Nonlinear Dyn (2010) 59: 535–543
DOI 10.1007/s11071-009-9560-6

ORIGINAL PAPER

# T–S fuzzy $\mathcal{H}_\infty$ synchronization for chaotic systems via delayed output feedback control

**Choon Ki Ahn**

Received: 22 May 2009 / Accepted: 3 July 2009 / Published online: 24 July 2009


**Abstract** In this paper, we propose a new $\mathcal{H}_\infty$ synchronization strategy, called a fuzzy delayed output feedback $\mathcal{H}_\infty$ synchronization (FDOFHS) strategy, for chaotic systems in the presence of external disturbance. Based on Lyapunov–Krasovskii theory, the T–S fuzzy model, and a delayed feedback control scheme, the FDOFHS controller can guarantee stable synchronization. Furthermore, this controller reduces the effect of external disturbance to an $\mathcal{H}_\infty$ norm constraint. The proposed controller can be obtained by solving the linear matrix inequality (LMI) problem. A simulation study is presented to demonstrate the validity of the proposed FDOFHS approach.

**Keywords** $\mathcal{H}_\infty$ synchronization · Chaotic systems · Linear matrix inequality (LMI) · Takagi–Sugeno (T–S) fuzzy model · Delayed output feedback control

## 1 Introduction

During the last two decades, synchronization in chaotic dynamic systems has received a great deal of interest among scientists from various research fields since Pecora and Carroll [1] introduced a method to synchronize two identical chaotic systems with different initial conditions. It has been widely explored in a variety of fields including physical, chemical and ecological systems [2]. Over the past several years, the delayed feedback control approach, proposed by Pyragas [3], has received considerable attention. The use of time delay in the feedback loop eliminates the need for explicitly determining any information about the underlying dynamics other than the period of the desired orbit. The papers in [4–6] used the linear part and an assumption on the nonlinearities of Lur'e chaotic systems to achieve synchronization. In spite of some advances in the delayed feedback control, it is hard in general to get the corresponding linearized models along the system trajectory during the drive–response procedure.

In recent years, fuzzy logic has received much attention as a powerful tool for the nonlinear control. Among various kinds of fuzzy methods, Takagi–Sugeno (T–S) fuzzy model provides a successful method to describe certain complex nonlinear systems using some local linear subsystems [7, 8]. These linear subsystems are smoothly blended together through fuzzy membership functions. It is therefore intuitive to believe that the T–S fuzzy model can be used to develop synchronization methods via the delayed feedback control without the assumption as in [4–6].

In real physical systems, one is faced with model uncertainties and a lack of statistical information on the signals. This had led in recent years to an interest

This paper was supported by Wonkwang University in 2009.

C.K. Ahn (✉)
Division of Electronic and Control Engineering,
Wonkwang University, Iksan, Korea
e-mail: hironaka@wonkwang.ac.kr

in mini-max control, with the belief that $\mathcal{H}_\infty$ control is more robust and less sensitive to disturbance variances and model uncertainties [9]. In order to reduce the effect of the disturbance, Hou et al. [10] firstly adopted the $\mathcal{H}_\infty$ control concept [9] for chaotic synchronization problem. Recently, a dynamic controller for the $\mathcal{H}_\infty$ synchronization was proposed in [11]. To the best of our knowledge, however, for the T–S fuzzy model based $\mathcal{H}_\infty$ chaos synchronization via the delayed feedback control, there has appeared no result in the literature so far, which still remains challenging.

In this paper, a new $\mathcal{H}_\infty$ synchronization method based on the T–S fuzzy model and the delayed feedback control is proposed. This method is called a fuzzy delayed output feedback $\mathcal{H}_\infty$ synchronization (FDOFHS) method. By the proposed scheme, the closed-loop error system is asymptotically synchronized and the $\mathcal{H}_\infty$ norm from the external disturbance to the synchronization error is reduced to a disturbance attenuation level. Based on the Lyapunov–Krasovskii method [14, 15] and the linear matrix inequality (LMI) approach, an existence criterion for the proposed controller is represented in terms of the LMI. The LMI problem can be solved efficiently by using recently developed convex optimization algorithms [12].

This paper is organized as follows. In Sect. 2, we formulate the problem. In Sect. 3, the FDOFHS controller is proposed for chaotic systems with external disturbance. In Sect. 4, an application example for a Lorenz system is given, and finally, conclusions are presented in Sect. 5.

## 2 Problem formulation

In system analysis and design, it is important to select an appropriate model representing a real system. As an expression model of a real plant, we use the fuzzy implications and the fuzzy reasoning method suggested by Takagi and Sugeno [7]. Let us represent the chaotic system as follows:

Fuzzy Rule $i$:

IF $\omega_1$ is $\mu_{i1}$ and $\ldots$ $\omega_s$ is $\mu_{is}$ THEN

$$\dot{x}(t) = A_i x(t) + \eta_i(t), \tag{1}$$

$$y(t) = Cx(t), \tag{2}$$

where $x(t) \in R^n$ is the state vector, $y(t) \in R^m$ is the output vector, $A_i \in R^{n\times n}$ and $C \in R^{m\times n}$ are known constant matrices, $\eta_i(t) \in R^n$ denotes a bias term which is generated by the fuzzy modeling procedure, $\omega_j$ $(j = 1, \ldots, s)$ is the premise variable, $\mu_{ij}$ $(i = 1, \ldots, r, j = 1, \ldots, s)$ is the fuzzy set that is characterized by a membership function, $r$ is the number of the IF–THEN rules, and $s$ is the number of the premise variables.

