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ORIGINAL PAPER

# Solitary waves for power-law regularized long-wave equation and $R(m,n)$ equation

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**Abstract** This paper integrates the regularized long-wave equation with power-law nonlinearity using the solitary-wave ansatz. A few of the conserved quantities are calculated by using the 1-soliton solution. This technique is then extended to obtain the solitary-wave solution of the $R(m,n)$ equation and a conserved quantity is also calculated for this generalized equation.

**Keywords** Solitons · Integrability · Integrals of motion

## 1 Introduction

The nonlinear evolution equations play a vital role in the advancement of Physics and Engineering problems. They appear in various areas in these two broad areas. These include Fluid Dynamics, Quantum Mechanics, Nonlinear Optics, Plasma Physics, and many more [1–10]. The question of integrability of these equations is another factor. Although numerically many of these equations can be solved to "visualize" the solution [2, 6], it is still important to obtain a closed analytical solution to these equations. These closed-form solutions are necessary to carry out further investigation into studying the properties of the solution and so forth.

In this paper, one such nonlinear evolution equation will be studied along with its generalized form. This is the regularized long-wave equation (RLW) with power-law nonlinearity. This equation arises in the study of shallow-water waves. The generalized version of the RLW equation is known as the $R(m,n)$ equation that will be studied and analyzed in the subsequent section. In particular, the focus on this paper is going to be on obtaining the solitary-wave solution of this equation. This solitary wave will be used to compute a couple of conserved quantities of these equations.

There are various techniques that are used to carry out the integration of these nonlinear evolution equations. The classical method of Inverse Scattering Transform (IST), that is the nonlinear analogue of Fourier Transform, is used to carry out the integration of equations of this kind. But, nowadays, there are various modern methods of integrability that are used to carry out the integration of these various nonlinear evolution equations which otherwise cannot be integrated by the IST. These techniques are the Adomian decomposition method, $G'/G$ method, tanh–coth method, sine–cosine method, exponential function method, and many more. However, one needs to be very careful in applying these various tech-

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niques of integration, especially the exponential function method. These could lead to multiple solutions of such nonlinear evolution equations, and many of such solutions are several forms of a particular solution. This fact was pointed out in 2009 [6]. In this paper, the solitary-wave ansatz [2, 3] is going to be used to carry out the integration of RLW and the $R(m,n)$ equation.

## 2 RLW equation

The dimensionless form of the RLW equation with power-law nonlinearity that is going to be studied in this paper is given by [1, 8]

$$q_t + aq_x + bq^m q_x + cq_{xxt} = 0. \tag{1}$$

Here $a$, $b$, and $c$ are constants, and the parameter $m$ dictates the power-law nonlinearity. The first term is the evolution term, and the third term is the nonlinear term, while the second and fourth terms are the dispersion terms. The solitons are the result of a delicate balance between dispersion and nonlinearity. In order to obtain a 1-soliton solution to (1), the solitary-wave ansatz is assumed as

$$q(x,t) = \frac{A}{\cosh^p \tau}, \tag{2}$$

where

$$\tau = B(x - vt). \tag{3}$$

Here $A$ is the soliton amplitude, $B$ is the inverse width of the soliton, and $v$ is the soliton velocity. The unknown index $p$ will be determined in terms of $m$ during the course of derivation of the solution of (1). From (2) it is possible to obtain

$$q_t = pvAB\frac{\tanh\tau}{\cosh^p \tau}, \tag{4}$$

$$q_x = -pAB\frac{\tanh\tau}{\cosh^p \tau}, \tag{5}$$

$$q^m = \frac{A^m}{\cosh^{mp} \tau}, \tag{6}$$

$$q_{xxt} = p^3vAB^3\frac{\tanh\tau}{\cosh^p \tau} - p(p+1)(p+2)vAB^3\frac{\tanh\tau}{\cosh^{p+2} \tau}. \tag{7}$$

Substituting (4)–(7) into (1) gives

$$\frac{pvAB}{\cosh^p \tau} - \frac{apAB}{\cosh^p \tau} - \frac{bpA^{m+1}B}{\cosh^{(m+1)p} \tau} + \frac{cp^3vAB^3}{\cosh^p \tau} - \frac{cp(p+1)(p+2)vAB^3}{\cosh^{p+2} \tau} = 0. \tag{8}$$

