Nonlinear Dyn (2010) 59: 515–527
DOI 10.1007/s11071-009-9558-0

ORIGINAL PAPER

A four-wing hyper-chaotic attractor and transient chaos generated from a new 4-D quadratic autonomous system

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Received: 23 March 2009 / Accepted: 29 June 2009 / Published online: 16 July 2009

Abstract This paper presents a new four-dimensional (4-D) smooth quadratic autonomous chaotic system, which can present periodic orbit, chaos, and hyper-chaos under the conditions on different parameters. Importantly, the system can generate a four-wing hyper-chaotic attractor and a pair of coexistent double-wing hyper-chaotic attractors with two symmetrical initial conditions. Furthermore, a four-wing transient chaos occurs in the system. The dynamic analysis approach- in the paper involves time series, phase portraits, Poincaré maps, bifurcation diagrams, and Lyapunov exponents, to investigate some basic dynamical behaviors of the proposed 4-D system.

Keywords Chaos · Hyper-chaos · Four-wing attractor · Transient chaos

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1 Introduction

Since Lorenz found a simple three-dimensional chaotic system in 1963 [1], chaos theory has greatly developed in the past forty years. Most importantly, it is found that chaos has wide applications in the fields of biological systems [2], electronic engineering [3], computer and information processing [4, 5] on account of its complicated dynamical behavior. Recently, in order to obtain a more important value of chaos in practical applications, much attention has been focused on effectively generating chaotic attractor with complicated topological structures such as n-stroll [6–8] and multi-wing chaotic attractors [9–13]. Summarizing the previous works, it can be seen that these n-stroll chaotic attractors are generated by the Chua's circuit equation [14] and its generalized systems [15–17], whereas those multi-wing attractors mainly come from the Lorenz system [1] or Lorenz-like systems [18–20].

As for these n-stroll chaotic attractors, the common characteristics of dynamical system are the following [9, 10]:

(i) The nonlinear terms of these systems include two types of functions. One type is PWL continuous or discontinuous functions such as the stair function, switching function, and hysteresis-series function [7]; another is smooth functions such as the cubic term [21], sine function [22], and other functions [23] that are anti-symmetrical about the origin by coordinate translation.

(ii) The basic technique is either increasing the number of equilibria by using some PWL functions, or realizing the equilibrium jumping by using, for example, the stair or hysteresis functions.
(iii) The number of scrolls equals the number of equilibria.
(iv) The fundamental shape of the attractors is a spiral, called a scroll.

It is well known that the Lorenz system can generate a double-wing chaotic attractor. Recently, some new chaotic systems based on the Lorenz system were proposed, including the Chen system [18], the generalized Lorenz system family [20], and the hyperbolic-type of generalized Lorenz canonical form [24]. All these systems have butterfly attractors with double wings.

There are some common characteristics for these Lorenz-like systems, which are distinct from the generalized Chua's circuit equations:

(i) The nonlinearities of these systems are continuous, either smooth functions, such as two quadratic terms or three quadratic terms [9–11], or some nonsmooth functions including PWL function [25], absolute function [26], and so on.
(ii) They have three equilibria but produce a double-wing attractor, so the number of wings dose not equal the number of equilibria.
(iii) The basic shape of the attractors is a butterfly, called a wing, where each wing wanders around a corresponding nonzero equilibrium point but does not wander around the origin.

So far, there are a number of methods of generating n-stroll chaotic attractors [6, 7], while it is more difficult to obtain a chaotic attractor with more than double wings from a smooth dynamical system. On account of its complicated topological structure, generating multi-wing chaotic attractors from these smooth systems becomes a very important research topic. Recently, a few dynamic systems with four-wing or three-wing butterfly attractor were reported. By embedding two state-controlled binary switches, a simple three-dimensional system can generate a four-wing or three-wing attractor [27], but the system is nonsmooth.

As for a three-dimensional or a four-dimensional system with smooth quadratic terms, if there are five equilibria, it may produce a four-wing chaotic attractor under the conditions on some parameters. For example, Qi 3-D four-wing chaotic system [10] and those systems proposed in [9, 11] can generate a real and symmetrical four-wing attractor, which is rigorously verified from theoretical analysis and numerical simulation. However, seemingly, a three-dimensional system proposed in [28] can produce a four-wing attractor, but this is only an illusion and not a real four-wing attractor. The error of numerical computation in the computer results in the emergence of the pseudo four-wing attractor, as has been proved by rigorous mathematical analysis in [9].

Furthermore, those smooth dynamic systems can only generate the four-wing chaotic attractors but not produce four-wing hyper-chaotic attractors. In order to obtain hyper-chaos, two important requisites are as follows:

(i) The minimum dimension of the phase space in which a hyper-chaotic attractor is embedded should be at least four, because it requires the minimum number of coupled first-order autonomous ordinary differential equations to be four.
(ii) The number of terms in the coupled equations giving rise to instability should be at least two, of which at least one should have a nonlinear function [29].

Hyper-chaos was firstly reported by Rössler in 1979. Since then, many other hyper-chaotic systems have been found [30], of which Qi hyper-chaotic system with much higher Lyapunov exponents and complicated dynamics was typically established [31]. Some hyper-chaotic systems comparing to these ordinary chaotic systems, when being applied in the practical engineering, are preferable for those applications that require the complexity of dynamics, such as the network security and the data encryption. From both theoretical and practical points of view, it is always desirable to generate an attractor with both multiple wings and hyper-chaos possessing both complicated topological structures and complex dynamics such as rich bifurcations and wider frequency bandwidths. It is no doubt that such a four-wing hyper-chaos system can help us to make chaos more important in both engineering applications and theoretical research.

In this paper, a new four-dimensional dynamical system with three smooth quadratic nonlinear terms and five equilibria is introduced. The new system can generate a four-wing hyper-chaos attractor and a pair of coexistent double-wing attractors determined by two symmetrical initial conditions. Moreover, the phenomenon that a four-wing transient chaos evolves into a periodic orbit is investigated profoundly. The basic properties about equilibria of the new system are given in detail. Bifurcation analysis by Lyapunov exponents and bifurcation diagrams [32–34] illustrates the evolution processes of the system among periodic orbits, a double-wing hyper-chaotic attractor and a four-wing hyper-chaotic attractor, and Poincaré map analysis shows that the system has extremely rich dynamics.