Using a standard fuzzy inference method (using a singleton fuzzifier, product fuzzy inference, and weighted average defuzzifier), the system (1)–(2) is inferred as follows:

$$\dot{x}(t) = \sum_{i=1}^{r} h_i(\omega)\big[A_i x(t) + \eta_i(t)\big], \tag{3}$$

$$y(t) = Cx(t), \tag{4}$$

where $\omega = [\omega_1, \ldots, \omega_s]$, $h_i(\omega) = \varpi_i(\omega)/\sum_{i=1}^{r}\varpi_j(\omega)$, $\varpi_i : R^s \to [0, 1]$ $(i = 1, \ldots, r)$ is the membership function of the system with respect to the fuzzy rule $i$. $h_i$ can be regarded as the normalized weight of each IF–THEN rule and it satisfies

$$h_i(\omega) \geq 0, \quad \sum_{i=1}^{r} h_i(\omega) = 1. \tag{5}$$

The system (3)–(4) is considered as a drive system.

The synchronization problem of system (3)–(4) is considered by using the drive–response configuration. According to the drive–response concept, the controlled fuzzy response system is described by the following rules:

Fuzzy Rule $i$:

IF $\omega_1$ is $\mu_{i1}$ and $\ldots$ $\omega_s$ is $\mu_{is}$ THEN

$$\dot{\hat{x}}(t) = A_i\hat{x}(t) + \eta_i(t) + u(t) + G_i d(t), \tag{6}$$

$$\hat{y}(t) = C\hat{x}(t), \tag{7}$$

where $\hat{x}(t) \in R^n$ is the state vector of the response system, $\hat{y}(t) \in R^m$ is the output vector of the response system, $u(t) \in R^n$ is the control input, $d(t) \in R^k$ is the external disturbance, and $G_i \in R^{n\times k}$ is a known constant matrix. The fuzzy response system can be inferred as

$$\dot{\hat{x}}(t) = \sum_{i=1}^{r} h_i(\omega)\big[A_i\hat{x}(t) + \eta_i(t) + u(t) + G_i d(t)\big], \tag{8}$$

$$\hat{y}(t) = C\hat{x}(t). \tag{9}$$

Define the synchronization error $e(t) = \hat{x}(t) - x(t)$. Then we obtain the synchronization error system

$$\dot{e}(t) = \sum_{i=1}^{r} h_i(\omega)\big[A_i e(t) + u(t) + G_i d(t)\big]. \tag{10}$$

**Definition 1** (Asymptotical synchronization) The error system (10) is asymptotically synchronized if the synchronization error $e(t)$ satisfies

$$\lim_{t\to\infty} e(t) = 0. \tag{11}$$

**Definition 2** ($\mathcal{H}_\infty$ synchronization) The error system (10) is $\mathcal{H}_\infty$ synchronized if the synchronization error $e(t)$ satisfies

$$\int_0^\infty e^{\mathrm{T}}(t)Se(t)\,dt < \gamma^2 \int_0^\infty d^{\mathrm{T}}(t)d(t)\,dt, \tag{12}$$

for a given level $\gamma > 0$ under zero initial condition, where $S$ is a positive symmetric matrix. The parameter $\gamma$ is called the $\mathcal{H}_\infty$ norm bound or the disturbance attenuation level.

*Remark 1* The $\mathcal{H}_\infty$ norm [9] is defined as

$$\|T_{ed}\|_\infty = \frac{\sqrt{\int_0^\infty e^{\mathrm{T}}(t)Se(t)\,dt}}{\sqrt{\int_0^\infty d^{\mathrm{T}}(t)d(t)\,dt}}$$

where $T_{ed}$ is a transfer function matrix from $d(t)$ to $e(t)$. For a given level $\gamma > 0$, $\|T_{ed}\|_\infty < \gamma$ can be restated in the equivalent form (12). If we define

$$H(t) = \frac{\int_0^t e^{\mathrm{T}}(\sigma)Se(\sigma)\,d\sigma}{\int_0^t d^{\mathrm{T}}(\sigma)d(\sigma)\,d\sigma}, \tag{13}$$

the relation (12) can be represented by

$$H(\infty) < \gamma^2. \tag{14}$$

In Sect. 4, through the plot of $H(t)$ versus time, the relation (14) is verified.

The purpose of this paper is to design the FDOFHS controller $u(t)$ guaranteeing the $\mathcal{H}_\infty$ synchronization if there exists the external disturbance $d(t)$. In addition, this controller $u(t)$ will be shown to guarantee the asymptotical synchronization when the external disturbance $d(t)$ disappears.

## 3 Main results

The LMI problem for achieving the FDOFHS is presented in the following theorem.

