Nonlinear Dyn (2010) 60: 403–411
DOI 10.1007/s11071-009-9604-y

ORIGINAL PAPER

Nonlocal symmetries of evolution equations

Renat Zhdanov

Received: 10 July 2009 / Accepted: 29 September 2009 / Published online: 7 November 2009

Abstract We suggest the method for group classification of evolution equations admitting nonlocal symmetries which are associated with a given evolution equation possessing nontrivial Lie symmetry. We apply this method to second-order evolution equations in one spatial variable invariant under Lie algebras of the dimension up to three. As a result, we construct the broad families of new nonlinear evolution equations possessing nonlocal symmetries which in principle cannot be obtained within the classical Lie approach.

Keywords Evolution equation · Lie symmetry · Nonlocal symmetry

1 Introduction

Classical Lie symmetries has become an inseparable part of the modern mathematical physics toolkit for analysis of linear and nonlinear differential equations [1–6]. However, with all its power, versatility and universality the Lie symmetry approach has its limits, which prompted numerous researchers to look for possible generalizations of the concept of classical symmetry in order to be able to handle differential equations that do not possess Lie symmetries.

R. Zhdanov (✉)
BIO-key International, Eagan, MN 55121, USA
e-mail: renat.zhdanov@bio-key.com

One of the promising generalizations of the concept of the classical symmetry is allowing for a generalized symmetry to depend on integrals of dependent variables, in contrast to Lie symmetry which can only involve independent, dependent variables and their derivatives. In this way nonlocal symmetries have been introduced into the modern mathematical physics.

There was significant progress in studying nonlocal symmetries of linear differential equations (see, e.g., [7] and the references therein). Much less is known about nonlocal symmetries of nonlinear differential equations.

A possible way to approach the problem of constructing nonlocal symmetries of partial differential equations is utilization of the concept of quasi-local symmetry introduced in [8, 9]. Quasi-local symmetry is the one obtained from a classical Lie symmetry through nonlocal transformation. Another approach is based on the concept of potential symmetry [10–12]. However, as we established recently, potential symmetries of evolution type equations are quasi-local in a sense that there are nonlocal transformations reducing those to point or contact symmetries [13].

The principal motivation for this paper was a need for a more systematic approach to classification of partial differential equations admitting quasi-local symmetries. It is natural to expect that such an approach should rely on Lie symmetries of equations under study. We demonstrate in the paper that it is indeed possible to develop the regular and systematic classification scheme using the results of group classification of nonlinear second-order equations [14, 15] and the ideas suggested in [13, 16].

We utilize the results of group classification of the general second-order evolution equation in one spatial variable

$$u_t = f(t, x, u, u_x, u_{xx}) \tag{1}$$

to construct nonlocal symmetries of the associated evolution equations. In a what follows, we denote the class of partial differential equations (1) as $\mathbb{E}$.

The paper is organized as follows. In Sect. 2 we give the necessary definitions and present the detailed description of our approach to classifying nonlocal symmetries of (1). Section 3 is devoted to application of the approach to equations from $\mathbb{E}$ invariant under the Lie algebras of the dimension $s \le 3$. The last section contains brief discussion of the obtained results.

2 Description of the method

We begin by introducing the differential transformation which plays a key role in our subsequent considerations:

$$\mathfrak{P}(t, x, u) = (\bar t, \bar x, \bar u) \equiv (t, x, u_x). \tag{2}$$

Note that the inverse of differential transformation (2) reads as

$$t = \bar t, \qquad x = \bar x, \qquad u = \partial_{\bar x}^{-1} \bar u\ . \tag{3}$$

Hereafter ∂_x^{-1} is the inverse of the differentiation operator ∂_x, i.e., $\partial_x\,\partial_x^{-1} \equiv \partial_x^{-1}\,\partial_x \equiv 1$.

Consider a subclass of evolution equations of the form (1)

$$u_t = f(t, x, u_x, u_{xx}) \tag{4}$$

admitting the group of translations by u.

Differentiating (4) with respect to x and applying $\mathfrak{P}$ yields

$$\bar u_{\bar t} = \partial_{\bar x} f(\bar t, \bar x, \bar u, \bar u_{\bar x}). \tag{5}$$

Consequently, $\mathfrak{P}$ maps a subclass of (4) from $\mathbb{E}$ into another subclass of (5) from $\mathbb{E}$. Remarkably, this very transformation is responsible for all potential symmetries of (1) [13].

Indeed, suppose that (4) admits Lie transformation group

$$t' = T(t, \theta), \qquad x' = X(t, x, u, \theta), \qquad u' = U(t, x, u, \theta), \tag{6}$$

where θ is the group parameter. Note that (6) is the most general one-parameter Lie group that can be admitted by an equation from the class $\mathbb{E}$ (see, e.g., [17]). Computing the first prolongation of formulas (6) we derive the transformation rule for the first derivative of u:

$$\frac{\partial u'}{\partial x'} = \frac{U_u u_x + U_x}{X_u u_x + X_x}.$$

Now the image of transformation group (5) under the action of $\mathfrak{P}$ takes the form

$$\bar t' = T(\bar t, \theta), \qquad \bar x' = X(\bar t, \bar x, u, \theta), \qquad \bar u' = \frac{U_u \bar u + U_{\bar x}}{X_u \bar u + X_{\bar x}} \tag{7}$$

with $u = \partial_{\bar x}^{-1} \bar u$ and $U = U(\bar t, \bar x, u, \theta)$. Consequently, if the right-hand sides of (7) contain nonlocal variable u, then (7) is nonlocal symmetry of (4). According to [13], transformations (7) include the variable u if and only if $X_u \ne 0$ or $U_{\bar x u}^2 + U_{uu}^2 \ne 0$.