Now, from (8) equating the exponents $(m+1)p$ and $p+2$ leads to

$$(m+1)p = p+2 \tag{9}$$

which gives

$$p = \frac{2}{m}. \tag{10}$$

From (8) setting the coefficients of $1/\cosh^{p+j}\tau$, $j = 0, 2$, to zero, since these are linearly independent functions, gives

$$v = \frac{am^2}{m^2+4cB^2} \tag{11}$$

and

$$A = \left[-\frac{2ac(m+1)(m+2)B^2}{b(m^2+4cB^2)}\right]^{\frac{1}{m}}. \tag{12}$$

It should be noted that in (12) the quantity in the square brackets should be negative if $m$ is even and could be positive if $m$ is odd. Thus, finally, the 1-soliton solution to (1) is given by

$$q(x,t) = \frac{A}{\cosh^{\frac{2}{m}} \tau}, \tag{13}$$

where the amplitude $A$ is related to the width $B$ of the soliton as given in (12), and the velocity of the soliton is given by (11).

### 2.1 Integrals of motion

The power-law RLW equation supports at least three integrals of motion. They are the mass ($M$), momentum ($P$) and the energy ($E$) that are respectively given by

$$M = \int_{-\infty}^{\infty} q\,dx = \frac{A}{B}\frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{m})}{\Gamma(\frac{1}{2}+\frac{1}{m})}, \tag{14}$$

$$P = \int_{-\infty}^{\infty} \left(q^2 - cq_x^2\right) dx = \frac{A^2(m^2+4m-4cB^2)}{m(m+4)B^2} \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{2}{m}\right)}{\Gamma\left(\frac{1}{2}+\frac{2}{m}\right)} \qquad (15)$$

and

$$E = \int_{-\infty}^{\infty} \left(q^4 + cq_x^2\right) dx = \frac{A^4}{B} \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{4}{m}\right)}{\Gamma\left(\frac{1}{2}+\frac{4}{m}\right)} + \frac{4cA^2B}{m(m+4)} \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{4}{m}\right)}{\Gamma\left(\frac{1}{2}+\frac{4}{m}\right)}, \qquad (16)$$

where in (14)–(16), $\Gamma(x)$ represents the Euler's gamma function. In order to compute these conserved quantities, the 1-soliton solution given by (13) is utilized.

## 3 $R(m,n)$ equation

The dimensionless form of the generalized RLW equation, known as the $R(m,n)$ equation, is given by [7]

$$q_t + aq_x + b\left(q^m\right)_x + c\left(q^n\right)_{xxt} = 0. \qquad (17)$$

The special case $n = 1$, $m = 3$ reduces (17) to the modified RLW equation that is already studied [8]. In this section, (17) will be studied for $m = n$. It needs to be noted that this generalization is along the same lines as $K(m,n)$ equation and $B(m,n)$ equation that are generalizations of the Korteweg-de Vries equation and Boussinesq equations, respectively [2, 3]. Now, in order to solve (17) for $m = n$ the same solitary-wave ansatz as in (2) will be used to start off. Thus,

$$\left(q^m\right)_x = mpA^mB \frac{\tanh\tau}{\cosh^{mp}\tau} \qquad (18)$$

and

$$\left(q^m\right)_{xxt} = m^3p^3vA^mB^3 \frac{\tanh\tau}{\cosh^{mp}\tau} - mp(mp+1) \times (mp+2)vA^mB^3 \frac{\tanh\tau}{\cosh^{mp+2}\tau}. \qquad (19)$$

Substituting (4)–(5) and (18)–(19) into (17) gives

$$\frac{pvAB}{\cosh^p\tau} - \frac{apAB}{\cosh^p\tau} - \frac{bmpA^mB}{\cosh^{mp}\tau} + \frac{cm^3p^3vA^mB^3}{\cosh^{mp}\tau} - \frac{cmp(mp+1)(mp+2)vA^mB^3}{\cosh^{mp+2}\tau} = 0. \qquad (20)$$