The paper is organized as follows. In Sect. 2, we briefly introduce an autonomous chaotic system with three smooth quadratic terms. In Sect. 3, several simulations are carried out to give a clear observation on the new chaotic attractor. In Sect. 4, some bifurcation analyses about the hyper-chaos are given. Section 5 introduces a four-wing transient chaos. Finally, we draw our conclusions.

2 The proposed 4-D dynamical system

Consider the following four-dimensional (4-D) dynamic system with quadratic coupling terms:

$$\begin{cases} \dot{x} = -ax - e\omega + yz \\ \dot{y} = by + xz \\ \dot{z} = cz + f\omega - xy \\ \dot{\omega} = d\omega - gz \end{cases} \tag{2.1}$$

where x, y, z, ω are the state variables, and a, b, c, d, e, f, g are real constant parameters. This system is found to be chaotic in a wide parameter range and has many interesting complex dynamic behaviors including periodic orbit, chaos, and hyper-chaos. Moreover, transient chaos will occur under certain conditions.

2.1 Symmetry and invariance

Generally, the symmetry property widely exists in dynamic system with an even number of attractors, such as the Lorenz system and the Chua's circuit equation. From system (2.1) it can be seen that system (2.1) has a natural symmetry under the coordinate transform $(x, y, z, \omega) \to (-x, y, -z, -\omega)$, which persists for all values of the system parameters. Also it is clear that the y-axis is a solution orbit. Moreover, if $x = z = \omega = 0$ and $b < 0, a, c, d, e, f, g > 0$, then the orbit on y-axis tends to the origin as $t \to +\infty$.

2.2 Dissipativity and the existence of attractor

Construct the following Lyapunov function:

$$V(x, y, z, \omega) = \frac{1}{2}x^2 + \frac{1}{2}y^2 + z^2 + \frac{f}{g}\omega^2 \tag{2.2}$$

So, we have

$$\begin{aligned} \dot{V}(x, y, z, \omega) &= x\dot{x} + y\dot{y} + 2z\dot{z} + \frac{2f}{g}\omega\dot{\omega} \\ &= -\left(a - \frac{e}{4}\right)x^2 - e\left(\omega + \frac{x}{2}\right)^2 + by^2 \\ &\quad + 2cz^2 + \left(\frac{2f}{g}d + e\right)\omega^2 \end{aligned} \tag{2.3}$$

It is easy to verify that system (2.1) is globally, uniformly, and asymptotically stable about its zero equilibrium if $0 < e < 4a$, $b < 0$, $c < 0$, $2fd < -ge$. At the same time, we note that

$$\nabla V = \frac{\partial \dot{x}}{\partial x} + \frac{\partial \dot{y}}{\partial y} + \frac{\partial \dot{z}}{\partial z} + \frac{\partial \dot{\omega}}{\partial x} = -a + b + c + d \tag{2.4}$$

So, with $a > b+c+d$, system (2.1) is dissipative, with an exponential contraction rate,

$$\frac{dV}{dt} = e^{-(a-b-c-d)} \tag{2.5}$$

That is, a volume element V_0 is contracted by the flow into a volume element $V_0e^{-(a-b-c-d)}$ in time t. This means that each volume containing the system trajectory shrinks to zero as $t \to +\infty$ at the exponential rate $b + c + d - a$, which is independent of state variables x, y, z, and ω. Therefore, all system orbits are ultimately confined to some subset of zero volume, and the asymptotic motion settles onto some attractors.

2.3 Equilibria and stability

It is clear that the origin is a trivial equilibrium of system (2.1). In order to obtain the other nonzero equilibria of system (2.1), let $\dot{x} = \dot{y} = \dot{z} = \dot{\omega} = 0$, and we can obtain the following expressions (2.6) by performing some algebraic operation:

$$\begin{cases} x^2 = -b\dfrac{cd+fg}{d} \\ y^2 - \dfrac{eg}{d}y - a\dfrac{cd+fg}{d} = 0 \\ z = \dfrac{d}{cd+fg}xy \\ \omega = \dfrac{g}{cd+fg}xy \end{cases} \tag{2.6}$$

Next, in order to solve (2.6), define

$$p = \frac{eg}{d}, \qquad q = \frac{cd+fg}{d}, \qquad m = \frac{cd+fg}{g} \tag{2.7}$$

We obtain

$$\begin{aligned} x_1 &= \sqrt{-bq}, \qquad x_2 = -\sqrt{-bq} \\ y_1 &= \frac{p+\sqrt{p^2+4aq}}{2} \\ y_2 &= \frac{p-\sqrt{p^2+4aq}}{2} \\ z_1 &= \frac{\sqrt{-bq}(p+\sqrt{p^2+4aq})}{2q} \\ z_2 &= \frac{-\sqrt{-bq}(p+\sqrt{p^2+4aq})}{2q} \\ z_3 &= \frac{\sqrt{-bq}(p-\sqrt{p^2+4aq})}{2q} \\ z_4 &= \frac{-\sqrt{-bq}(p-\sqrt{p^2+4aq})}{2q} \\ \omega_1 &= \frac{\sqrt{-bq}(p+\sqrt{p^2+4aq})}{2m} \\ \omega_2 &= \frac{-\sqrt{-bq}(p+\sqrt{p^2+4aq})}{2m} \\ \omega_3 &= \frac{\sqrt{-bq}(p-\sqrt{p^2+4aq})}{2m} \\ \omega_4 &= \frac{-\sqrt{-bq}(p-\sqrt{p^2+4aq})}{2m} \end{aligned} \tag{2.8}$$

Table 1 Eigenvalues of the Jacobian matrix for all the equilibria

λ	S_0	S_1 S_2	S_3 S_4
λ_1	1.0988	0.1741	0.2322
λ_2	9.1012	$2.274+63.2874i$	$-0.0845+44.4608i$
λ_3	−16	$2.274-63.2874i$	$-0.0845-44.4608i$
λ_4	−50	−60.5221	−55.8642

