W_b^\alpha)+\xi_b^ju_{ij}^\alpha$, and $W_b^\alpha$ is the Lie characteristic function defined as $W_b^\alpha=\eta_b^\alpha-\xi_b^ju_j^\alpha$. The characteristic form of the approximate Lie–Bäcklund symmetry operator (2.2) is

$$\chi=\xi^iD_i+\mathcal{W}^\alpha\frac{\partial}{\partial u^\alpha}+\sum_{s\ge1}D_{i_1}\cdots D_{i_s}(\mathcal{W}^\alpha)\frac{\partial}{\partial u^\alpha_{i_1\cdots i_s}},$$

where $\mathcal{W}=(\mathcal{W}^1,\mathcal{W}^2,\dots,\mathcal{W}^m)$ is the characteristic of $\chi$, and

$$\mathcal{W}^i=W_0^i+\varepsilon W_1^i+\cdots+\varepsilon^kW_k^i,\quad i=1,2,\dots,m. \tag{2.3}$$

We can use the abbreviated form of (2.2)

$$\chi = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha},$$

where $\xi^i$ and $\eta^\alpha$ are given respectively by

$$\xi^i = \xi_0^i + \varepsilon \xi_1^i + \cdots + \varepsilon^k \xi_k^i, \quad i = 1, \ldots, n,$$

and

$$\eta^\alpha = \eta_0^\alpha + \varepsilon \eta_1^\alpha + \cdots + \varepsilon^k \eta_k^\alpha, \quad \alpha = 1, \ldots, m.$$

**Definition 3** [11] The approximate Noether operator associated with an approximate Lie–Bäcklund operator $\chi$ is given by

$$\mathcal{N}^i = \xi^i + \mathcal{W}^\alpha \frac{\delta}{\delta u_i^\alpha} + \sum_{s \geq 1} D_{i_1} \cdots D_{i_s}\big(\mathcal{W}^\alpha\big) \frac{\delta}{\delta u^\alpha_{ii_1 \cdots i_s}}, \quad i = 1, 2, \ldots, n.$$

It was known [12] that the Euler–Lagrange, approximate Lie–Bäcklund and approximate Noether operators are connected by the operator identity

$$\chi + D_i\big(\xi^i\big) = \mathcal{W}^\alpha \frac{\delta}{\delta u^\alpha} + D_i \mathcal{N}^i. \tag{2.4}$$

**Definition 4** [14] A vector $\mathcal{T} = (\mathcal{T}^1, \ldots, \mathcal{T}^n)$ defined as

$$\mathcal{T}^i = T_0^i + \varepsilon T_1^i + \cdots + \varepsilon^k T_k^i$$

is an approximate conserved vector of (2.1) if the approximate equation

$$D_i \mathcal{T}^i = O\big(\varepsilon^{k+1}\big) \tag{2.5}$$

is satisfied for all approximate solutions of (2.1). Equation (2.5) is said to be an approximate conservation law for (2.1).

**Definition 5** [11] The equation $D_i \mathcal{T}^i = \mathcal{Q}_\beta E^\beta$ is referred to as the approximate characteristic form of the approximate conservation law (2.5), and the function $\mathcal{Q} = (\mathcal{Q}_1, \mathcal{Q}_2, \ldots, \mathcal{Q}_m)$ is the associated characteristic of the approximate conservation law.

Assume that (2.1) can be written as

$$\begin{aligned} E^\beta &= E_0^\beta(x, u, u_{(1)}, \ldots, u_{(r)}) \\ &\quad + \varepsilon E_1^\beta(x, u, u_{(1)}, \ldots, u_{(p)}), \\ \beta &= 1, 2, \ldots, m, \end{aligned} \tag{2.6}$$

where $E_1^\beta = E_1^{0\beta}(x, u, u_{(1)}, \ldots, u_{(p)}) + E_1^{1\beta}(x, u)$, $p < r$.

**Definition 6** If there exists a function $L = L(x, u, u_{(1)}, \ldots, u_{(l)})$, $l \leq r$ and nonzero functions $f_\gamma^\beta$ such that (2.6) can be written as $\delta L / \delta u^\beta = \varepsilon f_\gamma^\beta E_1^\gamma$, where $f_\gamma^\beta = f_\gamma^\beta(x, u, u_{(1)}, \ldots, u_{(r-1)})$, $\beta, \gamma = 1, 2, \ldots, m$ is an invertible matrix [12], $L$ is called a Lagrangian of (2.6). If there exists a function

$$\begin{aligned} \mathcal{L} &= L_0(x, u, u_{(1)}, \ldots, u_{(l)}) \\ &\quad + \varepsilon L_1(x, u, u_{(1)}, \ldots, u_{(l)}), \quad l \leq r, \end{aligned}$$

and nonzero functions $g_\gamma^\beta$ such that (2.6) can be written as $\delta \mathcal{L} / \delta u^\beta = \varepsilon g_\gamma^\beta E_1^{1\gamma}$, where $g_\gamma^\beta = g_\gamma^\beta(x, u, u_{(1)}, \ldots, u_{(r-1)})$, $\beta, \gamma = 1, 2, \ldots, m$ is an invertible matrix [12], $\mathcal{L}$ is called an approximate Lagrangian of (2.6). We call differential equations of the form

$$\frac{\delta L}{\delta u^\beta} = \varepsilon f_\gamma^\beta E_1^\gamma, \qquad \frac{\delta \mathcal{L}}{\delta u^\beta} = \varepsilon g_\gamma^\beta E_1^{1\gamma} \tag{2.7}$$

the approximate Euler–Lagrange-type equations.

**Definition 7** An approximate Lie–Bäcklund symmetry operator $\chi$ of the form (2.2) is called an approximate Noether symmetry operator corresponding to a Lagrangian $L$ (approximate Lagrangian $\mathcal{L}$) if there exists a vector $\mathcal{B} = (\mathcal{B}^1, \ldots, \mathcal{B}^m)$ defined by

$$\mathcal{B}^i = B_0^i + \varepsilon B_1^i + \cdots + \varepsilon^k B_k^i, \quad i = 1, \ldots, m,$$

such that

$$\begin{aligned} &\chi\big(L(\mathcal{L})\big) + L(\mathcal{L}) D_i\big(\xi^i\big) \\ &\quad = \mathcal{W}^\alpha \frac{\delta L(\mathcal{L})}{\delta u^\alpha} + D_i\big(\mathcal{B}^i\big) + O\big(\varepsilon^{k+1}\big), \end{aligned}$$

where $\mathcal{W} = (\mathcal{W}^1, \ldots, \mathcal{W}^m)$ is the characteristic of $\chi$ as given by (2.3).

