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# Nonlinear observer-based impulsive synchronization in chaotic systems with multiple attractors

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**Abstract** The issue of impulsive synchronization of the coupled Newton–Leipnik system is investigated. Based on the impulsive stability theory, nonlinear observer-based impulsive synchronization scheme is derived. A new and less conservative criteria for impulsive synchronization via nonlinear observer is proposed. The boundary of the stable regions is also estimated. One important advantage of the proposed method is that it is also applicable for the systems with more than one attractor. Numerical simulations on Newton–Leipnik system are illustrated to verify the theoretical results.

**Keywords** Impulsive synchronization · Nonlinear observer · Newton–Leipnik system

## 1 Introduction

Synchronization in two coupled chaotic systems has attracted much attention for both theoretical and practical applications in secure communication [1–11]. Recently, impulsive control has been widely used to stabilize and synchronize chaotic systems [12–16]. In many systems, such as frequency modulated signal processing systems, some flying object motions and optimal control of economic systems etc. are characterized by abrupt changes in the state at certain instants. The importance of such systems lies in that, in some cases, they cannot be continuously controlled. Moreover, impulsive control may give an efficient method to deal with systems that cannot create continuous disturbance. Furthermore, impulsive method can also greatly reduce the control cost. This type of impulsive phenomenon can also be found in the fields of automatic control systems, computer networks, telecommunications, robotics, electronics, etc. Many sudden and sharp changes occur instantaneously in these systems, in the form of impulses which cannot be well addressed by a pure continuous time or discrete time model.

Recently, T. Yang [17] achieved the synchronization of two identical chaotic systems, i.e. Chua circuits, using the state variable at the fixed instant time as the impulsive signal. C.D. Li et al. [18] discussed the impulsive synchronization of nonlinear coupled chaotic systems. Haeri and Dehghani [19] investigated impulsive synchronization of Chen's hyperchaotic systems. G. Zhang et al. [20] studied the synchronization in complex dynamical networks via impulsive control. The above methods can be applied for the systems which contain only one attractor. In this letter,

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we address the question of impulsive synchronization in chaotic system with multiple strange attractors.

Motivated by the aforementioned comments, the main aim of this paper is to further study the impulsive synchronization in chaotic systems via nonlinear observer design. Here, some new and less conservative criteria are proposed to synchronize the systems with varying impulsive intervals. Particularly, a simple and easily verified criteria is also derived with equivalent impulsive intervals. An illustrative example is included to show the effectiveness and feasibility of the proposed scheme. One important result of our numerical simulation is that impulsive synchronization occurs also when the initial conditions lie in two different basins of attraction.

The rest of the paper is organized as follows. In Sect. 2, a basic theory of impulsive differential equation is provided, which is used in our work. In Sect. 3, a new nonlinear observer-based impulsive synchronization scheme is described for chaotic systems. A new and less conservative criterion for impulsive synchronization is derived. Numerical simulations are discussed in Sect. 4. Finally, conclusions are given in Sect. 5.

## 2 A basic theory of impulsive differential equations

To make the present paper self-contained, we address some basic properties of impulsive differential equation which will be used in our work. These results are already discussed in [12, 17–19]. In the impulsive synchronization, the master system is described by

$$\dot{x} = f(x,t), \tag{1}$$

where $f : R^n \times R_+ \to R^n$ is a continuous function with respect to its arguments and $x \in R^n$ represents the state variables. Suppose that a discrete set $\{t_k\}$ of time instants satisfies

$$0 \le t_0 < t_1 < t_2 < \cdots < t_k < t_{k+1} < \cdots,$$
$$t_k \to \infty \text{ as } k \to \infty.$$

The slave system is characterized by

$$\left.\begin{aligned} &\dot{y} = f(y,t) - f(x,t), && t \ne t_k,\\ &\Delta y = B_k e, && t = t_k,\\ &y(t_0^+) = y_0, && \end{aligned}\right\} \tag{2}$$

where $f$ is the same as above and $e = [e_1, e_2, \dots, e_n]^T = [y_1 - x_1, y_2 - x_2, \dots, y_n - x_n]^T$. Subtracting (2) from (1), one gets the following result for synchronization error dynamics:

$$\left.\begin{aligned} &\dot{e} = f(y,t) - f(x,t), && t \ne t_k,\\ &\Delta e = B_k e, && t = t_k. \end{aligned}\right\} \tag{3}$$

The goal of impulsive synchronization is to find conditions of the control gains, $B_k$, and the impulsive distance, $\tau_k$, such that the impulsive controlled slave system (2) is globally asymptotically synchronous with the master system (1).