Therefore, there are four nonzero equilibria except for $S_0 = (0,0,0,0)$:

$$\begin{aligned} S_1 &= (x_1, y_1, z_1, \omega_1) = \left(x_1, y_1, \frac{x_1y_1}{q}, \frac{x_1y_1}{m}\right) \\ S_2 &= (x_2, y_1, z_2, \omega_2) = \left(-x_1, y_1, -\frac{x_1y_1}{q}, -\frac{x_1y_1}{m}\right) \\ S_3 &= (x_1, y_2, z_3, \omega_3) = \left(x_1, y_2, \frac{x_1y_2}{q}, \frac{x_1y_2}{m}\right) \\ S_4 &= (x_2, y_2, z_4, \omega_4) = \left(-x_1, y_2, -\frac{x_1y_2}{q}, -\frac{x_1y_2}{m}\right) \end{aligned} \tag{2.9}$$

In these equilibria, S_1 and S_2, S_3, and S_4 are symmetrically placed with respect to the y-axis. When $a = 50$, $b = -16$, $c = 10$, $d = 0.2$, $e = 10$, $f = 16$, $g = 0.5$, the five equilibria are

$S_0 = (0, 0, 0, 0)$

$S_1 = (28.2842, 64, 0389, 36.2285, 90.5644)$

$S_2 = (-28.2842, 64, 0389, -36.2285, -90.5644)$

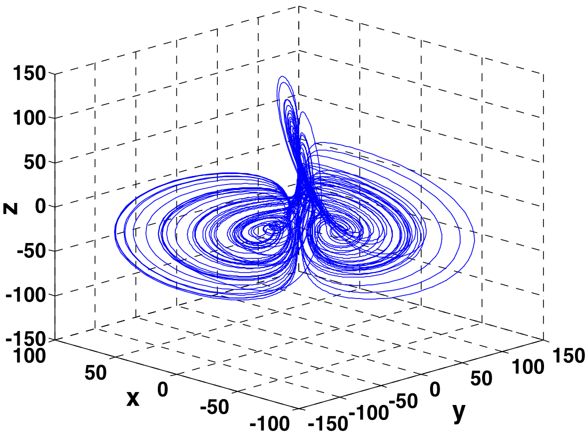
$S_3 = (28.2842, -39.0389, -22.0836, -55.2091)$

$S_4 = (-28.2842, -39.0389, 22.0836, 55.2091)$

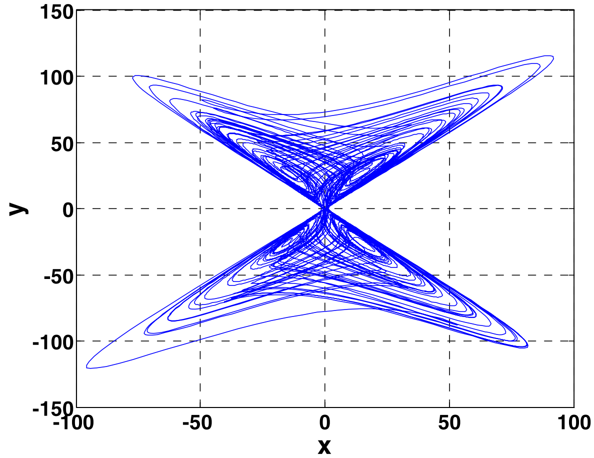
In order to investigate the stability of all the equilibria, we consider the Jacobian matrices with different equilibria and calculate their eigenvalues. The results are shown in Table 1. Based on the results, we can see that S_0 is a saddle-point and S_1, S_2, S_3, S_4 are all saddle-foci nodes.

3 Observation of new chaotic attractors

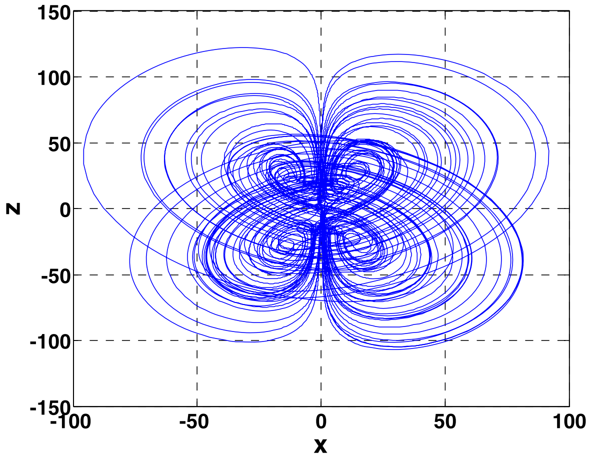
In this and the following sections, we set time step size 0.001, keep the absolute and relative error 0.00001,