**Theorem 1** *For given $\gamma > 0$, $1 > \zeta > 0$, and $S = S^{\mathrm{T}} > 0$, if there exist $P = P^{\mathrm{T}} > 0$, $R = R^{\mathrm{T}} > 0$, $Z = Z^{\mathrm{T}} > 0$, and $M_j$ such that*

$$\begin{bmatrix} [1,1] & -\zeta M_j C & 0 & PG_i & I & I \\ -\zeta C^{\mathrm{T}} M_j^{\mathrm{T}} & -R & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{\tau}Z & 0 & 0 & 0 \\ G_i^{\mathrm{T}} P & 0 & 0 & -\gamma^2 I & 0 & 0 \\ I & 0 & 0 & 0 & -\frac{1}{\tau}Z & 0 \\ I & 0 & 0 & 0 & 0 & -S^{-1} \end{bmatrix} < 0, \tag{15}$$

*where*

$$[1,1] = A_i^{\mathrm{T}} P + PA_i + M_j C + C^{\mathrm{T}} M_j^{\mathrm{T}} + R,$$

*for $i, j = 1, 2, \dots, r$, then the FDOFHS is achieved and the controller is given by*

$$u(t) = \sum_{j=1}^{r} h_j(\omega)P^{-1}M_j\big[\big(\hat{y}(t) - y(t)\big) \tag{16}$$

$$- \zeta\big(\hat{y}(t-\tau) - y(t-\tau)\big)\big], \tag{17}$$

*where $\tau > 0$ is the chosen time delay or the propagation as it is in [4, 5].*

*Proof* The FDOFHS controller can be constructed via the parallel distributed compensation. The controller is described by the following rules:

Fuzzy Rule $j$:

IF $\omega_1$ is $\mu_{j1}$ and $\dots$ $\omega_s$ is $\mu_{js}$ THEN

$$u(t) = K_j\big[\big(\hat{y}(t) - y(t)\big) - \zeta\big(\hat{y}(t-\tau) - y(t-\tau)\big)\big], \tag{18}$$

where $K_j \in R^{n\times m}$ is the gain matrix of the controller for the fuzzy rule $j$. The fuzzy controller can be inferred as

$$u(t) = \sum_{j=1}^{r} h_j(\omega)K_j\big[\big(\hat{y}(t) - y(t)\big) - \zeta\big(\hat{y}(t-\tau) - y(t-\tau)\big)\big]$$

$$= \sum_{j=1}^{r} h_j(\omega)K_j C\big[e(t) - \zeta e(t-\tau)\big]. \tag{19}$$

The closed-loop error system with the control input (19) can be written as

$$\dot{e}(t) = \sum_{i=1}^{r}\sum_{j=1}^{r} h_i(\omega)h_j(\omega)\big[(A_i + K_jC)e(t) - \zeta K_jCe(t-\tau) + G_id(t)\big]. \tag{20}$$

Consider the following Lyapunov–Krasovskii functional:

$$V\big(e(t)\big) = V_1\big(e(t)\big) + V_2\big(e(t)\big) + V_3\big(e(t)\big) \tag{21}$$

where

$$V_1\big(e(t)\big) = e^{\mathrm{T}}(t)Pe(t), \tag{22}$$

$$V_2\big(e(t)\big) = \int_{-\tau}^{0}\int_{t+\beta}^{t} e^{\mathrm{T}}(\alpha)Qe(\alpha)\,d\alpha\,d\beta \quad (Q = Q^{\mathrm{T}} > 0), \tag{23}$$

$$V_3\big(e(t)\big) = \int_{t-\tau}^{t} e^{\mathrm{T}}(\sigma)Re(\sigma)\,d\sigma. \tag{24}$$

The time derivative of $V_1(e(t))$ along the trajectory of (20) is

$$\begin{aligned}\dot{V}_1\big(e(t)\big) &= \dot{e}(t)^{\mathrm{T}}Pe(t) + e^{\mathrm{T}}(t)P\dot{e}(t)\\ &= \sum_{i=1}^{r}\sum_{j=1}^{r} h_i(\omega)h_j(\omega)\big\{e^{\mathrm{T}}(t)\big[A_i^{\mathrm{T}}P + PA_i + PK_jC + C^{\mathrm{T}}K_j^{\mathrm{T}}P\big]e(t)\\ &\quad - \zeta e^{\mathrm{T}}(t)PK_jCe(t-\tau) - \zeta e^{\mathrm{T}}(t-\tau)C^{\mathrm{T}}K_j^{\mathrm{T}}Pe(t)\\ &\quad + e^{\mathrm{T}}(t)PG_id(t) + d^{\mathrm{T}}(t)G_i^{\mathrm{T}}Pe(t)\big\}.\end{aligned}$$

If we use the inequality $X^{\mathrm{T}}Y + Y^{\mathrm{T}}X \le X^{\mathrm{T}}\Lambda X + Y^{\mathrm{T}}\Lambda^{-1}Y$, which is valid for any matrices $X \in R^{n\times m}$, $Y \in R^{n\times m}$, $\Lambda = \Lambda^{\mathrm{T}} > 0$, $\Lambda \in R^{n\times n}$, we have

$$e(t)^{\mathrm{T}}PG_id(t) + d^{\mathrm{T}}(t)G_i^{\mathrm{T}}Pe(t) \le \gamma^2 d^{\mathrm{T}}(t)d(t) + \frac{1}{\gamma^2}e(t)^{\mathrm{T}}PG_iG_i^{\mathrm{T}}Pe(t). \tag{25}$$