**Definition 8** Consider a system of $r$th-order approximate PDEs (2.6), its *standard*, *approximate adjoint* *equations* (I) and (II) are given respectively by

$$F^\alpha(x,u,v,u_{(1)},v_{(1)},\dots,u_{(r)},v_{(r)})=\frac{\delta(v^\beta E^\beta)}{\delta u^\alpha}=0,\quad \alpha=1,\dots,m,$$

$$G_0^\alpha(x,u,v,u_{(1)},v_{(1)},\dots,u_{(r)},v_{(r)})=\frac{\delta(v^\beta(E_0^\beta+\varepsilon E_1^{0\beta}))}{\delta u^\alpha}=0,\quad \alpha=1,\dots,m,$$

and

$$G_1^\alpha(x,u,v,u_{(1)},v_{(1)},\dots,u_{(r)},v_{(r)})=\frac{\delta(v^\beta E_0^\beta)}{\delta u^\alpha}=0,\quad \alpha=1,\dots,m,$$

where $v=(v^1,\dots,v^m)$ are new dependent variables, $v=v(x)$. Here, we call

$$\begin{cases} E_0^\beta+\varepsilon E_1^\beta=0, \quad \beta=1,\dots,m,\\ F^\alpha(x,u,v,u_{(1)},v_{(1)},\dots,u_{(r)},v_{(r)})=0,\\ \quad \alpha=1,\dots,m,\end{cases}$$

the *standard adjoint system*

$$\begin{cases} E_0^\beta+\varepsilon E_1^\beta=0, \quad \beta=1,\dots,m,\\ G_0^\alpha(x,u,v,u_{(1)},v_{(1)},\dots,u_{(r)},v_{(r)})=0,\\ \quad \alpha=1,\dots,m,\end{cases}$$

the *approximate adjoint system (I)*, and

$$\begin{cases} E_0^\beta+\varepsilon E_1^\beta=0, \quad \beta=1,\dots,m,\\ G_1^\alpha(x,u,v,u_{(1)},v_{(1)},\dots,u_{(r)},v_{(r)})=0,\\ \quad \alpha=1,\dots,m,\end{cases}$$

the *approximate adjoint system (II)*.

**Lemma 1** [11] *An approximate Lie–Bäcklund symmetry operator $\chi$ of the form* (2.2) *is an approximate Noether symmetry operator of a partial Lagrangian $L$ corresponding to an approximate Euler–Lagrange system of the form* (2.7) *iff the characteristic $\mathcal{W}=(\mathcal{W}^1,\dots,\mathcal{W}^m)$ of $\chi$ is also the characteristic of the conservation law $D_i\mathcal{T}^i=O(\varepsilon^{k+1})$, where*

$$\mathcal{T}^i=\mathcal{B}^i-\mathcal{N}^iL+O(\varepsilon^{k+1}),\quad i=1,2,\dots,n.$$

## 3 Main results

In this section, we present main results of deriving approximate conservation laws for perturbed nonlinear PDEs, which are given in the following theorems.

**Theorem 1** *For any system of perturbed differential equations* (2.6), *the standard adjoint system and the approximate adjoint system (I) has an approximate Lagrangian given by*

$$\mathcal{L}=v^\alpha E_0^\alpha+\varepsilon v^\alpha E_1^{0\alpha};\tag{3.1}$$

*while the standard adjoint system and approximate adjoint system (II) has a Lagrangian given by*

$$L=v^\alpha E_0^\alpha.\tag{3.2}$$

*Proof* Indeed, a direct computation shows that the variation of $\mathcal{L}$ given by (3.1) yields the following results: The standard adjoint system is equivalent to

$$\begin{cases} \frac{\delta\mathcal{L}}{\delta v^\beta}=-\varepsilon E_1^{1\beta}, \quad \beta=1,\dots,m,\\ \frac{\delta\mathcal{L}}{\delta u^\beta}=-\varepsilon\frac{\delta(v^\alpha E_1^{1\alpha})}{\delta u^\beta}, \quad \beta=1,\dots,m,\end{cases}$$

and the approximate adjoint system (I) is equivalent to

$$\begin{cases} \frac{\delta\mathcal{L}}{\delta v^\beta}=-\varepsilon E_1^{1\beta}, \quad \beta=1,\dots,m,\\ \frac{\delta\mathcal{L}}{\delta u^\beta}=0, \quad \beta=1,\dots,m.\end{cases}$$

Also, the variation of $L$ given by (3.2) shows that the standard adjoint system is equivalent to

$$\begin{cases} \frac{\delta L}{\delta v^\beta}=-\varepsilon E_1^{\beta}, \quad \beta=1,\dots,m,\\ \frac{\delta L}{\delta u^\beta}=-\varepsilon\frac{\delta(v^\alpha E_1^{\alpha})}{\delta u^\beta}, \quad \beta=1,\dots,m,\end{cases}$$

and the approximate adjoint system (II) is equivalent to

$$\begin{cases} \frac{\delta L}{\delta v^\beta}=-\varepsilon E_1^{\beta}, \quad \beta=1,\dots,m,\\ \frac{\delta L}{\delta u^\beta}=0, \quad \beta=1,\dots,m.\end{cases}$$

Clearly, $\mathcal{L}=v^\alpha E_0^\alpha+\varepsilon v^\alpha E_1^{0\alpha}$ is the approximate Lagrangian of the standard adjoint system and approximate adjoint system (I), while $L=v^\alpha E_0^\alpha$ is the Lagrangian of the standard adjoint system and approximate adjoint system (II) by Definition 6. □

**Theorem 2** *Let $X_0 + \varepsilon X_1$ be a first-order approximate Noether symmetry operator corresponding to the approximate Lagrangian $\mathcal{L} = L_0 + \varepsilon L_1$ of standard adjoint system, and $Y_0 + \varepsilon Y_1$ be first-order approximate Noether symmetry operator corresponding to the approximate Lagrangian $\mathcal{L} = L_0 + \varepsilon L_1$ of the approximate adjoint system (I). Then $X_0$ and $Y_0$ have the same forms. Furthermore, if $X_0 = Y_0$, then the following identity holds*

$$(X_1 - Y_1)L_0 + L_0 D_i\big(\xi_1^i - \xi_1^{i,y}\big) = -\big(\eta_0^\alpha - \xi_0^i u_i^\alpha\big)\frac{\delta(v^\beta E_1^{1\beta})}{\delta u^\alpha} + D_i\big(B_1^i - B_1^{i,y}\big), \tag{3.3}$$

*where $\xi_1^{i,y}$ and $B_1^{i,y}$ correspond to $Y_1$.*

*Proof* Since $X_0 + \varepsilon X_1$ is a first order approximate Noether symmetry operator of standard adjoint system and $Y_0 + \varepsilon Y_1$ is the first-order approximate Noether symmetry operator of the approximate adjoint system (I), then $X_0 + \varepsilon X_1$ and $Y_0 + \varepsilon Y_1$ satisfy the following equations:

$$(X_0 + \varepsilon X_1)(L_0 + \varepsilon L_1) + (L_0 + \varepsilon L_1) D_i\big(\xi_0^i + \varepsilon \xi_1^i\big) = -\varepsilon \mathcal{W}^\alpha \frac{\delta(v^\beta E_1^{1\beta})}{\delta u^\alpha} - \varepsilon \mathcal{V}^\alpha E_1^{1\alpha} + D_i\big(B_0^i + \varepsilon B_1^i\big) + O\big(\varepsilon^2\big),$$

$$(Y_0 + \varepsilon Y_1)(L_0 + \varepsilon L_1) + (L_0 + \varepsilon L_1) D_i\big(\xi_0^{i,y} + \varepsilon \xi_1^{i,y}\big) = \mathcal{V}^{\alpha,y}\big(-\varepsilon E_1^{1\alpha}\big) + D_i\big(B_0^{i,y} + \varepsilon B_1^{i,y}\big) + O\big(\varepsilon^2\big),$$