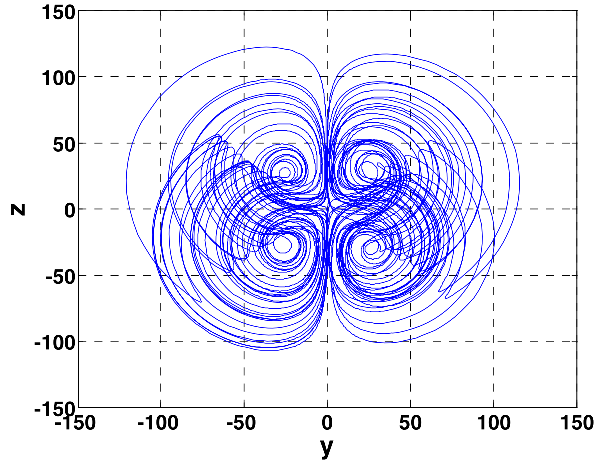
(a) 3D view in the *x*-*y*-*z* space



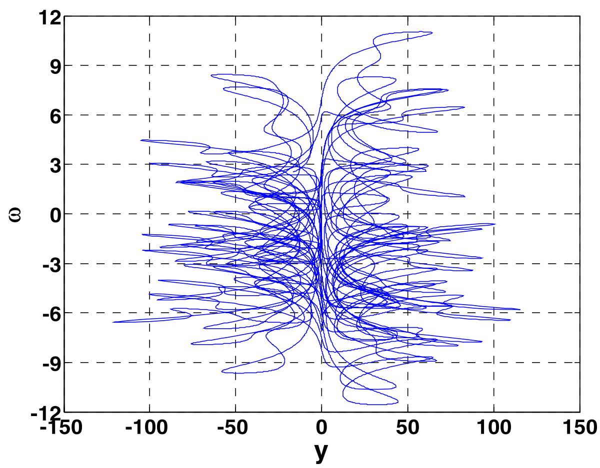
(b) Projection on the *x*-*y* plane



(c) Projection on the *x*-*z* plane

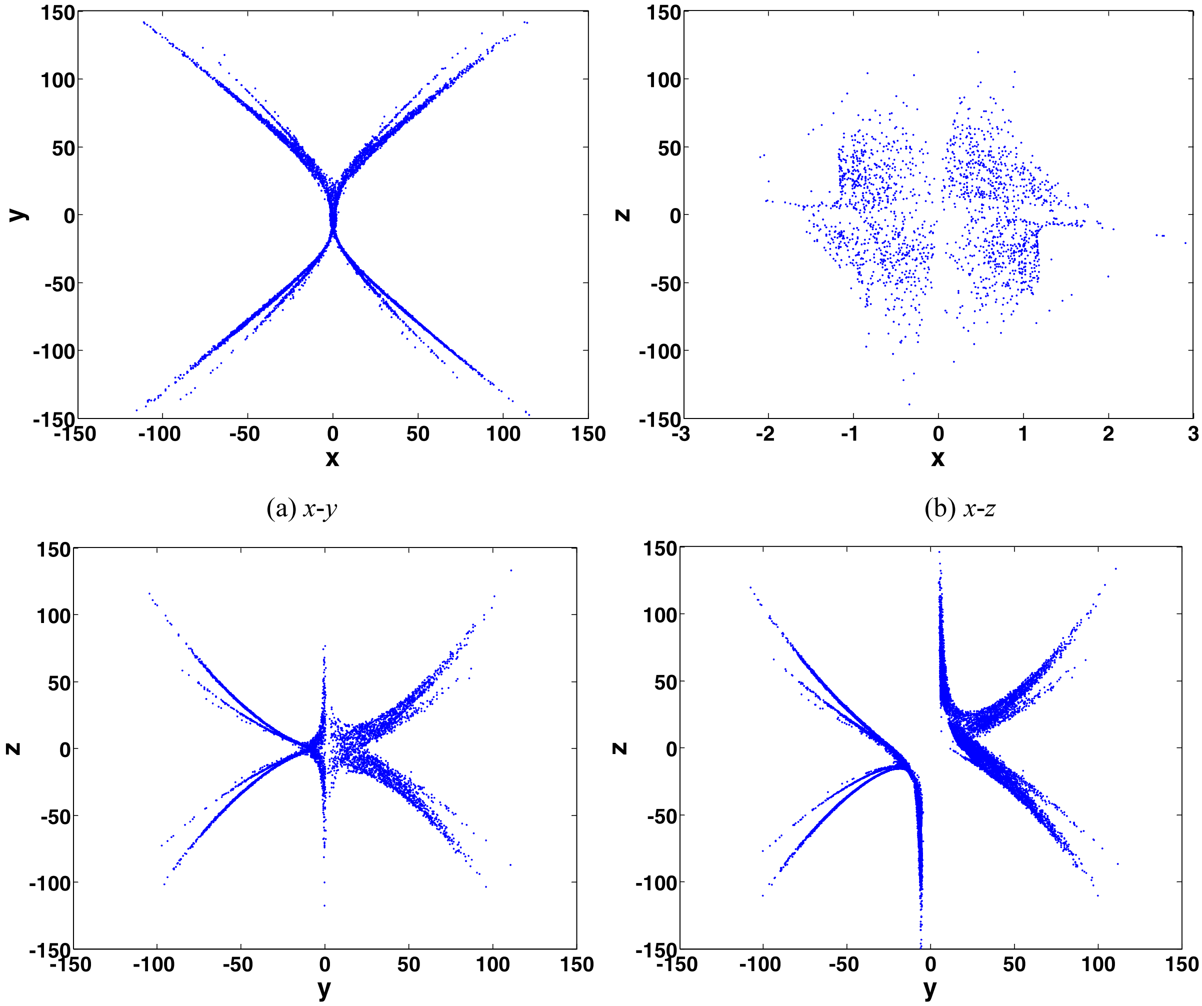


(d) Projection on the *y*-*z* plane



(e) Projection on the *y*- ω plane

Fig. 1 Phase portraits of system (2.1) with parameters $a = 50$, $b = -16$, $c = 10$, $d = 0.2$, $e = 10$, $f = 16$, $g = 0.5$



(a) x-y (b) x-z (c) y-z (d) y-z

Fig. 2 Poincaré section: (**a**) $z = 0$; (**b**) $y = 0$; (**c**) $x = 0$; (**d**) $x = 5$

choose the initial conditions $(1, 1, 1, 1)$, and adopt the fourth-order Runge–Kutta method to solve system (2.1) in MATLAB.

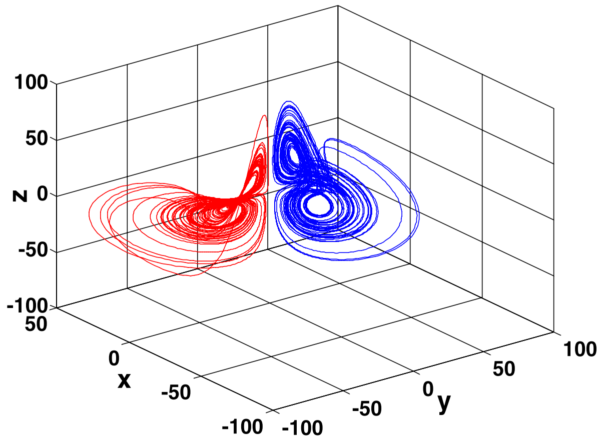
As for system (2.1), we keep parameter b negative and variable, and fix the values of other parameter as above. Via varying parameter b, a series of bifurcation phenomena will be obtained, such as periodic orbit, chaotic attractor, and hyper-chaotic attractor.