Summing up we conclude that provided symmetry group (6) of (4) obeys one of the constraints

$$X_u \ne 0 \quad \text{or} \quad U_{xu}^2 + U_{uu}^2 \ne 0, \tag{8}$$

then $\mathfrak{P}$ maps this symmetry into nonlocal symmetry (7) of (5). This nonlocal symmetry is called the potential symmetry of (5).

Constraints (8) can be written in terms of the coefficients of the infinitesimal operator of group (6):

$$S = \tau(t)\partial_t + \xi(t, x, u)\partial_x + \eta(t, x, u)\partial_u. \tag{9}$$

If

$$\xi_u \ne 0 \quad \text{or} \quad \eta_{xu}^2 + \eta_{uu}^2 \ne 0, \tag{10}$$

then symmetry S is mapped by $\mathfrak{P}$ into nonlocal symmetry [13].

Despite the fact that (4) is rather particular case of the general evolution equation (1), the scheme outlined above provides a very general framework for group classification of nonlocal symmetries.

Let the symbol $\mathcal{S}$ stand for the infinite-dimensional Lie algebra of differential operators (9) and the symbol $\mathcal{I}$ stand for the infinite-dimensional Lie algebra spanned by the operators of the form

$$Q = \xi(t,x,u)\partial_x + \eta(t,x,u)\partial_u. \qquad (11)$$

As direct computations shows, $[\mathcal{I},\mathcal{I}] \subset \mathcal{I}$ and $[\mathcal{I},\mathcal{S}] \subset \mathcal{I}$, which means that $\mathcal{I}$ is the ideal in the Lie algebra $\mathcal{S}$.

If an equation of the form (1) possesses Lie symmetry from $\mathcal{I}$, then it can always be reduced to the form (4) by a suitable local transformation of variables [2]. Consequently, the procedure for classification of nonlocal symmetries described above is applicable to any equation from the class $\mathbb{E}$ admitting at least one symmetry from $\mathcal{I}$.

In the paper [13] we suggest a generalization of the potential symmetry approach which allows for the use of more general differential transformations to construct nonlocal symmetries. Our generalization is based on the idea put forward by Sokolov [16]. He suggests that a differential transformation is to be looked for in the form

$$\mathfrak{D}(t,x,u) = (\bar{t},\bar{x},\bar{u}) \equiv \big(t, \omega_1(t,x,u,u_x,u_{xx},\ldots), \omega_2(t,x,u,u_x,u_{xx},\ldots)\big), \qquad (12)$$

ω_1 and ω_2 being differential invariants of the symmetry group of the evolution equation under study. For the case when $\mathfrak{D} = \mathfrak{P}$, we have $\omega_1 = x$ and $\omega_2 = u_x$, x and u_x being the simplest differential invariants of the group of displacements by u generated by the infinitesimal operator ∂_u. Evidently, evolution equation (4) admits symmetry ∂_u. Sokolov proves in [16] that for any contact symmetry of evolution equation of the form (1) preserving the temporal variable t the corresponding differential transformation maps equation in question into $\mathbb{E}$.

Now we are ready to formulate the group approach to classifying nonlocal symmetries of equations from the class $\mathbb{E}$.

Let (1) be invariant under the r-dimensional Lie algebra $\mathcal{L}_r = \langle e_1,\ldots,e_r\rangle$. Next, suppose that $\mathcal{L}_r$ contains a subalgebra $\mathcal{I}_s = \langle e_1,\ldots,e_s\rangle$, $s<r$, such that $\mathcal{I}_s \subset \mathcal{I}$. Saying it another way, the basis elements of the algebra $\mathcal{I}_s$ have the form (11). Let t,ω_1,ω_2 be functionally independent differential invariants of $\mathcal{I}_s$.

Making differential transformation (12) with so chosen ω_1,ω_2 we map the invariant equation (1) into another equation from $\mathbb{E}$. If the transformation group generated by the operators $e_{s+1},\ldots,e_r$ obeys one of the inequalities (12), then its image under the differential transformation $\mathfrak{D}$ is a nonlocal symmetry of the transformed equation.

Direct application of the transformation of the form (1), especially for the case when ω_1,ω_2 depend on higher derivatives, may become quite challenging computational problem by itself. That is why we suggest a modification of the approach above, which is to represent $\mathfrak{D}$ as a superposition of elementary differential transformations. These elementary transformations are obtained as a superposition of a local transformation reducing the individual basis operators of the Lie algebra $\mathcal{I}_s$ to the form ∂_u and of the differential transformation $\mathfrak{P}$.

We begin with the local change of variables, $\mathfrak{T}_1$, reducing the first basis operator e_1 to the form ∂_u and after that we apply the transformation $\mathfrak{P}$. Next, we recompute the remaining basis operators in the new variables t,x,u_x.