where $(\mathcal{W}^1, \ldots, \mathcal{W}^m, \mathcal{V}^1, \ldots, \mathcal{V}^m)$ is the characteristic of $X_0 + \varepsilon X_1$, and $(\mathcal{W}^{1,y}, \ldots, \mathcal{W}^{m,y}, \mathcal{V}^{1,y}, \ldots, \mathcal{V}^{m,y})$ is the characteristic of $Y_0 + \varepsilon Y_1$. Separation by the zeroth and first-orders of $\varepsilon$ for the above two equations gives rise to the following equations:

$$X_0 L_0 + L_0 D_i\big(\xi_0^i\big) = D_i B_0^i,$$

$$Y_0 L_0 + L_0 D_i\big(\xi_0^{i,y}\big) = D_i B_0^{i,y},$$

$$X_0 L_1 + X_1 L_0 + L_0 D_i\big(\xi_1^i\big) + L_1 D_i\big(\xi_0^i\big) = -W_0^\alpha \frac{\delta(v^\beta E_1^{1\beta})}{\delta u^\alpha} - V_0^\alpha E_1^{1\alpha} + D_i B_1^i,$$

$$Y_0 L_1 + Y_1 L_0 + L_0 D_i\big(\xi_1^{i,y}\big) + L_1 D_i\big(\xi_0^{i,y}\big) = -V_0^{\alpha,y} E_1^{1\alpha} + D_i B_1^{i,y}.$$

The above equations imply that the determining equations for $X_0$ and $Y_0$ are the same. So $X_0$ and $Y_0$ have the similar forms. Furthermore, substituting $X_0$ and $Y_0$ respectively into the determining equations of $X_1$ and $Y_1$, i.e. the third and fourth equations, and comparing the two equations, we arrive at the following identity

$$(X_1 - Y_1)L_0 + L_0 D_i\big(\xi_1^i - \xi_1^{i,y}\big) = -W_0^\alpha \frac{\delta(v^\beta E_1^{1\beta})}{\delta u^\alpha} + D_i\big(B_1^i - B_1^{i,y}\big),$$

where $W_0^\alpha = \eta_0^\alpha - \xi_0^i u_i^\alpha$. Consequently, the identity (3.3) holds. □

*Remark* The result similar to the above theorem is also true for the standard adjoint system, approximate adjoint system (II) and Lagrangian $L$.

**Theorem 3** *If the approximate Lie–Bäcklund symmetry operator $\chi$ of the form (2.2) is an approximate Noether symmetry operator of an approximate Lagrangian $\mathcal{L}$ corresponding to an approximate Euler–Lagrange-type system of the form (2.7), then the conserved vector of (2.7) is given by*

$$\mathcal{T}^i = \mathcal{B}^i - \mathcal{N}^i \mathcal{L} + O\big(\varepsilon^2\big), \quad i = 1, 2, \ldots, n.$$

*Proof* We prove the theorem along with the lines of [7]. Utilizing identity (2.4) and acting with it on $\mathcal{L}$, we obtain

$$\chi \mathcal{L} + D_i\big(\xi^i\big)\mathcal{L} = \mathcal{W}^\alpha \frac{\delta \mathcal{L}}{\delta u^\alpha} + D_i \mathcal{N}^i \mathcal{L}.$$

Since $\chi$ is an approximate Noether symmetry corresponding to $\mathcal{L}$, we have

$$\chi \mathcal{L} + D_i\big(\xi^i\big)\mathcal{L} = \mathcal{W}^\alpha g_\gamma^\alpha E_1^{1\gamma} + D_i\big(\mathcal{B}^i\big).$$

From the above two identities and the notion of conservation law, it is easy to see that $\mathcal{T}^i = \mathcal{B}^i - \mathcal{N}^i \mathcal{L} + O(\varepsilon^2)$, $i = 1, 2, \ldots, n$ are the conserved components. □

*Remark* Considering (2.6) together with its standard, approximate adjoint equations (I) or (II), we could obtain the approximate conservation laws by Lemma 1 and Theorem 3, which involves an arbitrary function $v(x)$ satisfying standard approximate adjoint equations (I) or (II).

## 4 Examples

For simplicity, we only consider the first-order approximate Noether symmetries, and we assume that $\xi_b^i$, $\eta_b^i$ and $B_b^i$ are functions of independent and dependent variables.

*Example 1* Consider the perturbed wave equation

$$u_{tt} - u_{xx} + \varepsilon\big(u^m u_t - au + bu^p\big) = 0. \tag{4.1}$$

Its standard adjoint system is

$$\begin{cases} u_{tt} - u_{xx} + \varepsilon(u^m u_t - au + bu^p) = 0, \\ v_{tt} - v_{xx} + \varepsilon(-u^m v_t - av + bpu^{p-1}v) = 0, \end{cases} \tag{4.2}$$

the approximate adjoint system (I) is

$$\begin{cases} u_{tt} - u_{xx} + \varepsilon(u^m u_t - au + bu^p) = 0, \\ v_{tt} - v_{xx} + \varepsilon(-u^m v_t) = 0, \end{cases}$$

and the approximate adjoint system (II) is

$$\begin{cases} u_{tt} - u_{xx} + \varepsilon(u^m u_t - au + bu^p) = 0, \\ v_{tt} - v_{xx} = 0. \end{cases}$$

First, we consider the standard adjoint system (4.2). It is easily seen that the system (4.2) has an approximate Lagrangian $\mathcal{L} = L_0 + \varepsilon L_1 = v(u_{tt} - u_{xx}) + \varepsilon(u^m u_t v)$ by Theorem 1. Then the approximate Euler–Lagrange-type equations are

$$\frac{\delta \mathcal{L}}{\delta u} = \varepsilon\big(a - bpu^{p-1}\big)v$$

and

$$\frac{\delta \mathcal{L}}{\delta v} = \varepsilon\big(au - bu^p\big).$$

It follows that the first-order approximate Noether symmetry operator $X_0 + \varepsilon X_1$ corresponding to the approximate Lagrangian $\mathcal{L}$ must satisfy the following equation

$$\begin{aligned} &(X_0 + \varepsilon X_1)\mathcal{L} + \mathcal{L} D_i\big(\xi_0^i + \varepsilon \xi_1^i\big) \\ &\quad = \varepsilon\big(a - bpu^{p-1}\big)v\big[\big(\eta_0^1 - \xi_0^1 u_t - \xi_0^2 u_x\big) \\ &\qquad + \varepsilon\big(\eta_1^1 - \xi_1^1 u_t - \xi_1^2 u_x\big)\big] \\ &\qquad + \varepsilon\big(au - bu^p\big)\big[\big(\eta_0^2 - \xi_0^1 v_t - \xi_0^2 v_x\big) \\ &\qquad + \varepsilon\big(\eta_1^2 - \xi_1^1 v_t - \xi_1^2 v_x\big)\big] \\ &\qquad + D_i\big(B_0^i + \varepsilon B_1^i\big) + O\big(\varepsilon^2\big). \end{aligned}$$

Separating by the zeroth and first-orders of $\varepsilon$ yields the following equations:

$$X_0 L_0 + L_0 D_i\big(\xi_0^i\big) = D_i B_0^i, \tag{4.3}$$

$$\begin{aligned} &X_1 L_0 + X_0 L_1 + L_1 D_i\big(\xi_0^i\big) + L_0 D_i\big(\xi_1^i\big) \\ &\quad = D_i B_1^i + \big(\eta_0^1 - \xi_0^1 u_t - \xi_0^2 u_x\big)\big(a - bpu^{p-1}\big)v \\ &\qquad + \big(\eta_0^2 - \xi_0^1 v_t - \xi_0^2 v_x\big)\big(au - bu^p\big). \end{aligned} \tag{4.4}$$