Case 1: $b = -16$

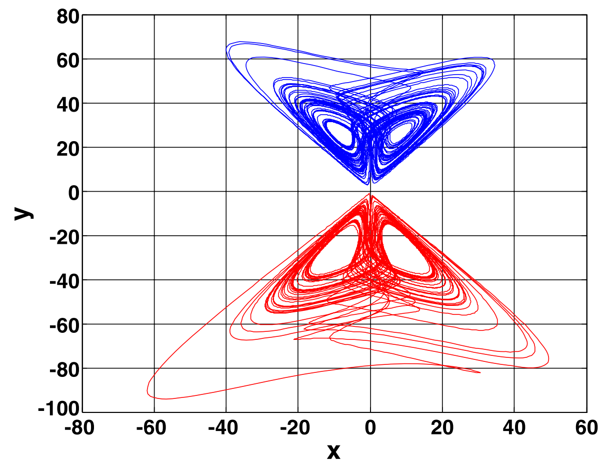
When $b = -16$, there exists a simple four-wing chaotic attractor shown in Fig. 1. The corresponding Lyapunov exponents are $l_1 = 2.4736$, $l_2 = 0.5175$, $l_3 = 0$, $l_4 = -58.7248$.

It is obvious that the new chaotic attractor is a rare four-wing hyper-chaotic attractor, which has not appeared in references. The 3-D four-wing hyper-chaotic attractor viewed in the x–y–z space is shown in Fig.1(a), and the projections of phase portrait on x–y, x–z, y–z, and y–ω planes are shown in Figs. 1(b), 1(c), 1(d), and 1(e), respectively. It is can be seen from Fig. 1 that the chaotic attractor is symmetrical with respect to y-axis, as is just used to verify the basic property of system (2.1) mentioned in Sect. 2.1.

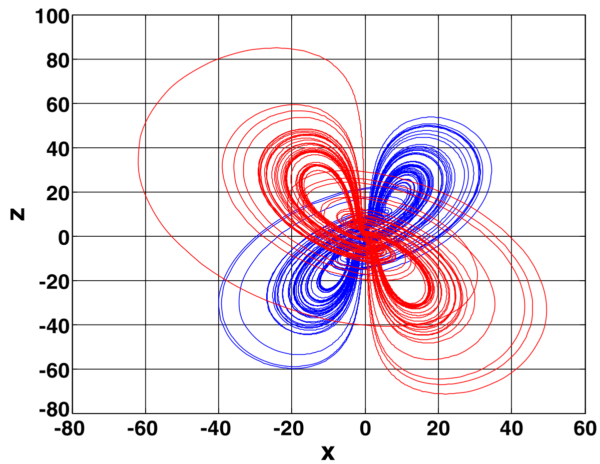
The Poincaré maps of system (2.1) projected on different planes are shown in Figs. 2(a), 2(b), 2(c),