Now, from (20) equating the exponents $mp+2$ and $p$ leads to

$$mp + 2 = p \qquad (21)$$

which gives

$$p = \frac{2}{1-m}. \qquad (22)$$

From (20) setting the coefficients of $1/\cosh^{mp+j}\tau$, $j = 0, 2$, to zero, since these are linearly independent functions, gives

$$v = \frac{a(1-m)^2}{(1-m)^2 - 2cm(m+1)A^{m-1}B^2} \qquad (23)$$

and

$$v = \frac{b}{cm^2p^2B^2}. \qquad (24)$$

Equating the two values of the velocity $v$ from (23) and (24) gives

$$B = \sqrt{\frac{b(1-m)^2}{4acm^2 + 2bcm(m+1)A^{m-1}}}. \qquad (25)$$

Thus, finally, the 1-soliton solution to (17) is given by

$$q(x,t) = \frac{A}{\cosh^{\frac{2}{1-m}}\tau}, \qquad (26)$$

where the amplitude $A$ is related to the width $B$ of the soliton as given in (25), and the velocity of the soliton is given by (23) or (24).

### 3.1 Integrals of motion

The $R(m,n)$ equation for $m = n$ supports at least one conserved quantity, that is, the mass ($M$) which is given by

$$M = \int_{-\infty}^{\infty} q\,dx = \frac{A}{B} \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{1-m}\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{1-m}\right)}. \qquad (27)$$

Here, the conserved quantity is calculated by using the 1-soliton solution that is given by (26).

## 4 Conclusions

This paper obtains the 1-soliton solution of the RLW equation with power-law nonlinearity. The conserved quantities are calculated using this solution. Also, this RLW is generalized to $R(m,n)$ equation that is solved for $m = n$, and a conserved quantity is calculated for this equation too. In both cases, the solitary-wave ansatz is used to carry out the integration.

In future, this soliton solution will be used to study the perturbed RLW and $R(m,n)$ equation, and the soliton perturbation theory will be developed for these respective equations. In addition to the deterministic perturbation terms, the stochastic perturbation terms will be studied, and the corresponding Langevin equations will be developed. Furthermore, these equations with time-dependent coefficients will be developed and solved.

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## References

1. Ali, A.H.A.: Spectral methods for solving the equal width equation based on Chebyshev polynomials. Nonlinear Dyn. **51**(1–2), 59–70 (2008)
2. Biswas, A.: 1-soliton solution of the $K(m,n)$ equation with generalized evolution. Phys. Lett. A **372**(25), 4601–4602 (2008)
3. Biswas, A.: 1-soliton solution of the $B(m,n)$ equation with generalized evolution. Commun. Nonlinear Sci. Numer. Simul. **14**(8), 3226–3229 (2009)
4. Dag, I., Ozer, M.N.: Approximation of the RLW equation by the least square cubic B-spline finite element method. Appl. Math. Model. **25**(3), 221–231 (2001)
5. Feng, D., Li, J., Lü, J., He, T.: New explicit and exact solutions for a system of variant RLW equations. Appl. Math. Comput. **198**, 715–720 (2008)
6. Kudryashov, N.A., Loguinova, N.B.: Be careful with Exp-function method. Commun. Nonlinear Sci. Numer. Simul. **14**(5), 1881–1890 (2009)
7. Mustafa, Inc.: New exact solitary pattern solutions of the nonlinearly dispersive $R(m,n)$ equations. Chaos Solitons Fractals **29**(2), 499–505 (2006)
8. Raslan, K.R., Hassan, S.M.: Solitary waves for the MRLW equation. Appl. Math. Lett. **22**(7), 984–989 (2009)
9. Soliman, A.A., Hussien, M.H.: Collocation solution for RLW equation with septic spline. Appl. Math. Comput. **161**(2), 623–636 (2005)
10. Wazwaz, A.-M.: Analytic study on nonlinear variants of the RLW and the PHI-four equations. Commun. Nonlinear Sci. Numer. Simul. **12**(3), 314–327 (2007)