The determining equation (4.3) can be written as

$$\begin{aligned} &\eta_0^2(u_{tt} - u_{xx}) + \zeta_{0,tt}^1 v - \zeta_{0,xx}^1 v \\ &\qquad + \big(D_t\xi_0^1 + D_x\xi_0^2\big)v(u_{tt} - u_{xx}) \\ &\quad = D_t B_0^1 + D_x B_0^2. \end{aligned} \tag{4.5}$$

Equating to zero the coefficients of the derivatives of $u$, $v$, and the free term, we obtain the following determining equations:

$$\begin{aligned} &\xi_0^1 = \xi_0^1(t,x), \qquad \xi_0^2 = \xi_0^2(t,x), \\ &\eta_0^1 = \eta_0^1(t,x,u), \\ &\xi_{0t}^1 = \xi_{0x}^2, \qquad \xi_{0x}^1 = \xi_{0t}^2, \\ &\eta_0^2 = -v\eta_{0u}^1, \qquad \eta_{0uu}^1 = 0, \\ &v\big(2\eta_{0tu}^1 - \xi_{0tt}^1 + \xi_{0xx}^1\big) = B_{0u}^1, \\ &v\big(2\eta_{0xu}^1 - \xi_{0xx}^2 + \xi_{0tt}^2\big) = -B_{0u}^2, \\ &B_{0v}^1 = 0, \qquad B_{0v}^2 = 0, \\ &v\big(\eta_{0tt}^1 - \eta_{0xx}^1\big) = B_{0t}^1 + B_{0x}^2. \end{aligned}$$

The solutions of the above system are

$$\begin{aligned} &\xi_0^1 = \xi_0^1(t,x), \qquad \xi_0^2 = \xi_0^2(t,x), \\ &\eta_0^1 = cu + d(t,x), \qquad \eta_0^2 = -cv, \\ &B_0^1 = B_0^1(t,x), \qquad B_0^2 = B_0^2(t,x), \end{aligned}$$

which satisfy

$$\begin{aligned} &\xi_{0t}^1 = \xi_{0x}^2, \qquad \xi_{0x}^1 = \xi_{0t}^2, \\ &d_{tt} = d_{xx}, \qquad B_{0t}^1 + B_{0x}^2 = 0, \end{aligned}$$

where $c$ is an arbitrary constant. Setting $B_0^1 = B_0^2 = 0$, the Lie-Bäcklund symmetry operator $X_0$ is given by

$$X_0^1 = \frac{\partial}{\partial t}, \qquad X_0^2 = \frac{\partial}{\partial x},$$

$$X_0^3 = t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x}, \qquad X_0^4 = u\frac{\partial}{\partial u} - v\frac{\partial}{\partial v}.$$

Next, we consider each case, respectively.

(1). $X_0 = \partial/\partial t$. Equation (4.4) becomes

$$\begin{aligned}&\eta_1^2(u_{tt} - u_{xx}) + \zeta^1_{1,tt}v - \zeta^1_{1,xx}v \\ &\quad + \big(D_t\xi_1^1 + D_x\xi_1^2\big)v(u_{tt} - u_{xx}) \\ &= D_tB_1^1 + D_xB_1^2 - u_t\big(a - bpu^{p-1}\big)v \\ &\quad - v_t\big(au - bu^p\big).\end{aligned}$$

Solving it, one gets $B_1^1 = (au - bu^p)v$, $B_1^2 = 0$ and $X_1$ is given by

$$X_1^1 = \frac{\partial}{\partial t}, \qquad X_1^2 = \frac{\partial}{\partial x},$$
$$X_1^3 = t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x}, \qquad X_1^4 = u\frac{\partial}{\partial u} - v\frac{\partial}{\partial v}.$$

Hence, we have the following approximate Noether symmetry operators for the system (4.2):

$$\chi^1 = \frac{\partial}{\partial t} + \varepsilon\frac{\partial}{\partial t}, \qquad \chi^2 = \frac{\partial}{\partial t} + \varepsilon\frac{\partial}{\partial x},$$
$$\chi^3 = \frac{\partial}{\partial t} + \varepsilon\left(t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x}\right),$$
$$\chi^4 = \frac{\partial}{\partial t} + \varepsilon\left(u\frac{\partial}{\partial u} - v\frac{\partial}{\partial v}\right).$$

Using Theorem 3, we obtain the approximate conserved vectors corresponding to $\chi^1$, $\chi^2$, $\chi^3$, and $\chi^4$ given respectively by

$$\begin{aligned}\mathcal{T}_1 = \Big(&(u_{xx}v - u_tv_t) \\ &+ \varepsilon\big[\big(au - bu^p\big)v + u_{xx}v - u_tv_t\big], \\ &(1+\varepsilon)(-u_{tx}v + u_tv_x)\Big),\end{aligned}$$

$$\begin{aligned}\mathcal{T}_2 = \Big(&u_{xx}v - u_tv_t + \varepsilon\big[u_{tx}v - u_xv_t + \big(au - bu^p\big)v\big], \\ &u_tv_x - u_{tx}v + \varepsilon(u_xv_x - u_{tt}v)\Big),\end{aligned}$$

$$\begin{aligned}\mathcal{T}_3 = \Big(&u_{xx}v - u_tv_t + \varepsilon\big[\big(au - bu^p\big)v + tu_{xx}v \\ &- (tu_t + xu_x)v_t + (u_t + xu_{tx})v\big], \\ &u_tv_x - u_{tx}v + \varepsilon\big[-xu_{tt}v + (tu_t + xu_x)v_x \\ &- (tu_{tx} + u_x)v\big]\Big),\end{aligned}$$

$$\begin{aligned}\mathcal{T}_4 = \Big(&u_{xx}v - u_tv_t + \varepsilon\big[\big(au - bu^p\big)v + uv_t - u_tv\big], \\ &u_tv_x - u_{tx}v + \varepsilon(u_xv - uv_x)\Big).\end{aligned}$$

(2). $X_0 = \partial/\partial x$. In this case, (4.4) becomes

$$\begin{aligned}&\eta_1^2(u_{tt} - u_{xx}) + \zeta^1_{1,tt}v - \zeta^1_{1,xx}v \\ &\quad + \big(D_t\xi_1^1 + D_x\xi_1^2\big)v(u_{tt} - u_{xx}) \\ &= D_tB_1^1 + D_xB_1^2 - u_x\big(a - bpu^{p-1}\big)v \\ &\quad - v_x\big(au - bu^p\big).\end{aligned}$$