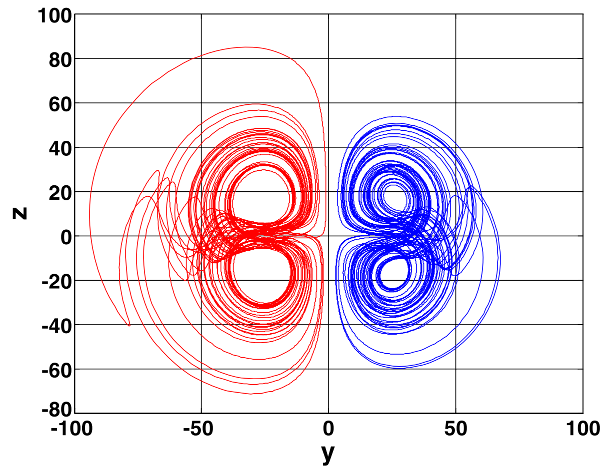
(a) 3D view in the *x*-*y*-*z* space



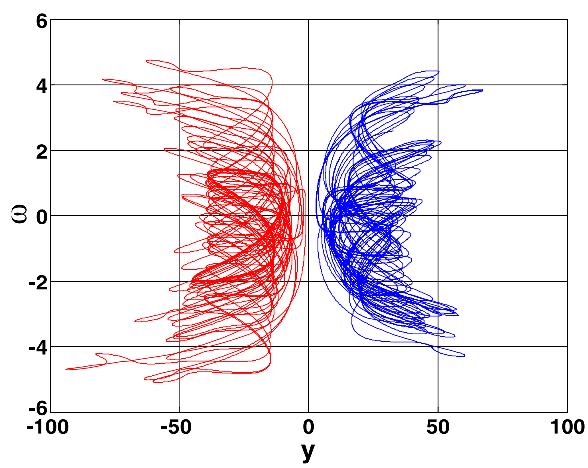
(b) Projection on the *x*-*y* plane



(c) Projection on the *x*-*z* plane

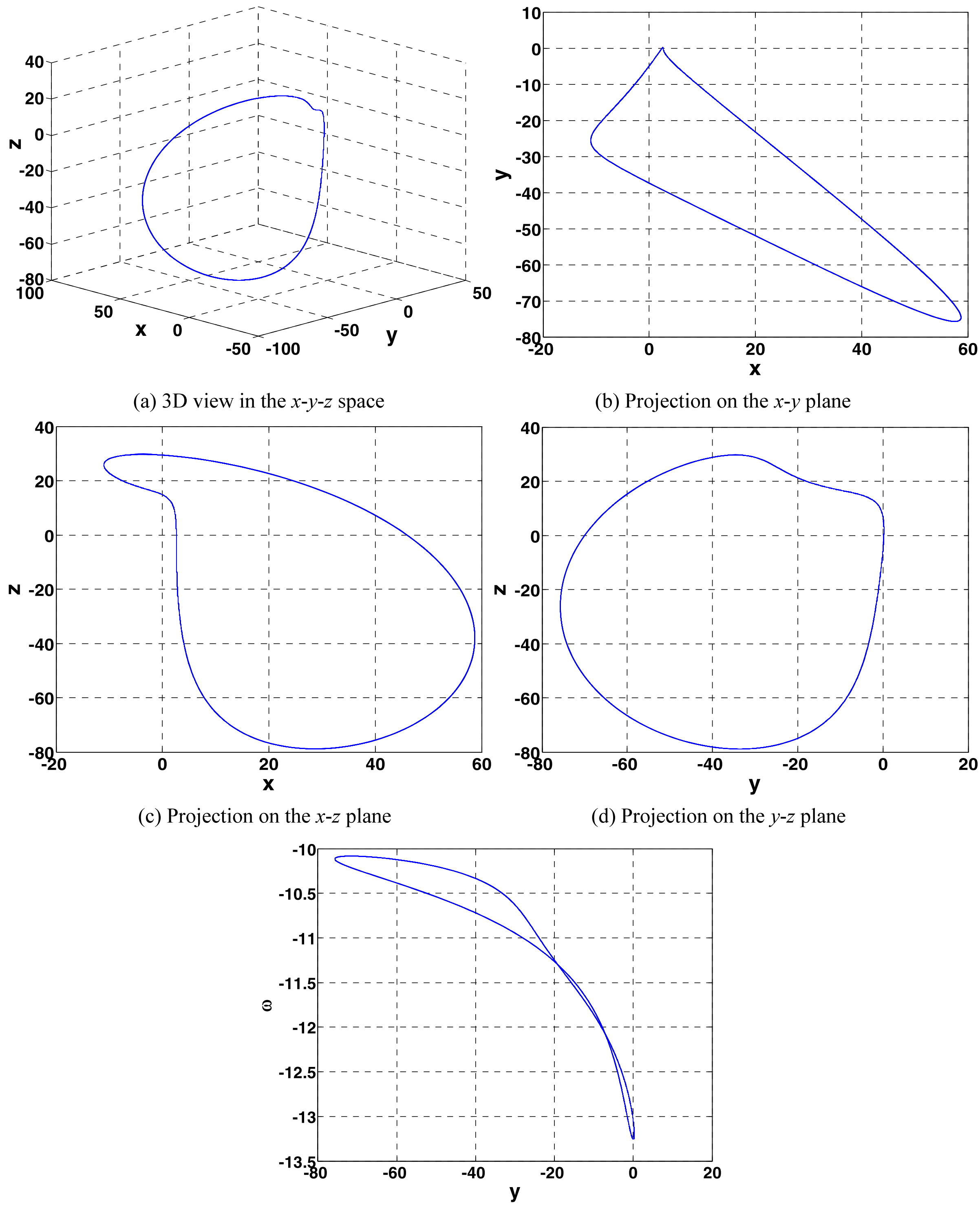


(d) Projection on the *y*-*z* plane



(e) Projection on the y-ω plane

Fig. 3 Phase portraits of system (2.1) with coexistence of two chaotic attractors under different initial conditions



(a) 3D view in the x-y-z space

(b) Projection on the x-y plane

(c) Projection on the x-z plane

(d) Projection on the y-z plane

(e) Projection on the y-ω plane

Fig. 4 Phase portraits of system (2.1) with parameters $a = 50$, $b = -20$, $c = 10$, $d = 0.2$, $e = 10$, $f = 16$, $g = 0.5$

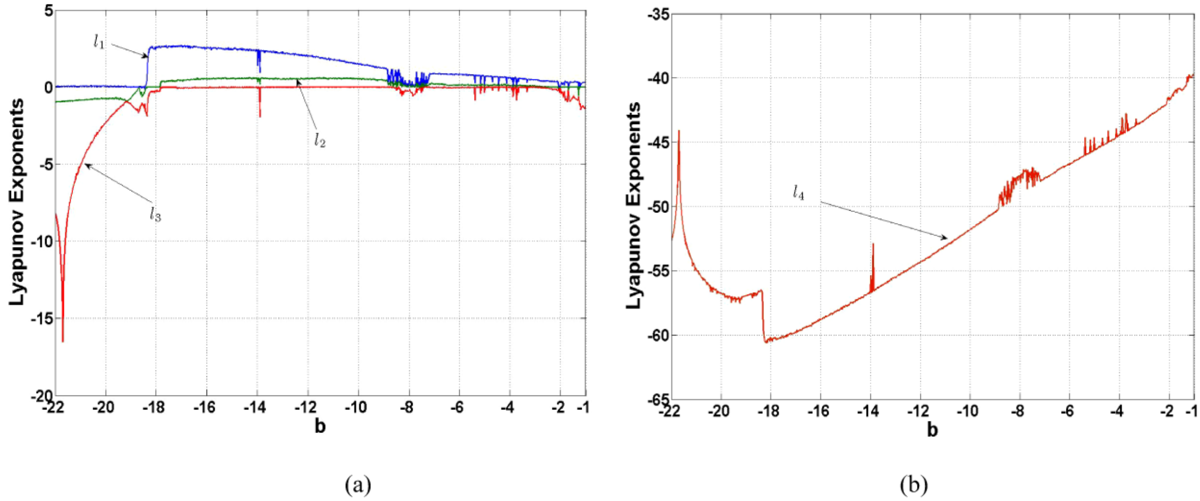


Fig. 5 The Lyapunov exponents versus parameter b of system (2.1)

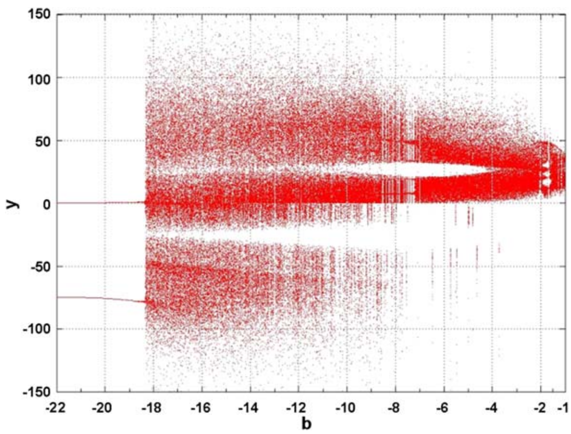


Fig. 6 Bifurcation diagram for increasing parameter b

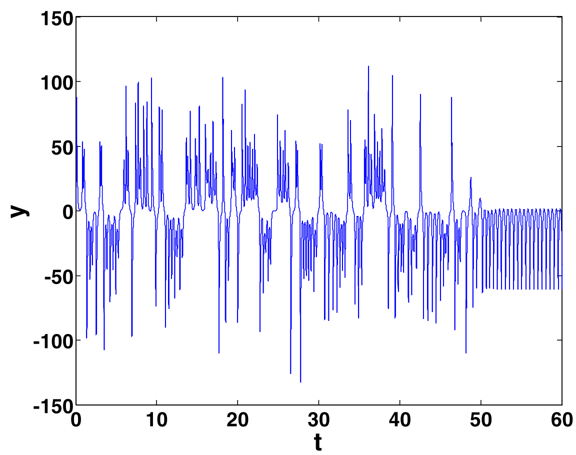


Fig. 7 Time series of (2.1) with $a = 50$, $b = -16$, $c = 10$, $d = 0.2$, $e = 20$, $f = 16$, $g = 0.5$

and 2(d). Since system (2.1) is in chaos, the return map is disordered, as can further explain the motion state.