Using (25), we obtain

$$\begin{aligned}\dot{V}_1\big(e(t)\big) &\le \sum_{i=1}^{r}\sum_{j=1}^{r} h_i(\omega)h_j(\omega)\bigg\{e^{\mathrm{T}}(t)\bigg[A_i^{\mathrm{T}}P + PA_i + PK_jC + C^{\mathrm{T}}K_j^{\mathrm{T}}P + \frac{1}{\gamma^2}PG_iG_i^{\mathrm{T}}P\bigg]e(t)\\ &\quad - \zeta e^{\mathrm{T}}(t)PK_jCe(t-\tau) - \zeta e^{\mathrm{T}}(t-\tau)C^{\mathrm{T}}K_j^{\mathrm{T}} \times Pe(t) + \gamma^2 d^{\mathrm{T}}(t)d(t)\bigg\}.\end{aligned} \tag{26}$$

The time derivative of $V_2(e(t))$ is

$$\dot{V}_2\big(e(t)\big) = \tau e^{\mathrm{T}}(t)Qe(t) - \int_{t-\tau}^{t} e^{\mathrm{T}}(\sigma)Qe(\sigma)\,d\sigma. \tag{27}$$

Using the inequality [13]

$$\bigg[\int_{t-\tau}^{t} e(\sigma)\,d\sigma\bigg]^{\mathrm{T}} Q \bigg[\int_{t-\tau}^{t} e(\sigma)\,d\sigma\bigg] \le \tau\int_{t-\tau}^{t} e(\sigma)^{\mathrm{T}}Qe(\sigma)\,d\sigma, \tag{28}$$

we have

$$\dot{V}_2\big(e(t)\big) \le \tau e^{\mathrm{T}}(t)Qe(t) - \frac{1}{\tau}\bigg[\int_{t-\tau}^{t} e(\sigma)\,d\sigma\bigg]^{\mathrm{T}} Q \times \bigg[\int_{t-\tau}^{t} e(\sigma)\,d\sigma\bigg]. \tag{29}$$

Since $\dot{V}_3(e(t))$ yields the relation

$$\dot{V}_3\big(e(t)\big) = e(t)^{\mathrm{T}}Re(t) - e^{\mathrm{T}}(t-\tau)Re(t-\tau), \tag{30}$$

we have the derivative of $V(e(t))$ as

$$\begin{aligned}\dot{V}\big(e(t)\big) &= \dot{V}_1\big(e(t)\big) + \dot{V}_2\big(e(t)\big) + \dot{V}_3\big(e(t)\big)\\ &\le \sum_{i=1}^{r}\sum_{j=1}^{r} h_i(\omega)h_j(\omega)\left\{\begin{bmatrix} e(t)\\ e(t-\tau)\\ \int_{t-\tau}^{t} e(\sigma)\,d\sigma\end{bmatrix}^{\mathrm{T}} \times \begin{bmatrix} (1,1) & -\zeta PK_jC & 0\\ -\zeta C^{\mathrm{T}}K_j^{\mathrm{T}}P & -R & 0\\ 0 & 0 & -\frac{1}{\tau}Q\end{bmatrix} \times \begin{bmatrix} e(t)\\ e(t-\tau)\\ \int_{t-\tau}^{t} e(\sigma)\,d\sigma\end{bmatrix} - e^{\mathrm{T}}(t)Se(t) + \gamma^2 d^{\mathrm{T}}(t)d(t)\right\},\end{aligned} \tag{31}$$

where

$$(1,1) = A_i^{\mathrm{T}} P + P A_i + P K_j C + C^{\mathrm{T}} K_j^{\mathrm{T}} P + \frac{1}{\gamma^2} P G_i G_i^{\mathrm{T}} P + \tau Q + R + S. \qquad (32)$$

If the following matrix inequality is satisfied:

$$\begin{bmatrix} (1,1) & -\zeta P K_j C & 0 \\ -\zeta C^{\mathrm{T}} K_j^{\mathrm{T}} P & -R & 0 \\ 0 & 0 & -\frac{1}{\tau} Q \end{bmatrix} < 0, \qquad (33)$$

for $i, j = 1, 2, \ldots, r$, we have

$$\dot{V}\bigl(e(t)\bigr) < \sum_{i=1}^{r} \sum_{j=1}^{r} h_i(\omega) h_j(\omega) \bigl\{ -e^{\mathrm{T}}(t) S e(t) + \gamma^2 d^{\mathrm{T}}(t) d(t) \bigr\} = -e^{\mathrm{T}}(t) S e(t) + \gamma^2 d^{\mathrm{T}}(t) d(t). \qquad (34)$$

Integrating both sides of (34) from 0 to $\infty$ gives

$$V\bigl(e(\infty)\bigr) - V\bigl(e(0)\bigr) < -\int_0^\infty e^{\mathrm{T}}(t) S e(t)\, dt + \gamma^2 \int_0^\infty d^{\mathrm{T}}(t) d(t)\, dt.$$