To construct the image of infinitesimal operator (11) under the transformation $\mathfrak{P}$, we compute the coefficient of ∂_{u_x} in its first prolongation:

$$\zeta(t,x,u,u_x) = \eta_x + u_x(\eta_u - \xi_x) - u_x^2\xi_{uu}.$$

Next, we replace u with $\partial_x^{-1}u$ and u_x with u so that the transformed infinitesimal operator, $\bar{Q}$, takes the form

$$\bar{Q} = \xi(t,x,u)\partial_x + \zeta\big(t,x,\partial_x^{-1}u,u\big)\partial_u. \qquad (13)$$

If there is a basis element, e_i, which does not depend on the nonlocal variable $\partial_x^{-1}u$, then we can repeat the procedure and reduce e_i to the form ∂_u and then apply $\mathfrak{P}$, and so on.

The above described procedure has two possible outcomes. The first one is when we are able to perform all s steps and get an evolution equation which is the image of the initial $\mathcal{I}_s$-invariant equation under transformation (12). The resulting differential transformation takes the form

$$\mathfrak{T} = \prod_{i=s}^{1}(\mathfrak{P}\circ\mathfrak{T}_i) \equiv \mathfrak{P}\circ\mathfrak{T}_s\circ\mathfrak{P}\circ\mathfrak{T}_{s-1}\circ\cdots\circ\mathfrak{P}\circ\mathfrak{T}_1. \qquad (14)$$

Applying $\mathfrak{T}$ to (1) yields the sequence of s evolution equations from the class $\mathbb{E}$ together with the sequence of their symmetries, some of which might be nonlocal.

The second outcome is when at some step $p < s$ all the remaining basis operators contain the nonlocal variable $\partial_x^{-1} u$ and we cannot proceed any further. In this case we obtain the sequence of transformations

$$\mathfrak{T} = \prod_{i=p}^{1} (\mathfrak{P} \circ \mathfrak{T}_i) \equiv \mathfrak{P} \circ \mathfrak{T}_p \circ \mathfrak{P} \circ \mathfrak{T}_{p-1} \circ \cdots \circ \mathfrak{P} \circ \mathfrak{T}_1 \tag{15}$$

that maps the initial $\mathcal{I}_s$-invariant evolution equation into equation which admits at least $s - p$ nonlocal symmetries of the form (13).

We summarize the above speculations in the form of the algorithm of classification of evolution equations possessing nonlocal symmetries.

The starting point of our classification algorithm is evolution equation (1) invariant under r-dimensional Lie algebra $\mathcal{L}_r \subset \mathcal{S}$. Suppose also that the algebra $\mathcal{L}_r$ contains a subalgebra $\mathcal{I}_s \subset \mathcal{I}$ of the dimension $1 \le s \le r$. Now, to classify evolution equations admitting nonlocal symmetries, one needs to:

- classify $\mathcal{I}_s$-inequivalent subalgebras $\mathcal{J}^1, \ldots, \mathcal{J}^N$ of the algebra $\mathcal{I}_s$,
- compute the sequence of transformations (14) mapping the initial invariant equation into $\mathbb{E}$ for each subalgebra $\mathcal{J}^i$, $i = 1, \ldots, N$, and generate the corresponding sequence of evolution equations and their invariance algebras,
- verify for each invariance algebra whether its basis elements satisfy one of the inequalities (10). If this is the case, then the remaining elements of the sequence of evolution equations possess nonlocal symmetry.

Upon completing the three steps of the algorithm above, ones obtains the set of evolution equations (1) that admit nonlocal symmetries.

3 Applications

In [14, 15, 17] we obtain exhaustive description of inequivalent Lie algebras, which are symmetry algebras of equations of the form (1). This fact facilitates application of the algorithm presented in the previous section in its full generality. However, in this paper we restrict our considerations to subalgebras of the symmetry algebra of (1) of the dimension $s \le 3$.

The list of inequivalent realizations of one-dimensional symmetry algebras of (1) consists of one representative

$$\mathcal{A}_1 = \langle \partial_u \rangle: \quad u_t = f(t, x, u_x, u_{xx}).$$

This case has already been considered in the previous section, the algebra $\mathcal{A}_1$ giving rise to the transformation $\mathfrak{P}$.

According to [17] the class of operators, $\mathcal{I}$, contains four inequivalent realizations of two-dimensional symmetry algebras of (1). These realizations are listed below together with the corresponding invariant equations:

$$\begin{aligned}
&\mathcal{A}_2^1 = \langle \partial_x, \partial_u \rangle: \quad u_t = f(t, u_x, u_{xx}),\\
&\mathcal{A}_2^2 = \langle x\partial_u, \partial_u \rangle: \quad u_t = f(t, x, u_{xx}),\\
&\mathcal{A}_2^3 = \langle -x\partial_x - u\partial_u, \partial_u \rangle: \quad u_t = x f(t, u_x, x u_{xx}),\\
&\mathcal{A}_2^4 = \langle -u\partial_u, \partial_u \rangle: \quad u_t = u_x f\big(t, x, u_x^{-1} u_{xx}\big).
\end{aligned}$$

We start by analyzing differential transformations associated with the algebra $\mathcal{A}_2^1$. Applying the transformation $\mathfrak{P}$ to the equation $u_t = f(t, u_x, u_{xx})$ we get

$$u_t = f_u + u_{xx} f_{u_x},$$

where $f = f(t, u, u_x)$. Note that we dropped the bars.