Case 2: $b = -5.5$

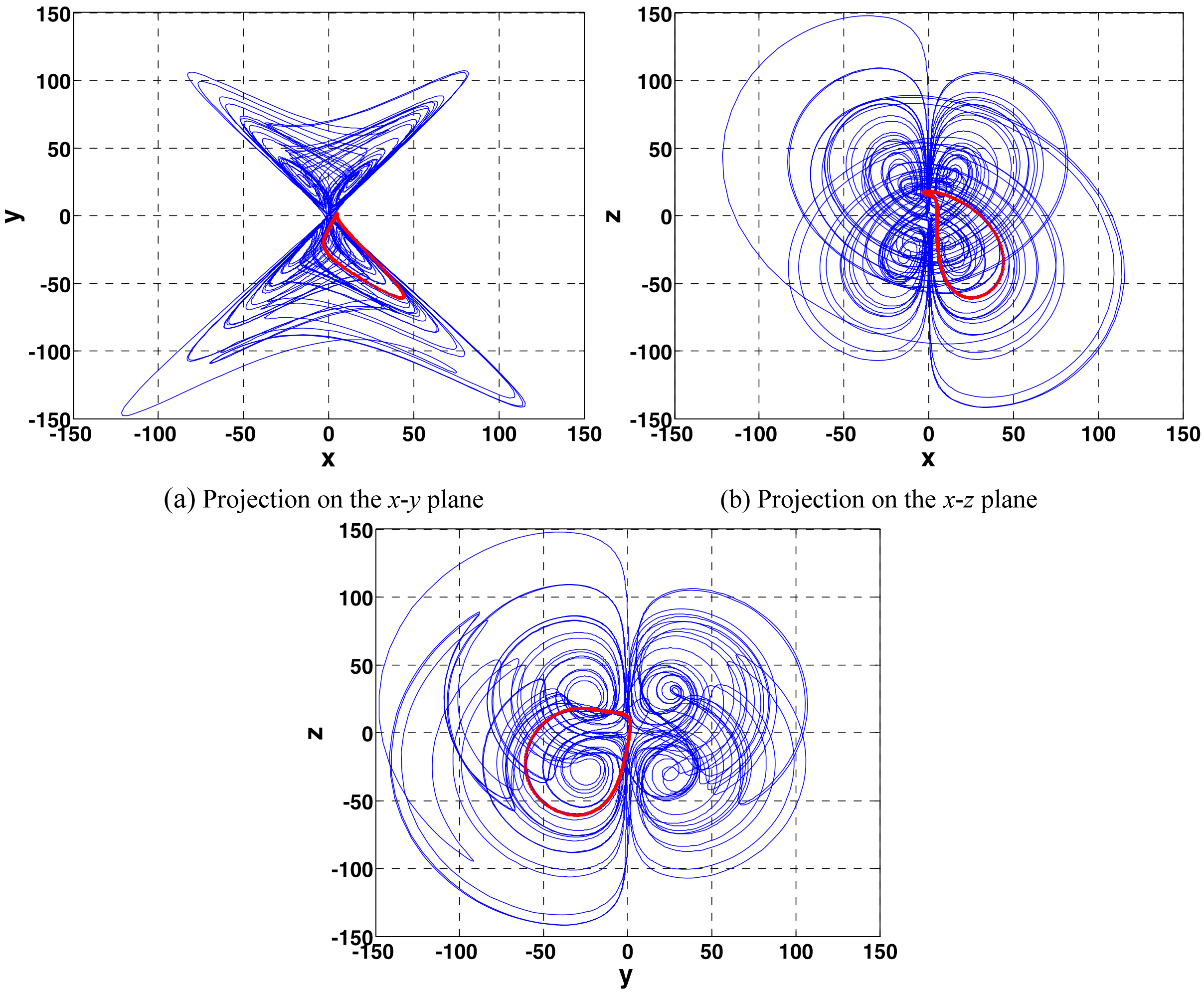
When $b = -5.5$, system (2.1) goes into another hyper-chaotic motion with Lyapunov exponents $l_1 = 0.7418$, $l_2 = 0.0961$, $l_3 = 0$, and $l_4 = -46.1368$, as shown in Fig. 3. Obviously, the hyper-chaotic attractor is different from the one shown in Fig. 1. As can be seen, Fig. 3 shows two coexisting double-wing attractors that are symmetrical with respect to y-axis. One is denoted by the red line, and another is denoted by the blue line. The red orbit and blue orbit shown in Fig. 3 originate from the initial conditions (1, 1, 1, 1) and (1, −1, 1, 1), respectively.

Case 3: $b = -20$

When $b = -20$, a periodic attractor occurs, as shown in Fig. 4. Obviously, the system motion is period-1. There is a great deal of difference in the shape between the periodic attractor and hyper-chaotic attractor and chaotic attractor shown in Figs. 1 and 3, respectively.

Table 2 Some typical parameter values of b that lead to different system portraits

b	l_1	l_2	l_3	l_4	Dynamics
−1.647	0	−0.0925	−0.4532	−40.9027	Periodic orbit (period-3)
−5.5	0.7418	0.0961	0	−46.1386	Hyper-chaos (2-wing)
−8.2	0.2281	0	−0.2523	−47.9758	Chaos (2-wing)
−12	1.9703	0.5421	0	−54.0216	Hyper-chaos (4-wing)
−16	2.4736	0.5175	0	−58.7248	Hyper-chaos (4-wing)
−20	0	−0.7408	−2.3169	−56.3892	Periodic orbit (period-1)



(a) Projection on the x-y plane

(b) Projection on the x-z plane

(c) Projection on the y-z plane

Fig. 8 Phase portraits of system (2.1) with parameters $a = 50$, $b = -16$, $c = 10$, $d = 0.2$, $e = 20$, $f = 16$, $g = 0.5$

According to the simulation results of Cases 1, 2, and 3, the system undergoes hyper-chaos (4-wing), hyper-chaos (2-wing), and periodic orbits when the parameter b varies.