Since $V(e(\infty)) \geq 0$ and $V(e(0)) = 0$, we have the relation (12). From the Schur complement, the matrix inequality (33) is equivalent to

$$\begin{bmatrix} \{1,1\} & -\zeta P K_j C & 0 & P G_i & I & I \\ -\zeta C^{\mathrm{T}} K_j^{\mathrm{T}} P & -R & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{\tau} Q & 0 & 0 & 0 \\ G_i^{\mathrm{T}} P & 0 & 0 & -\gamma^2 I & 0 & 0 \\ I & 0 & 0 & 0 & -\frac{1}{\tau} Q^{-1} & 0 \\ I & 0 & 0 & 0 & 0 & -S^{-1} \end{bmatrix} < 0, \qquad (35)$$

where

$$\{1,1\} = A_i^{\mathrm{T}} P + P A_i + P K_j C + C^{\mathrm{T}} K_j^{\mathrm{T}} P + R.$$

Pre- and post-multiplying (35) by $\mathrm{diag}(I, I, Q^{-1}, I, I, I)$ and introducing change of variables such as $M_j = P K_j$, and $Z = Q^{-1}$, (35) is equivalently changed into the LMI (15). Then the gain matrix of the control input $u(t)$ is given by $K_j = P^{-1} M_j$. This completes the proof. □

**Corollary 1** *Without the external disturbance, if we use the control input $u(t)$ proposed in Theorem 1, the asymptotical synchronization is obtained.*

*Proof* When $d(t) = 0$, we obtain

$$\dot{V}\bigl(e(t)\bigr) < -e^{\mathrm{T}}(t) S e(t) \leq 0 \qquad (36)$$

from (34). This guarantees

$$\lim_{t \to \infty} e(t) = 0 \qquad (37)$$

from Lyapunov–Krasovskii theory. This completes the proof. □

Based on Theorem 1, the optimal $\mathcal{H}_\infty$ norm bound for the $\mathcal{H}_\infty$ synchronization is obtained.

**Corollary 2** *For a given $S > 0$, the optimal $\mathcal{H}_\infty$ norm bound $\gamma$ is obtained by solving the following semi-definite programming problem:*

$$\min_{\gamma > 0} \gamma^2 \qquad (38)$$

*subject to the LMI (15), $P > 0$, $R > 0$, and $Z > 0$.*

*Remark 2* The LMI problem given in Theorem 1 is to determine whether the solution exists or not. It is called the feasibility problem. The LMI problem can be solved efficiently by using recently developed convex optimization algorithms [12]. In this paper, in order to solve the LMI problem, we utilize the MATLAB LMI Control Toolbox [16], which implements state-of-the-art interior-point algorithms.

*Remark 3* Because the LMI problem in Theorem 1 is the feasibility problem, we may find several solutions. However, we can find a unique optimal solution to the LMI problem in Corollary 2 because this problem is the convex optimization problem in terms of the LMI.

## 4 Numerical example

In this section, to verify and demonstrate the effectiveness of the proposed method, we discuss the simulation result for synchronizing Lorenz system under different initial conditions. Consider the following Lorenz system:

$$\dot{x}_1(t) = -10x_1(t) + 10x_2(t),$$
$$\dot{x}_2(t) = 28x_1(t) - x_2(t) - x_1(t)x_3(t), \qquad (39)$$
$$\dot{x}_3(t) = x_1(t)x_2(t) - \frac{8}{3}x_3(t).$$

In order to apply the proposed scheme, we need the T–S fuzzy model representation of the Lorenz system. By defining two fuzzy sets, we can obtain the following fuzzy drive system that exactly represents the nonlinear equation of the Lorenz system:

$$\dot{x}(t) = \sum_{i=1}^{2} h_i(\omega)\big[A_i x(t) + \eta_i\big], \qquad (40)$$

where

$$A_1 = \begin{bmatrix} -10 & 10 & 0 \\ 28 & -1 & -d \\ 0 & d & -\frac{8}{3} \end{bmatrix},$$

$$A_2 = \begin{bmatrix} -10 & 10 & 0 \\ 28 & -1 & d \\ 0 & -d & -\frac{8}{3} \end{bmatrix},$$

$$\eta_1 = \eta_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

The membership functions are

$$h_1(\omega) = \frac{1}{2}\left(1 + \frac{x_1}{d}\right), \qquad h_2(\omega) = \frac{1}{2}\left(1 - \frac{x_1}{d}\right). \qquad (41)$$

For the numerical simulation, we use the following parameters:

$$d = 30, \qquad \tau = 0.2, \qquad \zeta = 0.1,$$

$$C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, \qquad G_1 = G_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},$$

$$S = \begin{bmatrix} 0.1 & 0 & 0 \\ 0 & 0.1 & 0 \\ 0 & 0 & 0.1 \end{bmatrix}.$$

For the design objective (12), let the $\mathcal{H}_\infty$ performance be specified by $\gamma = 0.3$. Applying Theorem 1 to the fuzzy system (40) yields

$$P = \begin{bmatrix} 3.4970 & -0.8802 & -0.8532 \\ -0.8802 & 1.3224 & -0.0100 \\ -0.8532 & -0.0100 & 1.3149 \end{bmatrix},$$