The second basis operator of $\mathcal{A}_2^1$ reads now as ∂_x. To reduce it to the form ∂_u we apply the hodograph transformation

$$\mathfrak{T}_1(t, x, u) = (\bar{t}, \bar{x}, \bar{u}) \equiv (t, u, x). \tag{16}$$

The transformed equations has the form

$$u_t = -f_{z_1} + \frac{u_{xx}}{u_x} f_{z_2},$$

where $f = f(t, z_1, z_2) \equiv f(t, u, -u_x^{-1})$. Applying the transformation $\mathfrak{P}$, yields the final form of the transformed evolution equation:

$$u_t = \frac{\partial}{\partial x}\left(-f_{z_1} + \frac{u_x}{u} f_{z_2}\right).$$

Here $f = f(t, z_1, z_2) \equiv f(t, u, -u^{-1})$.

Turn now to the algebra $\mathcal{A}_2^2$. Applying the transformation $\mathfrak{P}$ to the invariant equation $u_t = f(t, x, u_{xx})$

yields

$$u_t = f_x + u_{xx} f_{u_x}$$

with $f = f(t, x, u_x)$. As usual, we drop the bars. The second basis element of $\mathcal{A}_2^2$ takes the form ∂_u and we can directly apply the transformation $\mathfrak{P}$ thus getting

$$u_t = f_{xx} + 2u_x f_{xu} + u_x^2 f_{uu} + f_u u_{xx}.$$

Consider next the algebra $\mathcal{A}_2^3$. Applying the transformation $\mathfrak{P}$ to the equation $u_t = x f(t, u_x, x u_{xx})$ we have

$$u_t = f + x u_x f_{z_1} + x(u_x + x u_{xx}) f_{z_2} \tag{17}$$

with $f = f(t, z_1, z_2) \equiv f(t, u, x u_x)$. The second basis operator of the algebra $\mathcal{A}_2^3$ reads now as $x\partial_x$. Applying the hodograph type local transformation

$$\mathfrak{T}_2(t, x, u) = (\bar{t}, \bar{x}, \bar{u}) \equiv (t, u, \ln x) \tag{18}$$

we reduce it to the form ∂_u. Transforming (17) accordingly, we get

$$u_t = -x f_{z_1} + \big(x^2 u_x^{-2} u_{xx} + x u_x^{-1} - x\big) f_{z_2},$$

where $f = f(t, z_1, z_2) \equiv f(t, x, u_x^{-1})$. Now we can apply the transformation $\mathfrak{P}$ thus getting

$$u_t = \frac{\partial}{\partial x}\big(-x f_{z_1} + \big(x^2 u^{-2} u_x + x u^{-1} - x\big) f_{z_2}\big)$$

with $f = f(t, z_1, z_2) \equiv f(t, x, u^{-1})$.

Finally, we turn to the algebra $\mathcal{A}_2^4$. Applying the transformation $\mathfrak{P}$ to the invariant equation $u_t = u_x f(t, x, u_x^{-1} u_{xx})$ yields

$$u_t = u_x f + u f_{z_1} + \big(u_{xx} - u^{-1} u_x^2\big) f_{z_2}, \tag{19}$$

where $f = f(t, z_1, z_2) \equiv f(t, x, u^{-1} u_x)$. The second basis element of the algebra $\mathcal{A}_2^4$ is still of the form $u\partial_u$, so we need to reduce it to the form ∂_u. To this end, we perform the local transformation

$$\mathfrak{T}_3(t, x, u) = (\bar{t}, \bar{x}, \bar{u}) \equiv (t, x, \ln u). \tag{20}$$

As a result, (19) takes the form

$$u_t = u_x f + f_{z_1} + u_{xx} f_{z_2}$$

with $f = f(t, z_1, z_2) \equiv f(t, x, u_x)$. Applying the transformation $\mathfrak{P}$ finally yields

$$u_t = u_x f + u f_x + u u_x f_u + f_{xx} + 2u_x f_{xu} + u_x^2 f_{uu} + u_{xx} f_u,$$

where $f = f(t, x, u)$.

Summing up, we present the four sequences of differential transformations that map four subclasses of the class of second-order evolution equations $\mathbb{E}$ into $\mathbb{E}$.

- Differential transformation $\mathfrak{P} \circ \mathfrak{T}_1 \circ \mathfrak{P}$ maps equation
$$u_t = f(t, u_x, u_{xx})$$
into
$$u_t = \frac{\partial}{\partial x}\left(-f_{z_1} + \frac{u_x}{u} f_{z_2}\right)$$
with $f = f(t, z_1, z_2) \equiv f(t, u, -1/u)$.
- Differential transformation $\mathfrak{P} \circ \mathfrak{P}$ maps equation
$$u_t = f(t, x, u_{xx})$$
into
$$u_t = f_{xx} + 2u_x f_{xu} + u_x^2 f_{uu} + f_u u_{xx}$$
with $f = f(t, x, u)$.
- Differential transformation $\mathfrak{P} \circ \mathfrak{T}_2 \circ \mathfrak{P}$ maps equation
$$u_t = x f(t, u_x, x u_{xx})$$
into
$$u_t = \frac{\partial}{\partial x}\big(-x f_{z_1} + \big(x^2 u^{-2} u_x + x u^{-1} - x\big) f_{z_2}\big)$$
with $f = f(t, z_1, z_2) = f(t, x, u^{-1})$.
- Differential transformation $\mathfrak{P} \circ \mathfrak{T}_3 \circ \mathfrak{P}$ maps equation
$$u_t = u_x f\big(t, x, u_x^{-1} u_{xx}\big)$$
into
$$u_t = u_x f + u f_x + u u_x f_u + f_{xx} + 2u_x f_{xu} + u_x^2 f_{uu} + u_{xx} f_u, \quad f = f(t, x, u).$$

In the above formulas the transformations $\mathfrak{T}_1, \mathfrak{T}_2, \mathfrak{T}_3$ are given by (16), (17) and (20), respectively.