## 3 Nonlinear observer-based impulsive synchronization scheme

Consider the general nonlinear system described by

$$\dot{x} = Ax + Bf(x) + C, \tag{4}$$

where $x \in R^n$, $A \in R^{n\times n}$, $B \in R^{n\times n}$, $C \in R^{n\times 1}$ and $f = (f_1(x), f_2(x), \dots, f_n(x))^T \in R^{n\times 1}$ is a nonlinear vector field. Given the dynamic system (4) with scalar output $z = s(x) \in R$, the dynamics

$$\dot{y} = Ay + Bf(y) + C + g\big(z - s(y)\big) \tag{5}$$

is said to be nonlinear observer of system (4) if $y - x \to 0$ as $t \to \infty$, where $g : R \to R^n$ is a suitable nonlinear function. We choose $g(z - s(y)) = B(s(x) - s(y)$ with $s(x) = f(x) + kx$, where $k \in R^{1\times n}$.

In order to achieve synchronization of chaotic system (4), the impulsive controller is designed as

$$\begin{aligned} \dot{y} &= Ay + Bf(y) + C + g(z - s(y))\\ &\quad + P_k\delta(t - t_k)(x - y), \end{aligned} \tag{6}$$

where $\delta(\cdot)$ is the Dirac impulsive function; i.e.,

$$\delta(t - t_k) = \begin{cases} 1, & t = t_k,\\ 0, & t \ne t_k. \end{cases} \tag{7}$$

The impulsively controlled state of (5) can then be described by the following impulsive differential equations:

$$\dot{y} = Ay + Bf(y) + C + g\big(z - s(y)\big), \quad t \ne t_k, \tag{8a}$$

$$\Delta y = P_k(x - y), \quad t = t_k. \tag{8b}$$

Then the dynamics (8) is said to be nonlinear impulsive observer of system (4). The error dynamics (3) can be written as

$$\left.\begin{aligned} \dot{e} &= Ae - Bke, \quad & t \neq t_k, \\ \Delta e &= P_k e, \quad & t = t_k, \end{aligned}\right\} \tag{9}$$

where $t_k$ denotes the moment when impulsive control occurs. For convenience, define the following notations:

$$\lambda = \lambda_{\max}\left(\frac{A + A^T}{2}\right)$$

$$\bar{\lambda} = \lambda_{\max}\left(\frac{K^T B^T + BK}{2}\right)$$

$$\beta_k = \lambda_{\max}\left[(I + P_k)^T (I + P_k)\right]$$

where $I$ is the $n \times n$ identity matrix and $\lambda_{\max}(A)$ is the maximal eigenvalue of matrix $A$. We have the following proposition which guarantees the impulsive control system to be global stable at origin.

**Proposition 1** *If there exists a constant $\xi > 1$ such that*

$$\ln(\xi \gamma_k) + 2(\lambda - \bar{\lambda})\tau_k \le 0, \quad k = 1, 2, 3, \ldots \tag{10}$$

*then the slave system (8) will be globally asymptotically synchronous with the master system (4).*