$$M_1 = M_2 = \begin{bmatrix} -759.4963 \\ -98.5322 \\ -32.9033 \end{bmatrix}.$$

Figure 1 shows the plot of $H(t)$ versus time when $d(t) = \sin(10t)$. Figure 1 verifies $H(\infty) < \gamma^2 = 0.09$. This means that the $\mathcal{H}_\infty$ norm from the external disturbance $d(t)$ to the synchronization error $e(t)$ is reduced within the $\mathcal{H}_\infty$ norm bound $\gamma$. Phase-space trajectories for drive and response systems are shown in Figs. 2 and 3, respectively, when the initial conditions are given by

$$\begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{bmatrix} = \begin{bmatrix} 15.8 \\ -17.48 \\ 15.64 \end{bmatrix}, \qquad \begin{bmatrix} \hat{x}_1(0) \\ \hat{x}_2(0) \\ \hat{x}_3(0) \end{bmatrix} = \begin{bmatrix} 13.8 \\ -14 \\ 13 \end{bmatrix}, \qquad (42)$$

and the external disturbance is given by $d(t) = w(t)$, where $w(t)$ means a Gaussian noise with mean 0 and variance 100. Figure 4 shows state trajectories for drive and response systems when the external disturbance $d(t)$ is given by

$$d(t) = \begin{cases} w(t), & 0 \le t \le 10, \\ 0, & \text{otherwise}. \end{cases}$$

From Fig. 4, it can be seen that drive and response systems are indeed achieving chaos synchronization. Figure 5 shows the proposed method reduces the effect of the external disturbance $d(t)$ on the synchronization error $e(t)$. In addition, it is shown that the synchronization error $e(t)$ goes to zero after the external disturbance $d(t)$ disappears.

## 5 Conclusion

In this paper, we have proposed the FDOFHS controller, which is a new $\mathcal{H}_\infty$ synchronization controller,

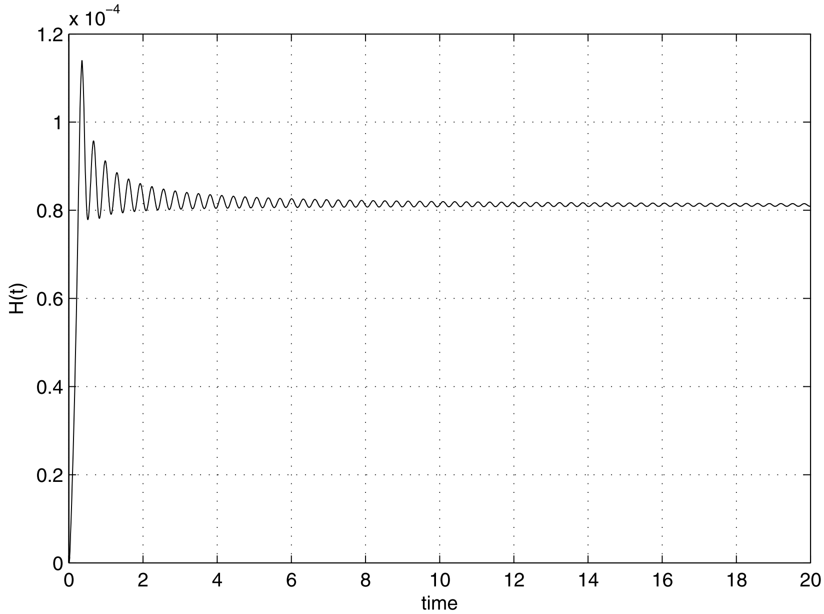



**Fig. 1** The plot of $H(t)$ versus time

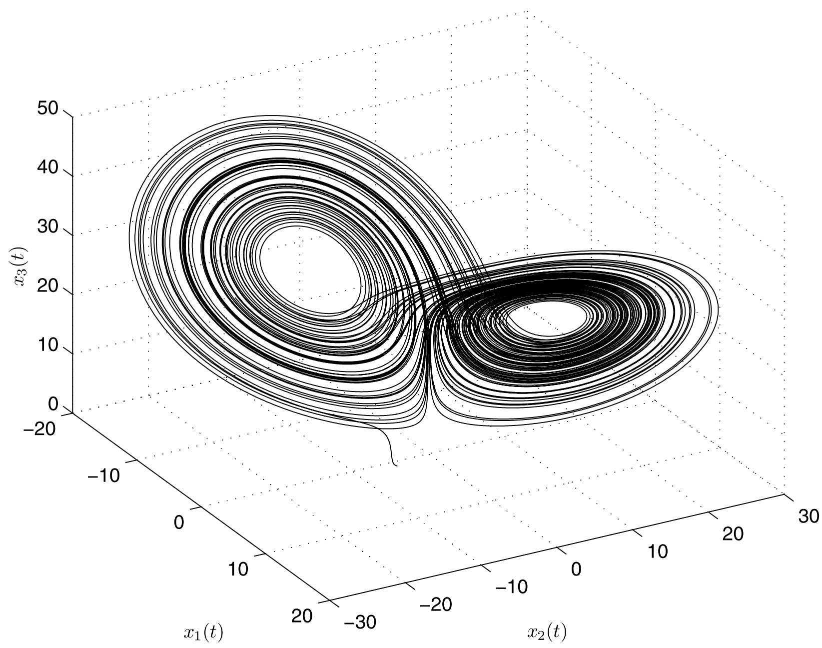



**Fig. 2** The chaotic behavior of the drive system

for chaotic systems with external disturbance. Based on Lyapunov–Krasovskii theory and the LMI approach, the proposed method guarantees the asymptotical synchronization and reduces the $\mathcal{H}_\infty$ norm from the disturbance to the synchronization error within a disturbance attenuation level. The synchronization for the Lorenz system is given to illustrate the effectiveness of the proposed scheme. Finally, the proposed FDOFHS scheme has the advantage that it can be effectively used to $\mathcal{H}_\infty$ control and synchronization of other nonlinear systems described by a T–S fuzzy model.