The class of first-order differential operators $\mathcal{I}$ contains twenty-two inequivalent realizations of three-dimensional symmetry algebras of (1) [17]. These realizations are listed below together with the corresponding invariant equations:

$$\mathcal{A}_3^1 = \langle -x\partial_x - u\partial_u, \partial_u, xt\partial_x\rangle: \quad u_t = -xt^{-1}u_x \ln u_x + xu_x f\big(t, xu_x^{-1}u_{xx}\big),$$

$$\mathcal{A}_3^2 = \langle -x\partial_x - u\partial_u, \partial_u, x\partial_x\rangle: \quad u_t = xu_x f\big(t, xu_x^{-1}u_{xx}\big),$$

$$\mathcal{A}_3^3 = \langle -x\partial_x - u\partial_u, \partial_u, x\partial_u\rangle: \quad u_t = xf(t, xu_{xx}),$$

$$\mathcal{A}_3^4 = \langle -u\partial_u, \partial_u, \partial_x\rangle: \quad u_t = u_x f\big(t, u_x^{-1}u_{xx}\big),$$

$$\mathcal{A}_3^5 = \langle \partial_u, \partial_x, x\partial_u + t\partial_x\rangle: \quad u_t = -\frac{1}{2}u_x^2 + f(t, u_{xx}),$$

$$\mathcal{A}_3^6 = \langle \partial_u, \partial_x, x\partial_u\rangle: \quad u_t = f(t, u_{xx}),$$

$$\mathcal{A}_3^7 = \big\langle x\partial_u, \partial_u, x^2\partial_x + xu\partial_u\big\rangle: \quad u_t = xf\big(t, x^3u_{xx}\big),$$

$$\mathcal{A}_3^8 = \big\langle \partial_x, \partial_u, (x+u)\partial_x + u\partial_u\big\rangle: \quad u_t = u_x \exp\big(u_x^{-1}\big) f\big(t, u_x^{-3}u_{xx}\exp\big(u_x^{-1}\big)\big),$$

$$\mathcal{A}_3^9 = \big\langle x\partial_u, \partial_u, x^2\partial_x + u(1+x)\partial_u\big\rangle: \quad u_t = x\exp\big(-x^{-1}\big) f\big(t, xu_{xx}\exp\big(x^{-1}\big)\big),$$

$$\mathcal{A}_3^{10} = \langle \partial_x, \partial_u, x\partial_x + u\partial_u\rangle: \quad u_t = u_{xx}^{-1} f(t, u_x),$$

$$\mathcal{A}_3^{11} = \langle \partial_x, \partial_u, x\partial_x - u\partial_u\rangle: \quad u_t = u_x^{1/2} f\big(t, u_x^{-3/2}u_{xx}\big),$$

$$\mathcal{A}_3^{12} = \langle x\partial_u, \partial_u, -2x\partial_x - u\partial_u\rangle: \quad u_t = x^{1/2} f\big(t, x^{3/2}u_{xx}\big),$$

$$\mathcal{A}_3^{13} = \langle \partial_x, \partial_u, x\partial_x + qu\partial_u\rangle: \quad u_t = u_x^{q/(q-1)} f\big(t, u_x^{(2-q)/(q-1)}u_{xx}\big), \quad 0 < |q| < 1,$$

$$\mathcal{A}_3^{14} = \big\langle x\partial_u, \partial_u, x(q-1)\partial_x + qu\partial_u\big\rangle: \quad u_t = x^{q/(q-1)} f\big(t, x^{(2-q)/(1-q)}u_{xx}\big), \quad q \neq 0, \pm 1,$$

$$\mathcal{A}_3^{15} = \langle \partial_x, \partial_u, u\partial_x - x\partial_u\rangle: \quad u_t = \big(1+u_x^2\big)^{1/2} f\big(t, \big(1+u_x^2\big)^{-3/2}u_{xx}\big),$$

$$\mathcal{A}_3^{16} = \big\langle x\partial_u, \partial_u, \big(1+x^2\big)\partial_x + xu\partial_u\big\rangle: \quad u_t = \big(1+x^2\big)^{1/2} f\big(t, \big(1+x^2\big)^{3/2}u_{xx}\big),$$

$$\mathcal{A}_3^{17} = \big\langle \partial_x, \partial_u, (qx+u)\partial_x + (-x+qu)\partial_u\big\rangle: \quad u_t = \big(1+u_x^2\big)^{1/2}\exp(-q\arctan u_x) \times f\big(t, \big(1+u_x^2\big)^{-3/2}\exp(-q\arctan u_x)u_{xx}\big), \quad q > 0,$$

$$\mathcal{A}_3^{18} = \big\langle x\partial_u, \partial_u, \big(1+x^2\big)\partial_x + (x+q)u\partial_u\big\rangle: \quad u_t = \big(1+x^2\big)^{1/2}\exp(\arctan x) \times f\big(t, \big(1+x^2\big)^{3/2}\exp(-q\arctan x)u_{xx}\big), \quad q \neq 0,$$