Applying the same procedure, we obtain the following approximate Noether symmetry operators for system (4.2):

$$\chi^5 = \frac{\partial}{\partial x} + \varepsilon\frac{\partial}{\partial t}, \qquad \chi^6 = \frac{\partial}{\partial x} + \varepsilon\frac{\partial}{\partial x},$$
$$\chi^7 = \frac{\partial}{\partial x} + \varepsilon\left(t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x}\right),$$
$$\chi^8 = \frac{\partial}{\partial x} + \varepsilon\left(u\frac{\partial}{\partial u} - v\frac{\partial}{\partial v}\right),$$

and $B_1^1 = 0$, $B_1^2 = (au - bu^p)v$. The corresponding approximate conserved vectors in this case are

$$\begin{aligned}\mathcal{T}_5 = \Big(&u_{tx}v - u_xv_t + \varepsilon\big[u_{xx}v - u_tv_t + u^mu_xv\big], \\ &u_xv_x - u_{tt}v + \varepsilon\big[\big(au - bu^p\big)v + u_tv_x \\ &- u_{tx}v - u^mu_tv\big]\Big),\end{aligned}$$

$$\begin{aligned}\mathcal{T}_6 = \Big(&u_{tx}v - u_xv_t + \varepsilon\big(u_{tx}v - u_xv_t + u^mu_xv\big), \\ &u_xv_x - u_{tt}v + \varepsilon\big[\big(au - bu^p\big)v + u_xv_x \\ &- u_{tt}v - u^mu_tv\big]\Big),\end{aligned}$$

$$\begin{aligned}\mathcal{T}_7 = \Big(&u_{tx}v - u_xv_t + \varepsilon\big[tu_{xx}v - (tu_t + xu_x)v_t \\ &+ \big(u_t + xu_{tx} + u^mu_x\big)v\big], \\ &u_xv_x - u_{tt}v + \varepsilon\big[\big(au - bu^p\big)v - xu_{tt}v \\ &+ (tu_t + xu_x)v_x - \big(tu_{tx} + u_x + u^mu_t\big)v\big]\Big),\end{aligned}$$

$$\mathcal{T}_8 = \Big(u_{tx}v - u_xv_t + \varepsilon\big(uv_t - u_tv + u^mu_xv\big),$$

$$u_x v_x - u_{tt} v + \varepsilon\big[(au - bu^p)v - uv_x + u_x v - u^m u_t v\big]\Big).$$

(3). $X_0 = t\partial/\partial t + x\partial/\partial x$. Equation (4.4) turns to be

$$\begin{aligned}
&\eta_1^2(u_{tt} - u_{xx}) + \zeta^1_{1,tt} v - \zeta^1_{1,xx} v \\
&\quad + \big(D_t\xi_1^1 + D_x\xi_1^2\big)v(u_{tt} - u_{xx}) + u^m u_t v \\
&= D_t B_1^1 + D_x B_1^2 - (tu_t + xu_x)\big(a - bpu^{p-1}\big)v \\
&\quad - (tv_t + xv_x)\big(au - bu^p\big).
\end{aligned}$$

A similar computation shows that the above determining equations have solutions if and only if $m = 0$, $au = bu^p$. The solution is now given by

$$\begin{aligned}
X_1^1 &= \frac{\partial}{\partial t} - \frac{1}{2}tu\frac{\partial}{\partial u} + \frac{1}{2}tv\frac{\partial}{\partial v}, \\
X_1^2 &= \frac{\partial}{\partial x} - \frac{1}{2}t\left(u\frac{\partial}{\partial u} - v\frac{\partial}{\partial v}\right), \\
X_1^3 &= t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} - \frac{1}{2}t\left(u\frac{\partial}{\partial u} - v\frac{\partial}{\partial v}\right), \\
X_1^4 &= \left(1 - \frac{1}{2}t\right)\left(u\frac{\partial}{\partial u} - v\frac{\partial}{\partial v}\right),
\end{aligned}$$

and $B_1^1 = B_1^2 = 0$. Hence, for the system,

$$\begin{cases} u_{tt} - u_{xx} + \varepsilon u_t = 0, \\ v_{tt} - v_{xx} - \varepsilon v_t = 0, \end{cases}$$

we obtain the following approximate Noether symmetry operators

$$\begin{aligned}
\chi^9 &= t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + \varepsilon\left[\frac{\partial}{\partial t} - \frac{1}{2}t\left(u\frac{\partial}{\partial u} - v\frac{\partial}{\partial v}\right)\right], \\
\chi^{10} &= t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + \varepsilon\left[\frac{\partial}{\partial x} - \frac{1}{2}t\left(u\frac{\partial}{\partial u} - v\frac{\partial}{\partial v}\right)\right], \\
\chi^{11} &= t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} \\
&\quad + \varepsilon\left[t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} - \frac{1}{2}t\left(u\frac{\partial}{\partial u} - v\frac{\partial}{\partial v}\right)\right], \\
\chi^{12} &= t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + \varepsilon\left(1 - \frac{1}{2}t\right)\left(u\frac{\partial}{\partial u} - v\frac{\partial}{\partial v}\right).
\end{aligned}$$

The corresponding approximate conserved vectors are

$$\begin{aligned}
\mathcal{T}_9 = \Bigg(& tu_{xx}v - (tu_t + xu_x)v_t + (u_t + xu_{tx})v \\
&+ \varepsilon\left[u_{xx}v - \left(\frac{1}{2}tu + u_t\right)v_t + \frac{1}{2}(u + tu_t)v + xu_xv\right], \\
& x(u_xv_x - u_{tt}v) + t(u_tv_x - u_{tx}v) - u_xv \\
&+ \varepsilon\left[\frac{1}{2}t(uv_x - vu_x) + u_tv_x - u_{tx}v - xu_tv\right]\Bigg),
\end{aligned}$$

$$\begin{aligned}
\mathcal{T}_{10} = \Bigg(& tu_{xx}v - (tu_t + xu_x)v_t + (u_t + xu_{tx})v \\
&+ \varepsilon\left[-\left(\frac{1}{2}tu + u_x\right)v_t + \left(\frac{1}{2}u + \frac{1}{2}tu_t + u_{tx}\right)v + xu_xv\right], \\
& x(u_xv_x - u_{tt}v) + t(u_tv_x - u_{tx}v) - u_xv \\
&+ \varepsilon\left[u_xv_x - u_{tt}v + \frac{1}{2}t(uv_x - vu_x) - xu_tv\right]\Bigg),
\end{aligned}$$

$$\begin{aligned}
\mathcal{T}_{11} = \Bigg(& tu_{xx}v - (tu_t + xu_x)v_t + (u_t + xu_{tx})v \\
&+ \varepsilon\left[tu_{xx}v - \left(\frac{1}{2}tu + tu_t + xu_x\right)v_t + \left(\frac{1}{2}u + \frac{1}{2}tu_t + u_t + xu_{tx}\right)v + xu_xv\right], \\
& -xu_{tt}v + (tu_t + xu_x)v_x - (tu_{tx} + u_x)v \\
&+ \varepsilon\left[-xu_{tt}v + \left(\frac{1}{2}tu + tu_t + xu_x\right)v_x - \left(\frac{1}{2}tu_x + tu_{tx} + u_x\right)v - xu_tv\right]\Bigg),
\end{aligned}$$

$$\mathcal{T}_{12} = \Big(tu_{xx}v - (tu_t + xu_x)v_t + (u_t + xu_{tx})v$$

$$+\varepsilon\Big[\Big(1-\frac{1}{2}t\Big)uv_t + \Big(\frac{1}{2}u+\frac{1}{2}tu_t-u_t\Big)v + xu_x v\Big],$$
$$-xu_{tt}v + (tu_t+xu_x)v_x - (tu_{tx}+u_x)v + \varepsilon\Big[\Big(\frac{1}{2}t-1\Big)(uv_x-u_x v) - xu_t v\Big]\Big).$$