**Fig. 3** The chaotic behavior of the response system

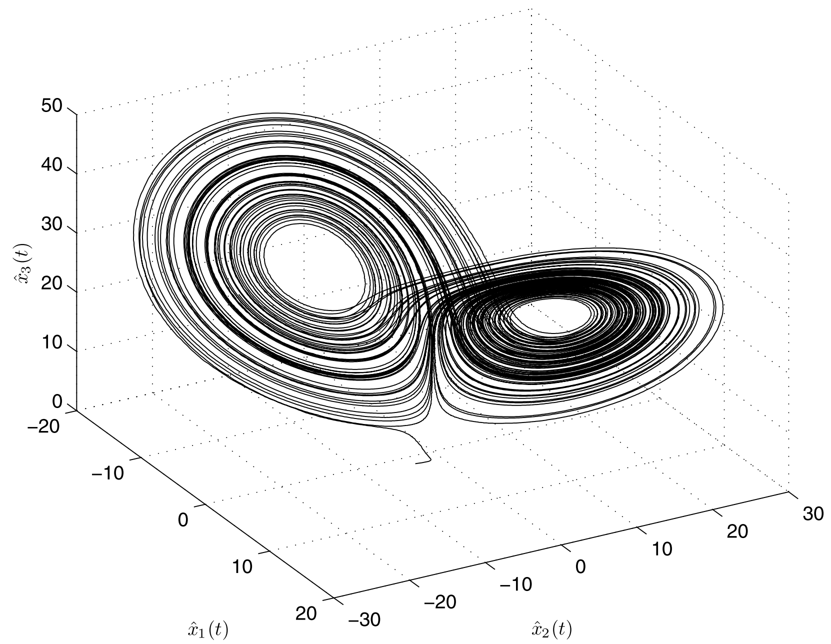



**Fig. 4** State trajectories

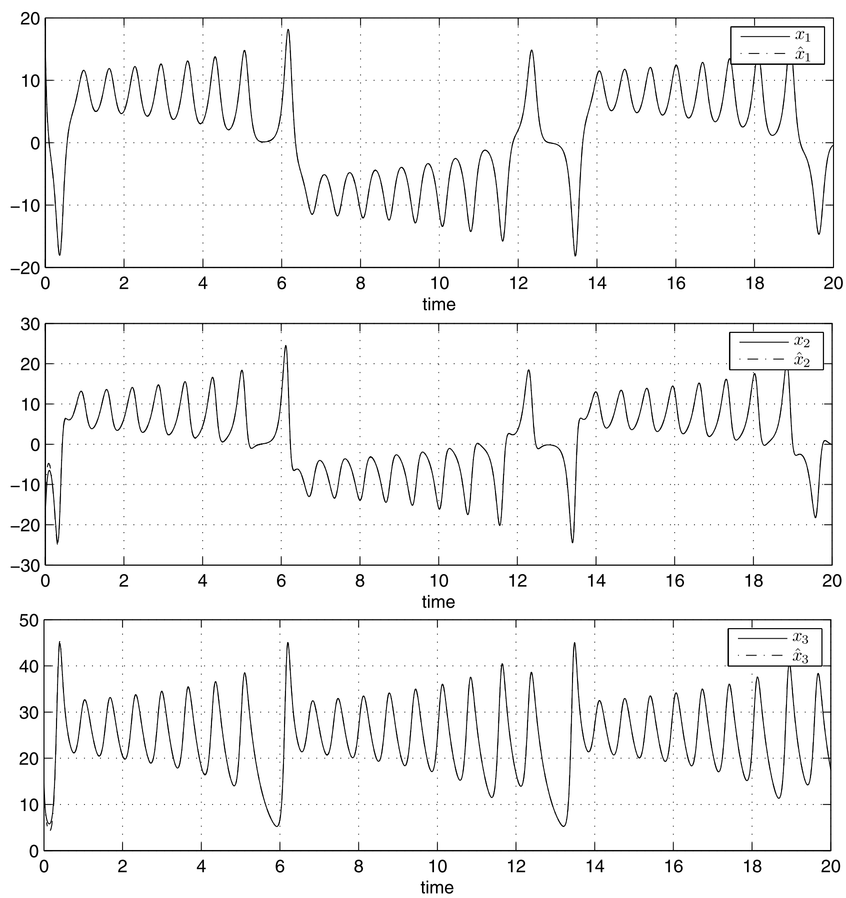
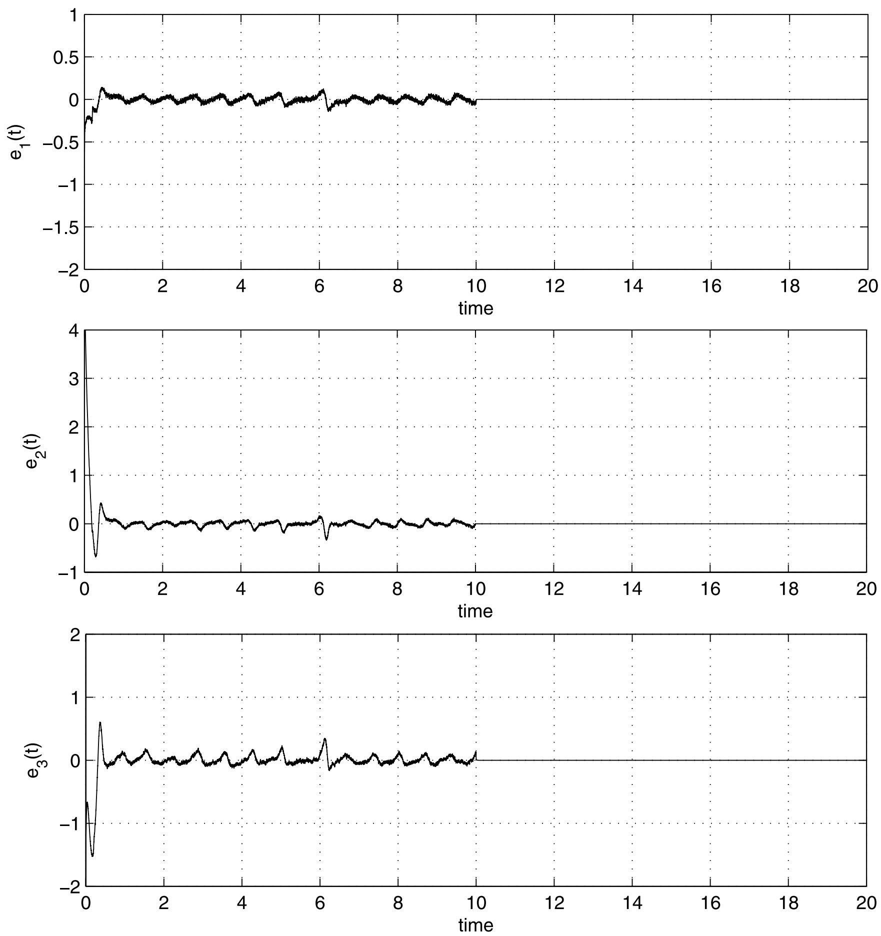