$$\mathcal{A}_3^{19} = \langle \partial_u, \cos u\,\partial_x + \tan x \sin u\,\partial_u, -\sin u\,\partial_x + \tan x\cos u\,\partial_u\rangle:$$
$$u_t = \big(\sec^2 x + u_x^2\big)^{1/2} \times f\big(t, \big(u_{xx}\cos x - \big(2 + u_x^2\cos^2 x\big)u_x\sin x\big) \times \big(1+u_x^2\cos^2 x\big)^{-3/2}\big),$$

$$\mathcal{A}_3^{20} = \big\langle 2u\partial_u - x\partial_x, -u^2\partial_u + xu\partial_x, \partial_u\big\rangle: \quad u_t = xu_x f\big(t, x^{-5}u_x^{-3}u_{xx} + 2x^{-6}u_x^{-2}\big),$$

$$\mathcal{A}_3^{21} = \big\langle 2u\partial_u - x\partial_x, \big(x^{-4} - u^2\big)\partial_u + xu\partial_x, \partial_u\big\rangle:$$
$$u_t = x^{-2}\big(4 + x^6u_x^2\big)^{1/2} \times f\left(t, \big(4+x^6u_x^2\big)^{-3/2} \times \left(x^4u_{xx} + 5x^3u_x + \frac{1}{2}x^9u_x^3\right)\right),$$

$$\mathcal{A}_3^{22} = \big\langle 2u\partial_u - x\partial_x, -\big(x^{-4} + u^2\big)\partial_u + xu\partial_x, \partial_u\big\rangle:$$
$$u_t = x^{-2}\big(x^6u_x^2 - 4\big)^{1/2} \times f\left(t, \big(x^6u_x^2 - 4\big)^{-3/2} \times \left(x^4u_{xx} + 5x^3u_x - \frac{1}{2}x^9u_x^3\right)\right).$$

Analysis of the sequences of differential transformations associated with the above algebras is similar to the case of two-dimensional algebras. We skip the derivation details and give the final result, the differential transformations (14), (15).

For the algebras $\mathcal{A}_3^1$-$\mathcal{A}_3^6$, $\mathcal{A}_3^{10}$-$\mathcal{A}_3^{14}$ the sequence of transformations $\mathfrak{T}$ reads as

$$\mathfrak{T} = \mathfrak{P} \circ \mathfrak{T}_2 \circ \mathfrak{P} \circ \mathfrak{T}_1 \circ \mathfrak{P} \tag{21}$$

where

$\mathcal{A}_3^1$: $\mathfrak{T}_1(t, x, u) = (t, u, -\ln x)$,

$\mathfrak{T}_2(t, x, u) = \left(t, xu, t^{-1}\ln x\right)$,

$\mathcal{A}_3^2$: $\mathfrak{T}_1(t, x, u) = (t, u, -\ln x)$,

$\mathfrak{T}_2(t, x, u) = (t, xu, \ln x)$,

$\mathcal{A}_3^3$: $\mathfrak{T}_1(t, x, u) = (t, x, u)$,

$\mathfrak{T}_2(t, x, u) = (t, xu, \ln x)$,

$\mathcal{A}_3^4$: $\mathfrak{T}_1(t, x, u) = (t, x, -\ln u)$,

$\mathfrak{T}_2(t, x, u) = (t, xu, \ln x)$,

$\mathcal{A}_3^5$: $\mathfrak{T}_1(t, u, x) = (t, u, x)$,

$\mathfrak{T}_2(t, x, u) = (t, u, x)$,

$\mathcal{A}_3^6$: $\mathfrak{T}_1(t, x, u) = (t, x, u)$,

$\mathfrak{T}_2(t, x, u) = (t, u, x)$,

$\mathcal{A}_3^{10}$: $\mathfrak{T}_1(t, x, u) = (t, u, x)$,

$\mathfrak{T}_2(t, x, u) = (t, x, \ln u)$,

$\mathcal{A}_3^{11}$: $\mathfrak{T}_1(t, x, u) = (t, u, x)$,

$\mathfrak{T}_2(t, x, u) = \left(t, x^3u^2, -\dfrac{1}{3}\ln u\right)$,

$\mathcal{A}_3^{12}$: $\mathfrak{T}_1(t, x, u) = (t, u, x)$,

$\mathfrak{T}_2(t, x, u) = \left(t, x^3u^2, -\dfrac{1}{3}\ln u\right)$,

$\mathcal{A}_3^{13}$: $\mathfrak{T}_1(t, x, u) = (t, u, x)$,

$\mathfrak{T}_2(t, x, u) = \left(t, x^{q-2}u^{q-1}, \dfrac{1}{q-2}\ln u\right)$,

$q \neq 2$,

$\mathfrak{T}_1(t, x, u) = (t, u, x)$,

$\mathfrak{T}_2(t, x, u) = (t, u, \ln u)$, $\quad q = 2$,

$\mathcal{A}_3^{14}$: $\mathfrak{T}_1(t, x, u) = (t, x, u)$,

$\mathfrak{T}_2(t, x, u) = \left(t, x^{q-2}u^{q-1}, \dfrac{1}{q-2}\ln u\right)$,

$q \neq 2$,

$\mathfrak{T}_1(t, x, u) = (t, x, u)$,

$\mathfrak{T}_2(t, x, u) = (t, u, \ln u)$, $\quad q = 2$.