*Proof* Let the Lyapunov function be in the form

$$V\bigl(e(t)\bigr) = \frac{1}{2} e^T e.$$

The time derivative of $V(t)$ along the solution of (9) is

$$\begin{aligned} \dot{V}\bigl(e(t)\bigr) &= \frac{1}{2}(Ae - Bke)^T e + \frac{1}{2} e^T (Ae - Bke) \\ &= \frac{1}{2} e^T \bigl(A^T + A\bigr) e - \frac{1}{2} e^T \bigl(k^T B^T + Bk\bigr) e \\ &\le 2\lambda V\bigl(e(t)\bigr) - 2\bar{\lambda} V\bigl(e(t)\bigr) \\ &= 2(\lambda - \bar{\lambda}) V\bigl(e(t)\bigr), \\ & t \in (t_{k-1}, t_k] \text{ for } k = 1, 2, 3, \ldots. \end{aligned} \tag{11}$$

This implies that

$$V\bigl(e(t)\bigr) \le V\bigl(e(t_{k-1}^+)\bigr) e^{2(\lambda - \bar{\lambda})(t - t_{k-1})}, \quad t \in (t_{k-1}, t_k] \text{ for } k = 1, 2, 3, \ldots. \tag{12}$$

From (9), when $t = t_k$,

$$\begin{aligned} V\bigl(e(t_k^+)\bigr) &= \frac{1}{2}\bigl[(I + P_k) e(t_k)\bigr]^T (I + P_k) e(t_k) \\ &= \frac{1}{2} e^T(t_k)\bigl[(I + P_k)^T (I + P_k)\bigr] e(t_k) \\ &\le \frac{1}{2} \gamma_k e^T(t_k) e(t_k) \\ &= \gamma_k V\bigl(e(t_k)\bigr) \quad \text{for } k = 1, 2, 3, \ldots. \end{aligned} \tag{13}$$

When $k = 1$ in inequality (12), then for any $t \in (t_0, t_1]$,

$$V\bigl(e(t)\bigr) \le V\bigl(e(t_0^+)\bigr) e^{2(\lambda - \bar{\lambda})(t - t_0)}. \tag{14}$$

This leads to

$$V\bigl(e(t_1)\bigr) \le V\bigl(e(t_0^+)\bigr) e^{2(\lambda - \bar{\lambda})(t_1 - t_0)}. \tag{15}$$

Also from (13) we have

$$\begin{aligned} V\bigl(e(t_1^+)\bigr) &\le \gamma_1 V\bigl(e(t_1)\bigr) \\ &\le \gamma_1 V\bigl(e(t_0^+)\bigr) e^{2(\lambda - \bar{\lambda})(t_1 - t_0)}. \end{aligned} \tag{16}$$

In the same way for $t \in (t_1, t_2]$, we have

$$\begin{aligned} V\bigl(e(t)\bigr) &\le V\bigl(e(t_1^+)\bigr) e^{2(\lambda - \bar{\lambda})(t - t_1)} \\ &\le \gamma_1 V\bigl(e(t_0^+)\bigr) e^{2(\lambda - \bar{\lambda})(t - t_0)}. \end{aligned} \tag{17}$$

In general, for any $t \in (t_k, t_{k+1}]$, one finds then

$$V\bigl(e(t)\bigr) \le \gamma_1 \gamma_2 \ldots \gamma_k V\bigl(e(t_0^+)\bigr) e^{2(\lambda - \bar{\lambda})(t - t_0)}. \tag{18}$$

From the assumption given in the theorem, we have

$$\xi \gamma_k e^{2(\lambda - \bar{\lambda})\tau_k} \le 1, \quad k = 1, 2, 3, \ldots. \tag{19}$$

Thus, for $t \in (t_k, t_{k+1}]$, $k = 1, 2, 3, \ldots$, we have

$$\begin{aligned} V\bigl(e(t)\bigr) &\le \gamma_1 \gamma_2 \ldots \gamma_k V\bigl(e(t_0^+)\bigr) e^{2(\lambda - \bar{\lambda})(t - t_0)} \\ &= V\bigl(e(t_0^+)\bigr)\bigl[\gamma_1 e^{2(\lambda - \bar{\lambda})\tau_1}\bigr]\bigl[\gamma_2 e^{2(\lambda - \bar{\lambda})\tau_2}\bigr] \\ &\quad \cdots \bigl[\gamma_k e^{2(\lambda - \bar{\lambda})\tau_k}\bigr] e^{2(\lambda - \bar{\lambda})(t - t_k)} \\ &\le V\bigl(e(t_0^+)\bigr) \frac{1}{\xi_k} e^{2(\lambda - \bar{\lambda})(t - t_0)}. \end{aligned} \tag{20}$$