(4). $X_0 = u\partial/\partial u - v\partial/\partial v$. In this case, (4.4) becomes

$$\eta_1^2(u_{tt}-u_{xx}) + \zeta^1_{1,tt}v - \zeta^1_{1,xx}v + \big(D_t\xi_1^1 + D_x\xi_1^2\big)v(u_{tt}-u_{xx}) + mu^m u_t v = D_t B_1^1 + D_x B_1^2 + b(1-p)u^p v.$$

It follows that the above determining equation has solutions if and only if $m=b=0$. The solution is as follows:

$$X_1^1 = \frac{\partial}{\partial t}, \qquad X_1^2 = \frac{\partial}{\partial x},$$
$$X_1^3 = t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x}, \qquad X_1^4 = u\frac{\partial}{\partial u} - v\frac{\partial}{\partial v},$$

and $B_1^1 = B_1^2 = 0$. Hence, the system

$$\begin{cases} u_{tt}-u_{xx}+\varepsilon(u_t-au)=0, \\ v_{tt}-v_{xx}-\varepsilon(v_t+av)=0, \end{cases}$$

has the following approximate Noether symmetry operators:

$$\chi^{13} = u\frac{\partial}{\partial u} - v\frac{\partial}{\partial v} + \varepsilon\frac{\partial}{\partial t},$$
$$\chi^{14} = u\frac{\partial}{\partial u} - v\frac{\partial}{\partial v} + \varepsilon\frac{\partial}{\partial x},$$
$$\chi^{15} = u\frac{\partial}{\partial u} - v\frac{\partial}{\partial v} + \varepsilon\Big(t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x}\Big),$$
$$\chi^{16} = u\frac{\partial}{\partial u} - v\frac{\partial}{\partial v} + \varepsilon\Big(u\frac{\partial}{\partial u} - v\frac{\partial}{\partial v}\Big).$$

Thus, the corresponding approximate conserved vectors are

$$\mathcal{T}_{13} = \big(uv_t - u_t v + \varepsilon(u_{xx}v - u_t v_t - uv), \; u_x v - uv_x + \varepsilon(u_t v_x - u_{tx}v)\big),$$
$$\mathcal{T}_{14} = \big(uv_t - u_t v + \varepsilon(u_{tx}v - u_x v_t - uv), \; u_x v - uv_x + \varepsilon(u_x v_x - u_{tt}v)\big),$$
$$\mathcal{T}_{15} = \big(uv_t - u_t v + \varepsilon\big[tu_{xx}v - (tu_t+xu_x)v_t + (u_t+xu_{tx})v - uv\big], \; u_x v - uv_x + \varepsilon\big[-xu_{tt}v + (tu_t+xu_x)v_x - (tu_{tx}+u_x)v\big]\big),$$
$$\mathcal{T}_{16} = \big(uv_t - u_t v + \varepsilon(uv_t - u_t v - uv), \; u_x v - uv_x + \varepsilon(u_x v - uv_x)\big).$$

Notice that the above approximate conserved vectors are nonlocal since they contain an auxiliary variable $v$. However, instituting a special approximate solution of the equation $v_{tt} - v_{xx} + \varepsilon(-u^m v_t - av + bpu^{p-1}v) = 0$ into the above conserved vectors, we could obtain local conserved vectors of the original equation.

For approximate adjoint system (I), we can obtain the approximate conservation laws by a similar manner, and the conclusion of Theorem 2 can be verified.

*Example 2* Consider the perturbed Korteweg–de Vries equation

$$u_t - uu_x + u_{xxx} + \varepsilon\big(u^2 u_x + cu\big) = 0. \tag{4.6}$$

For simplicity, we only consider its standard adjoint system given by

$$\begin{cases} u_t - uu_x + u_{xxx} + \varepsilon(u^2u_x + cu) = 0, \\ -v_t + uv_x - v_{xxx} + \varepsilon(cv - u^2 v_x) = 0. \end{cases} \tag{4.7}$$

For approximate adjoint system (I) and (II), the approach is quite similar. The Lagrangian of the system (4.7) is

$$L = v(u_t - uu_x + u_{xxx}),$$

and the approximate Euler–Lagrange-type equations are

$$\frac{\delta L}{\delta u} = \varepsilon\big(u^2 v_x - cv\big) \quad \text{and} \quad \frac{\delta L}{\delta v} = -\varepsilon\big(cu + u^2 u_x\big).$$

For the first-order approximate Noether symmetry operator $X_0 + \varepsilon X_1$, we have

$$(X_0+\varepsilon X_1)L + LD_i\big(\xi_0^i + \varepsilon\xi_1^i\big) = \big(\big(\eta_0^1 - \xi_0^1 u_t - \xi_0^2 u_x\big)$$

$$
\begin{aligned}
&+\varepsilon\left(\eta_1^1-\xi_1^1u_t-\xi_1^2u_x\right)\varepsilon\left(u^2v_x-cv\right)\\
&+\left(\left(\eta_0^2-\xi_0^1v_t-\xi_0^2v_x\right)\right.\\
&+\varepsilon\left(\eta_1^2-\xi_1^1v_t-\xi_1^2v_x\right)\left.\right)\left(-\varepsilon\left(cu+u^2u_x\right)\right)\\
&+D_i\left(B_0^i+\varepsilon B_1^i\right)+O\left(\varepsilon^2\right).
\end{aligned}
$$

It gives the following determining equations:

$$X_0L+LD_i\left(\xi_0^i\right)=D_iB_0^i, \tag{4.8}$$

$$
\begin{aligned}
&X_1L+LD_i\left(\xi_1^i\right)\\
&\quad=\left(\eta_0^1-\xi_0^1u_t-\xi_0^2u_x\right)\left(u^2v_x-cv\right)\\
&\qquad-\left(\eta_0^2-\xi_0^1v_t-\xi_0^2v_x\right)\left(cu+u^2u_x\right)+D_iB_1^i.
\end{aligned} \tag{4.9}
$$

The determining equation (4.8) can be written as

$$
\begin{aligned}
&\eta_0^2(u_t-uu_x+u_{xxx})+\left(\zeta_{0,t}^1-\eta_0^1u_x-\zeta_{0,x}^1u\right)v\\
&\qquad+\zeta_{0,xxx}^1v+\left(D_t\xi_0^1+D_x\xi_0^2\right)v(u_t-uu_x+u_{xxx})\\
&\quad=D_tB_0^1+D_xB_0^2.
\end{aligned}
$$

The same procedure as given in Example 1 yields the following equations:

$$
\begin{aligned}
&\xi_0^1=\xi_0^1(t),\qquad \xi_0^2=\xi_0^2(t,x),\qquad \eta_0^1=\eta_0^1(t,x,u),\\
&\eta_0^2+\left(\eta_{0u}^1+\xi_{0t}^1-2\xi_{0x}^2\right)v=0,\\
&\eta_{0uu}^1=0,\qquad \eta_{0xu}^1=\xi_{0xx}^2,\\
&v\eta_0^1+u\eta_0^2+v\xi_{0t}^2-v\left(3\eta_{0xxu}^1-\xi_{0xxx}^2\right)\\
&\qquad+uv\left(\xi_{0t}^1+\eta_{0u}^1\right)-B_{0u}^2=0,\\
&B_{0v}^1=0,\qquad B_{0v}^2=0,\\
&\eta_0^2+v\left(\eta_{0u}^1+\xi_{0x}^2\right)=B_{0u}^1,\\
&v\eta_{0t}^1-uv\eta_{0x}^1+v\eta_{0xxx}^1=B_{0t}^1+B_{0x}^2.
\end{aligned}
$$