Now if an equation invariant under one of the algebras from the list, $\mathcal{A}_3^1$-$\mathcal{A}_3^6$, $\mathcal{A}_3^{10}$-$\mathcal{A}_3^{14}$, admits additional Lie symmetry satisfying (10), then every member of the sequence of equations obtained by successive application of transformations $\mathfrak{P} \circ \mathfrak{T}_2 \circ \mathfrak{P} \circ \mathfrak{T}_1 \circ \mathfrak{P}$ possesses nonlocal symmetry. Moreover, if we are able to construct an exact solution of some equation from the sequence, we can map this solution into solutions of the remaining equations from the sequence.

For the algebras $\mathcal{A}_3^7$-$\mathcal{A}_3^9$, $\mathcal{A}_3^{15}$-$\mathcal{A}_3^{18}$, and $\mathcal{A}_3^{20}$-$\mathcal{A}_3^{22}$ the algorithm stops after the second iteration, since the third symmetry operator turns into a nonlocal one. Below we give the corresponding sequences of transformations:

$\mathcal{A}_3^7$: $\mathfrak{P} \circ \mathfrak{P}$,

$\mathcal{A}_3^8$: $\mathfrak{P} \circ \mathfrak{T}_1 \circ \mathfrak{P}$,

$\mathcal{A}_3^9$: $\mathfrak{P} \circ \mathfrak{P}$,

$\mathcal{A}_3^{15}$: $\mathfrak{P} \circ \mathfrak{T}_1 \circ \mathfrak{P}$,

$\mathcal{A}_3^{16}$: $\mathfrak{P} \circ \mathfrak{P}$,

$\mathcal{A}_3^{17}$: $\mathfrak{P} \circ \mathfrak{T}_1 \circ \mathfrak{P}$,

$\mathcal{A}_3^{18}$: $\mathfrak{P} \circ \mathfrak{P}$,

$\mathcal{A}_3^{20}$: $\mathfrak{P} \circ \mathfrak{T}_2 \circ \mathfrak{P}$,

$\mathcal{A}_3^{21}$: $\mathfrak{P} \circ \mathfrak{T}_2 \circ \mathfrak{P}$,

$\mathcal{A}_3^{22}$: $\mathfrak{P} \circ \mathfrak{T}_2 \circ \mathfrak{P}$,

where

$\mathfrak{T}_1(t, x, u) = (t, u, x)$,

$\mathfrak{T}_2(t, x, u) = \left(t, x^3u, \dfrac{1}{3}\ln u\right)$.

Applying the above sequences of transformations to the corresponding invariant equations yields evolution equations of the form (1) that admit nonlocal symmetries. The explicit form of those symmetries is very cumbersome, therefore we present the nonlocal symmetries obtained through the action of $\mathfrak{P}$ only. In addition, we give the corresponding transformed equations.

$\mathcal{A}_3^7$: $e_3 = x^2\partial_x + (v - xu)\partial_u$,

$$u_t = \frac{\partial}{\partial x}\big(xf\big(t, x^3u_x\big)\big),$$

$\mathcal{A}_3^8$: $e_3 = (x + v)\partial_x - u\partial_u$,

$$u_t = \frac{\partial}{\partial x}\big(u\exp\big(u^{-1}\big)f\big(t, u^{-3}\exp\big(u^{-1}\big)u_x\big)\big),$$

$\mathcal{A}_3^9$: $e_3 = x^2\partial_x + \big((1 - x)u + v\big)\partial_u$,

$$u_t = \frac{\partial}{\partial x}\big(x\exp\big(-x^{-1}\big)f\big(t, x\exp\big(x^{-1}\big)u_x\big)\big),$$

$\mathcal{A}_3^{15}$: $e_3 = v\partial_x - (u + 1)\partial_u$,

$$u_t = \frac{\partial}{\partial x}\big(\big(1 - u^2\big)^{1/2}f\big(t, \big(1 + u^2\big)^{-3/2}u_x\big)\big)$$

$\mathcal{A}_3^{16}$: $e_3 = \big(1 + x^2\big)\partial_x + (v - xu)\partial_u$,

$$u_t = \frac{\partial}{\partial x}\big(\big(1 + x^2\big)^{1/2}f\big(t, \big(1 + x^2\big)^{3/2}u_x\big)\big),$$

$\mathcal{A}_3^{17}$: $e_3 = (qx + v)\partial_x - (u + 1)\partial_u$,

$$u_t = \frac{\partial}{\partial x}\big(\big(1 + u^2\big)^{1/2}\exp(-q\arctan u) \times f\big(t, \big(1 + u^2\big)^{-3/2} \times \exp(-q\arctan u)u_x\big)\big),$$

$\mathcal{A}_3^{18}$: $\big(1 + x^2\big)\partial_x + \big((q - x)u + v\big)\partial_u$,

$$u_t = \frac{\partial}{\partial x}\big(\big(1 + x^2\big)^{1/2}\exp(\arctan x) \times f\big(t, \big(1 + x^2\big)^{3/2} \times \exp(-q\arctan x)u_x\big)\big),$$

$\mathcal{A}_3^{20}$: $e_2 = xv\partial_x - (xu + 3uv)\partial_u$,

$$u_t = \frac{\partial}{\partial x}\big(xuf\big(t, x^{-5}u^{-3}u_x + 2x^{-6}u^{-2}\big)\big),$$

$\mathcal{A}_3^{21}$: $e_2 = xu\partial_x - \big(4x^{-5} + xu + 3uv\big)\partial_u$,

$$u_t = \frac{\partial}{\partial x}\left(x^{-2}\big(4 + x^6u^2\big)^{1/2} \times f\left(t, \big(4 + x^6u^2\big)^{-3/2} \times \left(x^4u_x + 5x^3u + \frac{1}{2}x^9u^3\right)\right)\right),$$

$\mathcal{A}_3^{22}$: $e_2 = xu\partial_x + \big(4x^{-5} - xu - 3uv\big)\partial_u$,

$$u_t = \frac{\partial}{\partial x}\left(x^{-2}\big(x^6u^2 - 4\big)^{1/2} \times f\left(t, \big(x^6u^2 - 4\big)^{-3/2} \times \left(x^4u_x + 5x^3u - \frac{1}{2}x^9u^3\right)\right)\right). \tag{22}$$

Here $v = \partial_x^{-1}u$ is the nonlocal variable.