This implies that the origin of system (9) is globally asymptotically stable or the slave system is synchronized with the master system asymptotically for any initial condition. By this we conclude the proof of the proposition. □

**Corollary** *For* $\tau_k = \tau > 0$, *the gain matrix* $P_k = P$ $(k = 1, 2, \ldots)$, *if there exists a constant* $\xi > 0$ *such that*

$$\ln(\xi\gamma) + 2(\lambda - \bar{\lambda})\tau \le 0. \tag{21}$$

## 4 An illustrative example

In order to demonstrate and verify the performance of the proposed method, some numerical simulations are presented in this section. Meanwhile, some interesting and surprising numerical results are also discussed.

*Example* Here we investigate the impulsive synchronization of the Newton–Leipnik system [21]. The system is described as follows:

$$\left.\begin{aligned} \dot{x}_1 &= -ax_1 + y_1 + 10y_1z_1, \\ \dot{y}_1 &= -x_1 - 0.4y_1 + 5x_1z_1, \\ \dot{z}_1 &= bz_1 - 5x_1y_1, \end{aligned}\right\} \tag{22}$$

where $a$ and $b$ are two system parameters. The Newton–Leipnik system originates from the Euler rigid body equations modified with the addition of a linear feedback. The model contains two strange attractors depicting chaotic motion in rigid body dynamics. Stability and control of Newton–Leipnik system was investigated by Richter [22] with a static nonlinear feedback law based on the Lyapunov function. This system was also controlled by Wang and Tian [23] using a simple linear controller. The system (22) shows chaotic behavior when $a = 0.4$, $b = 0.175$ with initial conditions $(0.349, 0, -0.160)$ and $(0.349, 0, -0.180)$; the system exhibits two different chaotic attractors. In Fig. 1, two different chaotic attractors are shown. The upper attractor is given by an initial condition $(0.349, 0, -0.160)$ and the lower attractor by $(0.349, 0, -0.180)$. First we decompose the system dynamics into its linear and nonlinear parts. Thus (22) is rewritten as

$$\begin{pmatrix} \dot{x}_1 \\ \dot{y}_1 \\ \dot{z}_1 \end{pmatrix} = \begin{pmatrix} -a & 1 & 0 \\ -1 & -0.4 & 0 \\ 0 & 0 & b \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + \begin{pmatrix} 10 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & -5 \end{pmatrix} \begin{pmatrix} y_1z_1 \\ x_1z_1 \\ x_1y_1 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Then

$$\frac{1}{2}(A + A^T) = \begin{pmatrix} -a & 0 & 0 \\ 0 & -0.4 & 0 \\ 0 & 0 & b \end{pmatrix}.$$

The eigenvalues of this matrix are $-0.4$, $-0.4$ and $0.175$. Thus $\lambda = 0.175$. We choose $k = (\alpha, \alpha, \alpha) \in$