**Fig. 5** Synchronization errors




## References


1. Pecora, L.M., Carroll, T.L.: Synchronization in chaotic systems. Phys. Rev. Lett. **64**, 821–824 (1996)
2. Chen, G., Dong, X.: From Chaos to Order. World Scientific, Singapore (1998)
3. Pyragas, K.: Continuous control of chaos by self-controlling feedback. Phys. Lett. A **170**, 421–428 (1992)
4. Yalcin, M.E., Suykens, J.A.K., Vandewalle, J.: Master-slave synchronization of Lur'e systems with time-delay. Int. J. Bifurc. Chaos **11**, 1707–1722 (2001)
5. Liao, X.X., Chen, G.R.: Chaos synchronization of general Lur'e systems via time-delay feedback control. Int. J. Bifurc. Chaos **13**, 207–213 (2003)
6. Zhu, F.L.: Full-order and reduced-order observer-based synchronization for chaotic systems with unknown disturbances and parameters. Phys. Lett. A **372**, 223–232 (2008)
7. Takagi, T., Sugeno, M.: Fuzzy identification of systems and its applications to modeling and control. IEEE Trans. Syst. Man Cybern. **15**, 116–132 (1985)
8. Tanaka, K., Sugeno, M.: Stability analysis and design of fuzzy control systems. Fuzzy Sets Syst. **45**, 135–156 (1992)
9. Stoorvogel, A.: The $\mathcal{H}_\infty$ Control Problem: A State-space Approach. Prentice Hall International, Englewood Cliffs (1992)
10. Hou, Y.Y., Liao, T.L., Yan, J.J.: $\mathcal{H}_\infty$ synchronization of chaotic systems using output feedback control design. Physica A **379**, 81–89 (2007)
11. Lee, S.M., Ji, D.H., Park, J.H., Won, S.C.: $\mathcal{H}_\infty$ synchronization of chaotic systems via dynamic feedback approach. Phys. Lett. A **372**, 4905–4912 (2008)
12. Boyd, S., Ghaoui, L.E., Feron, E., Balakrishinan, V.: Linear Matrix Inequalities in Systems and Control Theory. SIAM, Philadelphia (1994)
13. Noldus, E.: Stabilization of a class of distributional convolutional equations. Int. J. Control **41**, 947–960 (1985)
14. Krasovskii, N.N.: Application of Lyapunov's second method for equations with time delay. Prikl. Mat. Mek. **20**, 315–327 (1956)
15. Kolmanovskii, V., Nosov, A.: Stability of Functional Differential Equations. Academic Press, San Diego (1986)
16. Gahinet, P., Nemirovski, A., Laub, A.J., Chilali, M.: LMI Control Toolbox. The Mathworks Inc. (1995)