4 Bifurcation analysis

It does not actually have a systematic methodology for purposefully designing a hyper-chaotic system, but a

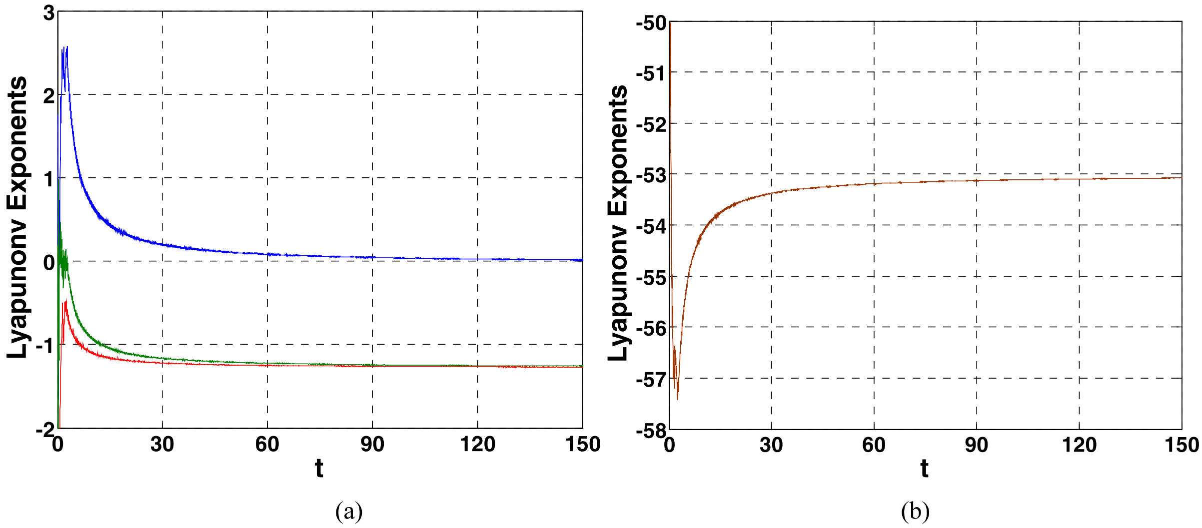


Fig. 9 The Lyapunov exponents versus time t of system (2.1) with $a = 50$, $b = -16$, $c = 10$, $d = 0.2$, $e = 20$, $f = 16$, $g = 0.5$

dynamic system that can generate hyper-chaos must satisfy the two basic rules proposed by Rössler [29]. Therefore, the investigation relies on a combination of mathematical analysis and numerical simulations.

Denoting by l_1, l_2, l_3, and l_4 the Lyapunov exponents of system (2.1), system (2.1) evolves into different modes in some typical different intervals of parameter b. The Lyapunov exponent spectrum of system (2.1) with respect to parameter b in the interval $[-22, -1]$ illustrates the dynamics of state variables, as shown in Fig. 5. The corresponding bifurcation diagram of state variable y is shown in Fig. 6. A detailed evolution among periodic orbits, hyper-chaos (2-wing), chaos (2-wing), hyper-chaos (4-wing), and periodic orbits are shown in Table 2.

5 Four-wing transient chaos

The appearance of chaos on finite time scales is known as transient chaos. The phenomenon of transient chaos accompanying boundary crisis is rather frequently encountered in dynamical systems. In particular, such phenomenon can be observed in many dynamical systems such as the logistic maps [35], the Rössler system [36], and the Lorenz system [37].

In a boundary crisis, as a control parameter p is raised, the distance between a strange attractor and the boundary of its basin of attraction in the phase space decreases until they touch each other at a critical value ($p = p_c$). At this point the attractor also touches an unstable periodic orbit, and the chaotic attractor exhibits a crisis. For $p > p_c$, the chaotic attractor no longer exists and will be replaced by a chaotic transient. In the initial stage of this regime, the system behavior is virtually undistinguishable from chaotic, but then the system rapidly passes to another stable state (attractor) that can be stationary, periodic, or chaotic as well [38].

As for system (2.1), let $a = 50$, $b = -16$, $c = 10$, $d = 0.2$, $f = 16$, $g = 0.5$ and keep parameter e variable. Via adjusting the size of e, the phenomenon of transient chaos will occur. Figure 7 shows the time series of state variable y in system (2.1) with parameter $e = 20$. Obviously, the time series is different from time series generated by a chaotic system. The first part in the interval $t \in [0, 48]$ is so disordered and chaotic, and the second part after the time $t = 48$ is in order and periodic.

In order to further analyze the dynamical property of system (2.1) under the conditions, the phase portrait can be used to recognize the dynamical behavior. The projections of phase portrait on x–y, x–z, y–z, and y–ω planes are shown in Figs. 8(a), 8(b), and 8(c), respectively.

The red lines in Fig. 8 represent the formed periodic orbits ultimately after a chaotic transient, while the blue lines illustrating a 4-wing chaotic attractor reflect the basic dynamic behavior on a finite time scale.

According to Fig. 8, it is not hard to see that the orbit originated from initial values $(1, 1, 1, 1)$ will return to a periodic attractor with period-1 eventually through the evolution in 50 seconds.

The phenomenon also can be illustrated in the Lyapunov exponent spectrum versus time to show the changes of dynamic behavior of system (2.1) under the conditions of fixed parameters, as shown in Fig. 9.

6 Conclusion

This paper presents a new four-dimensional (4-D) smooth quadratic autonomous chaotic system with four-wing hyper-chaotic attractor and coexistence of two double-wing hyper-chaotic attractors under different initial conditions. The four-wing hyper-chaotic system rarely appears in the existing chaotic systems. The proposed system is investigated via both numerical simulations and bifurcation analysis. Furthermore, the transient chaos phenomenon of variable y is illustrated. On account of the complicated topological structures, the proposed four-wing hyper-chaotic system may be further study theoretically and has a promising application in the field of information technology such as secure communication and encryption.

Acknowledgements This work was supported by the Natural Science Foundation of China (Grant No. 60774088) and the Foundation of the Application Base and Frontier Technology Research Project of Tianjin (Grant No. 08JCZDJC21900).

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