Finally, when computing transformations associated with the algebra $\mathcal{A}_3^{19}$ we have to stop after the first iteration, since the second and third basis operators are mapped by $\mathfrak{P}$ into nonlocal symmetries

$$\begin{aligned} e_2 &= \cos u\partial_x + \big(u\sin v + \sec^2 x\sin v + u\cos v\tan x\big)\partial_u, \\ e_3 &= -\sin u\partial_x + \big(u\cos v + \cos v\sec^2 x - u\sin v\tan x\big)\partial_u, \end{aligned} \tag{23}$$

where $v = \partial_x^{-1}u$. So that the sequence of differential transformations boils down to a single transformation $\mathfrak{T} = \mathfrak{P}$. And what is more, the evolution equation invariant under the nonlocal symmetries e_2 and e_3 reads as

$$u_t = \big(\sec^2 x\tan x + uu_x\big)\big(\sec^2 x + u^2\big)^{-1/2}f + \big(\sec^2 x + u^2\big)^{1/2}\frac{\partial f}{\partial x}$$

with

$$f = f\big(t, \big(u_x\cos x - \big(2 + u^2\cos^2 x\big)u\sin x\big) \times \big(1 + u^2\cos^2 x\big)^{-3/2}\big).$$

Let us emphasize that symmetries (22), (23) contain nonlocal variable and, consequently, cannot be obtained within the framework of the traditional infinitesimal Lie method.

4 Concluding remarks

We develop the regular method for group classification of evolution equations admitting nonlocal symmetries. The starting point of the method is an evolution equation that possesses nontrivial Lie symmetry. The source of nonlocal symmetries is classical symmetries subjected to nonlocal transformations.

One of the by-products of our approach is the sequences of evolution equations related to the initial invariant equation. So that if we succeed in constructing a (general or particular) solution of the sequence member then it is possible to map this solution into solutions of the remaining equations from the sequence.

Of special interest is the case when one of the members of the sequence is the linear evolution equation, since this means that the remaining equations are linearizable. Moreover, a direct application of the method to linear evolution equations admitting nontrivial Lie symmetries is also of importance, since it yields sequences of linearizable evolution equations.

These and related problems are under study now and will be reported elsewhere.

References

1. Lie, S., Engel, F.: Theorie der Transformationsgruppen. Teubner, Leipzig (1890)
2. Ovsyannikov, L.V.: Group Analysis of Differential Equations. Academic Press, San Diego (1982)
3. Olver, P.: Applications of Lie Groups to Differential Equations. Springer, New York (1987)
4. Bluman, G., Kumei, S.: Symmetries and Differential Equations. Springer, New York (1989)
5. Ibragimov, N.H.: Transformation Groups Applied to Mathematical Physics. Reidel, Dordrecht (1985)
6. Fushchych, W.I., Zhdanov, R.Z.: Symmetries of Nonlinear Dirac Equations. Mathematical Ukraina Publishers, Kyiv (1997)
7. Fushchych, W.I., Nikitin, A.G.: Symmetries of Equations of Quantum Mechanics. Allerton Press, New York (1994)
8. Akhatov, I.S., Gazizov, R.K. Ibragimov, N.H.: Nonlocal symmetries: A heuristic approach. J. Sov. Math. **55**, 1401–1450 (1991)
9. Meirmanov, A.M., Pukhnachov, V.M., Shmarev, S.I.: Evolution equations and Lagrangian coordinates. de Gruyter, Berlin (1997)
10. Bluman, G.W., Kumei, S., Reid, G.J.: New classes of symmetries for partial differential equations. J. Math. Phys. **29**, 806–811 (1988)
11. Bluman, G.W.: Use and construction of potential symmetries. Math. Comput. Model. **18**, 1–14 (1993)
12. Blumen, G.W., Cheviakov, A.F.: Framework for potential systems and nonlocal symmetries: Algorithmic approach. J. Math. Phys. **46**, 123506 (2005)
13. Zhdanov, R.Z.: On relation between potential and contact symmetries of evolution equations. J. Math. Phys. **50**, 053522 (2009)
14. Zhdanov, R.Z., Lahno, V.I.: Group classification of the general evolution equation: local and quasilocal symmetries. SIGMA **1**, 009 (2005)
15. Zhdanov, R., Lahno, V.: Group classification of the general second-order evolution equation: Semi-simple invariance groups. J. Phys. A: Math. Theor. **40**, 5083–5103 (2007)
16. Sokolov, V.V.: Symmetries of evolution equations. Russ. Math. Surv. **43**, 165–204 (1988)
17. Basarab-Hormath, P., Lahno, V., Zhdanov, R.: The structure of Lie algebras and the classification problem for partial differential equations. Acta Appl. Math. **69**, 43–94 (2001)