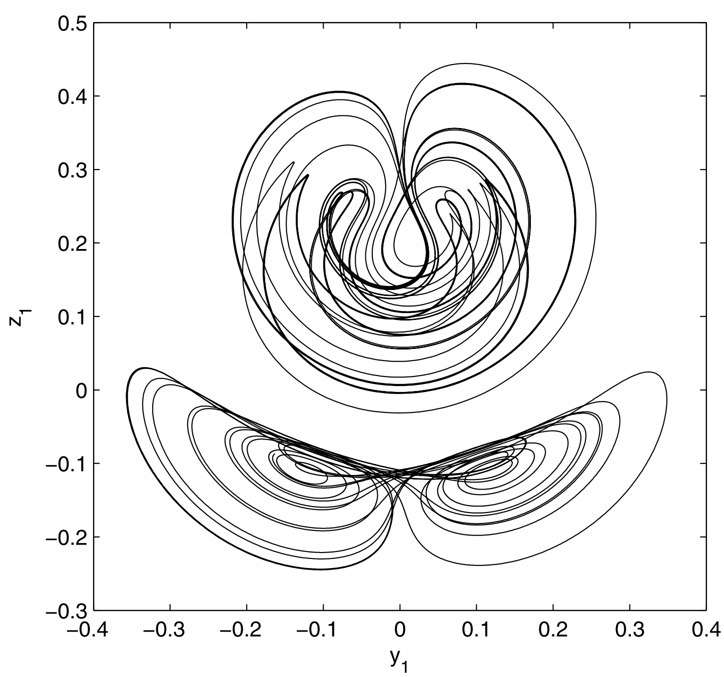


**Fig. 1** Chaotic attractors of the Newton–Leipnik system (22): $(0.349, 0, -0.160)$ is for upper chaotic attractor and $(0.349, 0, -0.180)$ for lower chaotic attractor