Solutions of the above equations are

$$
\begin{aligned}
&\xi_0^1=-\frac{3}{2}c_1t+c_4,\\
&\xi_0^2=-\frac{1}{2}c_1x-c_2t+c_3,\\
&\eta_0^1=c_1u+c_2,\qquad \eta_0^2=-\frac{1}{2}c_1v,\\
&B_0^1=B_0^1(t,x),\qquad B_0^2=B_0^2(t,x),
\end{aligned}
$$

where $(B_0^1,B_0^2)$ satisfies $B_{0t}^1+B_{0x}^2=0$, $c_1$, $c_2$, $c_3$, $c_4$ are arbitrary constants. Setting $B_0^1=B_0^2=0$, $X_0$ is given by

$$
\begin{aligned}
&X_0^1=-\frac{3}{2}t\frac{\partial}{\partial t}-\frac{1}{2}x\frac{\partial}{\partial x}+u\frac{\partial}{\partial u}-\frac{1}{2}v\frac{\partial}{\partial v},\\
&X_0^2=-t\frac{\partial}{\partial x}+\frac{\partial}{\partial u},\qquad X_0^3=\frac{\partial}{\partial x},\qquad X_0^4=\frac{\partial}{\partial t}.
\end{aligned} \tag{4.10}
$$

It is easy to verify that (4.9) has no solutions if $X_0=\frac{\partial}{\partial t}$ or $X_0=-\frac{3}{2}t\frac{\partial}{\partial t}-\frac{1}{2}x\frac{\partial}{\partial x}+u\frac{\partial}{\partial u}-\frac{1}{2}v\frac{\partial}{\partial v}$. Next, we consider $X_0=\frac{\partial}{\partial x}$ and $X_0=-t\frac{\partial}{\partial x}+\frac{\partial}{\partial u}$, respectively.

(1). $X_0=\partial/\partial x$. Equation (4.9) turns to be

$$
\begin{aligned}
&X_1L+L\left(D_t\xi_1^1+D_x\xi_1^2\right)\\
&\quad=u_x\left(cv-u^2v_x\right)+v_x\left(cu+u^2u_x\right)\\
&\qquad+D_tB_1^1+D_xB_1^2.
\end{aligned}
$$

Solving the above system implies that $B_1^1=0$, $B_1^2=-cuv$ and $X_1$ is given by

$$
\begin{aligned}
&X_1^1=-\frac{3}{2}t\frac{\partial}{\partial t}-\frac{1}{2}x\frac{\partial}{\partial x}+u\frac{\partial}{\partial u}-\frac{1}{2}v\frac{\partial}{\partial v},\\
&X_1^2=-t\frac{\partial}{\partial x}+\frac{\partial}{\partial u},\qquad X_1^3=\frac{\partial}{\partial x},\qquad X_1^4=\frac{\partial}{\partial t}.
\end{aligned}
$$

Hence, the approximate Noether symmetry operators are

$$
\begin{aligned}
&\chi^1=\frac{\partial}{\partial x}+\varepsilon\left(-\frac{3}{2}t\frac{\partial}{\partial t}-\frac{1}{2}x\frac{\partial}{\partial x}+u\frac{\partial}{\partial u}-\frac{1}{2}v\frac{\partial}{\partial v}\right),\\
&\chi^2=\frac{\partial}{\partial x}+\varepsilon\left(-t\frac{\partial}{\partial x}+\frac{\partial}{\partial u}\right),\\
&\chi^3=\frac{\partial}{\partial x}+\varepsilon\frac{\partial}{\partial x},\\
&\chi^4=\frac{\partial}{\partial x}+\varepsilon\frac{\partial}{\partial t}.
\end{aligned}
$$

The corresponding approximate conserved densities are as follows:

$$
\begin{aligned}
\mathcal{T}_1=\Bigg(&u_xv-\varepsilon\left[u+\frac{1}{2}xu_x+\frac{3}{2}t(uu_x-u_{xxx})\right]v,\\
&u_xv_{xx}-u_{xx}v_x-u_tv\\
&+\varepsilon\left[\frac{1}{2}(x+3tu)u_tv+(u-c)uv\right.
\end{aligned}
$$

$$- \left(u + \frac{3}{2}tu_t + \frac{1}{2}xu_x\right)v_{xx}$$

$$+ \frac{1}{2}(3u_x + 3tu_{tx} + xu_{xx})v_x$$

$$- \left(2u_{xx} + \frac{3}{2}tu_{txx}\right)v\Big]\Big),$$

$$\mathcal{T}_2 = \Big(u_x v - \varepsilon(1 + tu_x)v,$$

$$u_x v_{xx} - u_{xx}v_x - u_t v + \varepsilon\big[(1 - c)uv - v_{xx}$$

$$+ t(u_t v - u_x v_{xx} + u_{xx}v_x)\big]\Big),$$

$$\mathcal{T}_3 = \big(u_x v + \varepsilon u_x v,$$

$$u_x v_{xx} - u_{xx}v_x - u_t v$$

$$+ \varepsilon(-cuv - u_t v + u_x v_{xx} - u_{xx}v_x)\big),$$

$$\mathcal{T}_4 = \big(u_x v - \varepsilon(u_{xxx} - uu_x)v,$$

$$u_x v_{xx} - u_{xx}v_x - u_t v + \varepsilon(-cuv + u_t v_{xx}$$

$$- uu_t v - u_{tx}v_x + u_{txx}v)\big).$$

(2). $X_0 = -t\partial/\partial x + \partial/\partial u$. In this case, we obtain the following approximate Noether symmetry operators:

$$\chi^5 = -t\frac{\partial}{\partial x} + \frac{\partial}{\partial u}$$

$$+ \varepsilon\left[\frac{1}{2}ct^2\frac{\partial}{\partial x} + (2u - ct)\frac{\partial}{\partial u} - 2v\frac{\partial}{\partial v}\right],$$

$$\chi^6 = -t\frac{\partial}{\partial x} + \frac{\partial}{\partial u} + \varepsilon\left[3t\frac{\partial}{\partial t} + \left(x + \frac{1}{2}ct^2 - t\right)\frac{\partial}{\partial x}\right.$$

$$\left. + (1 - ct)\frac{\partial}{\partial u} - v\frac{\partial}{\partial v}\right],$$

$$\chi^7 = -t\frac{\partial}{\partial x} + \frac{\partial}{\partial u} + \varepsilon\left[3t\frac{\partial}{\partial t} + \left(x + \frac{1}{2}ct^2 + 1\right)\frac{\partial}{\partial x}\right.$$

$$\left. - ct\frac{\partial}{\partial u} - v\frac{\partial}{\partial v}\right],$$

$$\chi^8 = -t\frac{\partial}{\partial x} + \frac{\partial}{\partial u} + \varepsilon\left[(3t + 1)\frac{\partial}{\partial t} + \left(x + \frac{1}{2}ct^2\right)\frac{\partial}{\partial x}\right.$$

$$\left. - ct\frac{\partial}{\partial u} - v\frac{\partial}{\partial v}\right],$$

and the corresponding approximate conserved vectors:

$$\mathcal{T}_5 = \left(-(1 + tu_x)v + \varepsilon\left(\frac{1}{2}ct^2u_x + ct - 2u\right)v,\right.$$

$$uv - v_{xx} + t(u_t v - u_x v_{xx} + u_{xx}v_x)$$

$$+ \varepsilon\left[\left(ctu - u^2 - \frac{1}{2}ct^2u_t\right)v\right.$$

$$- (2u - ct)(v_{xx} - uv)$$

$$+ \frac{1}{2}ct^2(u_x v_{xx} - u_{xx}v_x)$$

$$\left.\left. + 2(u_x v_x - u_{xx}v)\right]\right),$$

$$\mathcal{T}_6 = \left(-(1 + tu_x)v + \varepsilon\left[3t(uu_x - u_{xxx}) + ct\right.\right.$$

$$\left. + \left(x - t + \frac{1}{2}ct^2\right)u_x - 1\right]v,$$

$$uv - v_{xx} + t(u_t v - u_x v_{xx} + u_{xx}v_x)$$

$$+ \varepsilon\left[ctuv - u^2v\right.$$

$$+ (1 - ct - 3tu_t)(uv - v_{xx})$$

$$+ \left(x - t + \frac{1}{2}ct^2\right)(u_x v_{xx} - u_{xx}v_x - u_t v)$$

$$\left.\left. - (3tu_{tx} + u_x)v_x + (3tu_{txx} + 2u_{xx})v\right]\right),$$

$$\mathcal{T}_7 = \left(-(1 + tu_x)v + \varepsilon\left[3t(uu_x - u_{xxx}) + ct\right.\right.$$

$$\left. + \left(1 + x + \frac{1}{2}ct^2\right)u_x\right]v,$$

$$uv - v_{xx} + t(u_t v - u_x v_{xx} + u_{xx}v_x)$$

$$+ \varepsilon\left[ctuv - u^2v + (ct + 3tu_t)(v_{xx} - uv)\right.$$

$$+ \left(1 + x + \frac{1}{2}ct^2\right)(u_x v_{xx} - u_{xx}v_x - u_t v)$$

$$\left.\left. - (3tu_{tx} + u_x)v_x + (3tu_{txx} + 2u_{xx})v\right]\right),$$

$$\mathcal{T}_8 = \left(-(1 + tu_x)v + \varepsilon\left[(3t + 1)(uu_x - u_{xxx}) + ct\right.\right.$$

$$\left. + \left(x + \frac{1}{2}ct^2\right)u_x\right]v,$$

$$(u+tu_t)v-(1+tu_x)v_{xx}+tu_{xx}v_x + \varepsilon\Big[ctuv-u^2v + \big(ct+(3t+1)u_t\big)(v_{xx}-uv) + \left(x+\frac{1}{2}ct^2\right)(u_xv_{xx}-u_{xx}v_x-u_tv) - \big((3t+1)u_{tx}+u_x\big)v_x + \big((3t+1)u_{txx}+2u_{xx}\big)v\Big]\Big).$$

Notice that $\mathcal{T}_1, \mathcal{T}_2, \ldots, \mathcal{T}_8$ are nonlocal conservation vectors since they contain an auxiliary variable $v(t,x)$, which satisfies the equation $-v_t+uv_x-v_{xxx}+\varepsilon(cv-u^2v_x)=0$. Obviously, the above equation has the solution $v=1+\varepsilon ct$. Inserting it into $\mathcal{T}_1$ and ignoring the terms involving the second order of $\varepsilon$ results in the approximate local conserved vector

$$\Big(u_x+\varepsilon\Big[ctu_x-u-\frac{1}{2}xu_x-\frac{3}{2}t(uu_x-u_{xxx})\Big], -u_t+\varepsilon\Big[(u-c)u-ctu_t+\frac{1}{2}(x+3tu)u_t - 2u_{xx}-\frac{3}{2}tu_{txx}\Big]\Big).$$

Indeed, by a direct computation we have

$$D_t\Big(u_x+\varepsilon\Big[ctu_x-u-\frac{1}{2}xu_x-\frac{3}{2}t(uu_x-u_{xxx})\Big]\Big) + D_x\Big((u-c)u-u_t+\varepsilon\Big[-ctu_t+\left(\frac{1}{2}x+\frac{3}{2}tu\right)u_t - \left(2u_{xx}+\frac{3}{2}tu_{txx}\right)\Big]\Big)\Big|_{(4.6)} = O(\varepsilon^2),$$

which gives an approximate conservation laws of (4.6).

**Acknowledgements** We are grateful to the referees for their constructive suggestions and comments. This work was supported by the Chinese National Science Fund for Distinguished Young Scholars (Grant No. 10925104) and the Chinese National Science Fund (Grant No. 10671156).

## References


1. Laplace, P.S.: Celestrial Mechanics, p. 1798. New York (1966)
2. Noether, E.: Invariante Variationsprobleme. Nacr. Konig. Gesell. Wissen., Gottingen, Math.-Phys. KI. Heft **2**, 235–275 (1918) (English translation in Transp. Theory Stat. Phys. **1**, 186–207 (1971))
3. Steudel, H.: Uber die Zuordnung zwischen Invarianzeigenschaften und Erhaltungssatzen. Z. Naturforsch. A **17**, 129–132 (1962)
4. Olver, P.J.: Applications of Lie Groups to Differential Equations. Springer, New York (1993)
5. Anco, S.C., Bluman, G.W.: Direct construction method for conservation laws of partial differential equations, part I: examples of conservation law classifications. Eur. J. Appl. Math. **13**, 545–566 (2002)
6. Anco, S.C., Bluman, G.W.: Direct construction method for conservation laws of partial differential equations, part II: general treatment. Eur. J. Appl. Math. **9**, 567–585 (2002)
7. Kara, A.H., Mahomed, F.M.: Noether-type symmetries and conservation laws via partial Lagrangians. Nonlinear Dyn. **45**, 367–383 (2006)
8. Ibragimov, N.H.: A new conservation theorem. J. Math. Anal. Appl. **333**, 311–328 (2007)
9. Kara, A.H., Mahomed, F.M., Ünal, G.: Approximate symmetries and conservation laws with applications. Int. J. Theor. Phys. **38**, 2389–2999 (1999)
10. Johnpillai, A.G., Kara, A.H., Mahomed, F.M.: A basis of approximate conservation laws for PDEs with a small parameter. Int. J. Non-Linear Mech. **41**, 830–837 (2006)
11. Johnpillai, A.G., Kara, A.H., Mahomed, F.M.: Approximate Noether-type symmetries and conservation laws via partial Lagrangians for PEDs with a small parameter. J. Comput. Appl. Math. **223**, 508–518 (2009)
12. Ibragimov, N.H., Kara, A.H., Mahomed, F.M.: Lie–Bäcklund and Noether symmetries with applications. Nonlinear Dyn. **15**, 115–136 (1998)
13. Johnpillai, A.G., Kara, A.H.: Variational formulation of approximate symmetries and conservation laws. Int. J. Theor. Phys. **40**, 1501–1509 (2001)
14. Baikov, V.A., Ibragimov, N.H.: In: Ibragimov, N.H. (ed.) CRC Handbook of Lie Group of Analysis of Differential Equations, vol. 3. CRC Press, Boca Raton (1996)