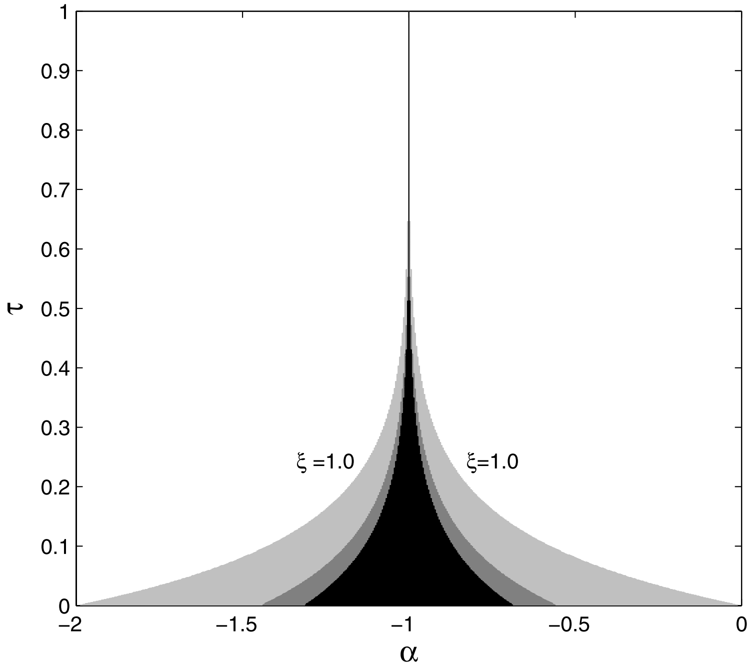


**Fig. 2** The boundaries of stable region for synchronization with different values of $\xi$

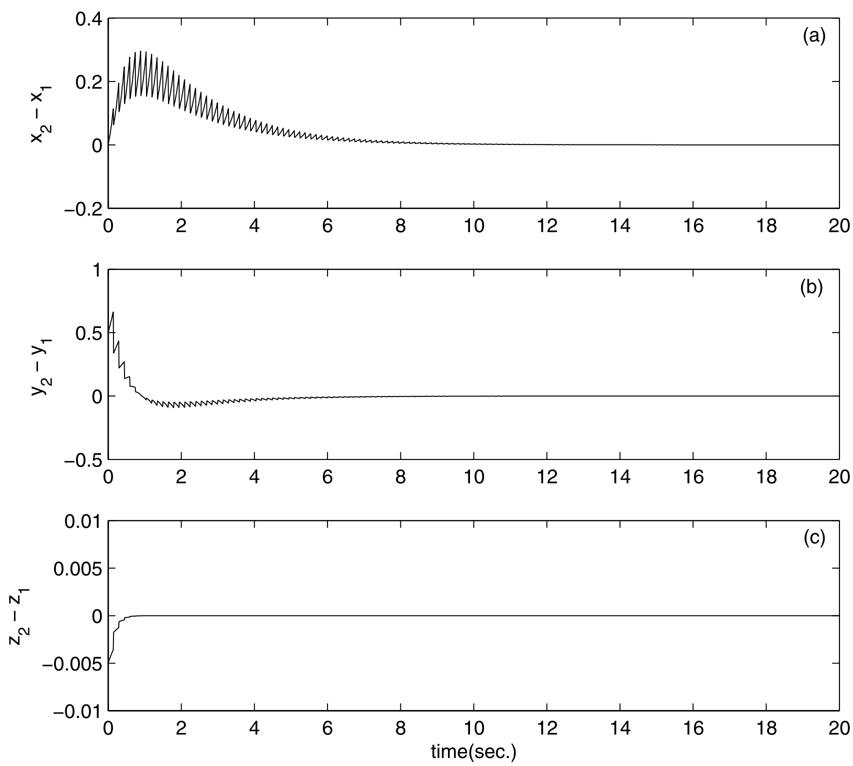


**Fig. 3** Impulsive synchronization errors for $\alpha = -0.5$, $\xi = 1.0$ and $\tau = 0.01$

$R^{1\times n}$. Then the eigenvalues of $\frac{1}{2}(K^T B^T + BK)$ are $10\alpha$, $5\alpha$ and $-5\alpha$. Thus $\bar{\lambda} = 10\alpha$. We choose the gain matrices $P_k, k = 1, 2, 3, \ldots$, as a constant matrix $P = \mathrm{diag}(\alpha, \alpha, \alpha)$. It is easy to see that $\gamma = (1+\alpha)^2$. Thus we obtain the boundary of the stable region as

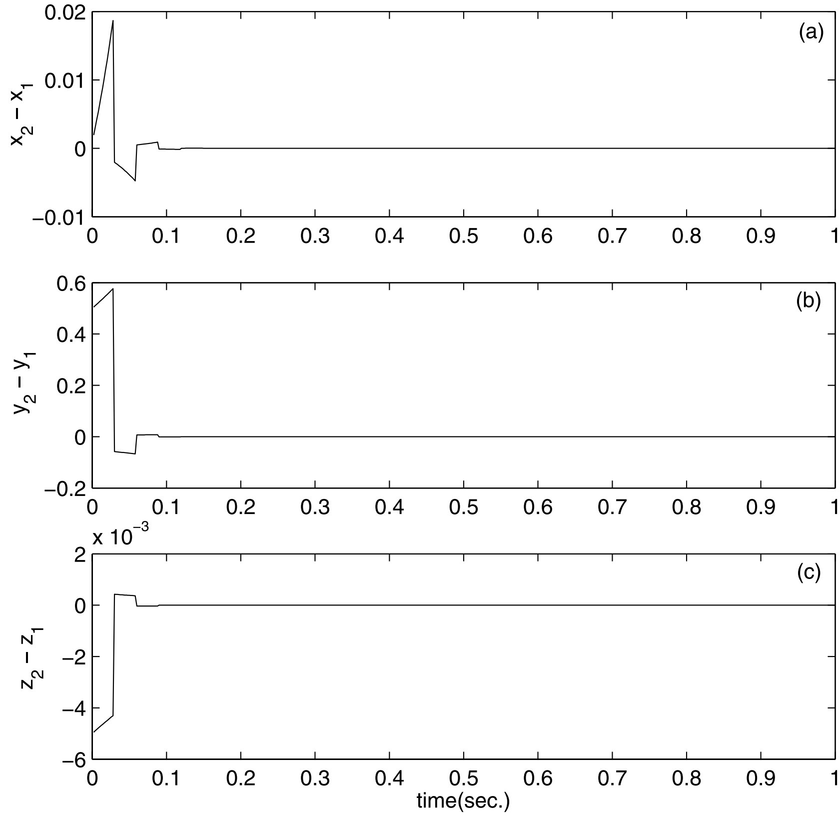


**Fig. 4** Impulsive synchronization errors for $\alpha = -1.1$ and $\tau = 0.002$

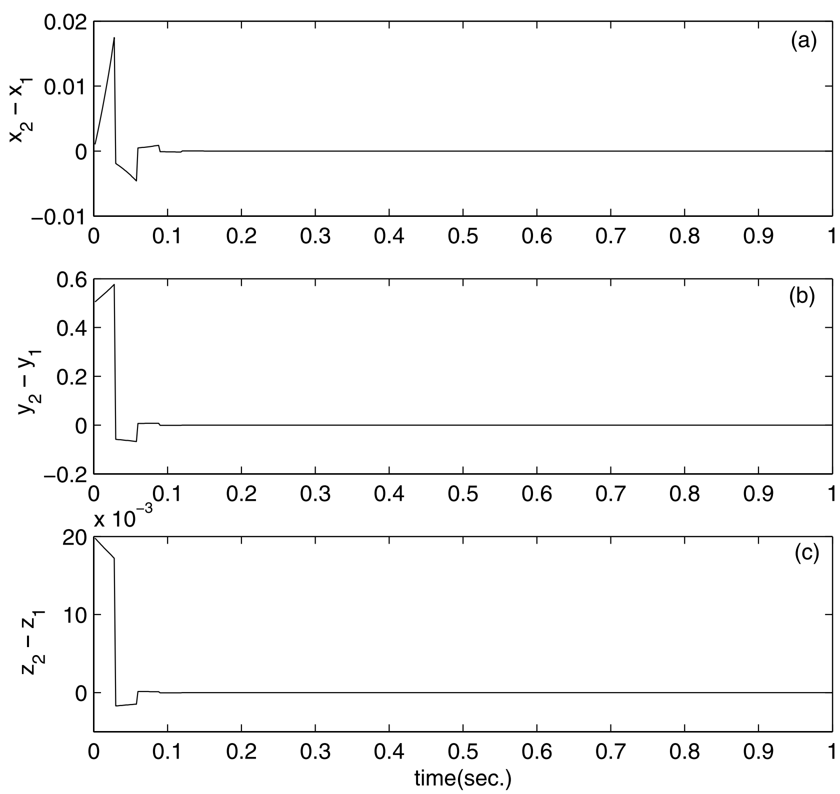


**Fig. 5** Impulsive synchronization errors for $\alpha = -1.1$ and $\tau = 0.002$. The initial condition for master system is $(x_1(0), y_1(0), z_1(0)) = (0.349, 0.0, -0.160)$ and response system $(x_2(0), y_2(0), z_2(0)) = (0.349, 0.0, -0.180)$

$$0 \le \tau \le -\frac{\ln(\xi) + \ln(\alpha+1)^2}{2(\lambda - \bar{\lambda})}. \qquad (23)$$

Figure 2 shows the stable regions for different values of $\xi$ and $\alpha$. The entire region under the curve corresponding to $\xi = 1$ is the predicted stable region. The stable region shrinks to a line $\alpha = -1.0$ when $\xi \to +\infty$. We choose $\alpha = -1.1$ and select $\xi = 1.0$, then $0 \le \tau \le 4.8532$. Numerical simulation results with $\alpha = -0.5$, $\xi = 1.0$ and the constant impulsive distance $\tau = 0.01$ are shown in Fig. 3. For a larger value $\xi = 5.0$, the impulsive synchronization errors are shown in Fig. 4 with $\alpha = -1.1$ and $\tau = 0.002$. In our last simulation we take the initial condition of drive system as $(x_1(0), y_1(0), z_1(0)) = (0.349, 0.0, -0.160)$ and response system as $(x_2(0), y_2(0), z_2(0)) = (0.349, 0.0, -0.180)$; the impulsive synchronization is shown in Fig. 5.

## 5 Conclusions

A nonlinear observer-based impulsive control procedure is developed for the synchronization of nonlinear systems with multiple attractors. It is interesting to note that the synchronization remains valid even when the two systems remain in the vicinity of two different attractors. In this respect we can mention that there are other nonlinear systems where one can have more than one such attractor. One important example is that of the system derived from chaotic Alfven wave in plasma [24].

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