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ORIGINAL PAPER

Nonlinear active observer-based generalized synchronization in time-delayed systems

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Abstract When two different chaotic oscillators are coupled, generalized synchronization can occur. It may imply a very complicated relation between the states of drive and response systems. We propose a method that can be used to detect and characterize the generalized synchronization in modulated time-delayed systems. Using Krasovskii–Lyapunov theory, sufficient condition for generalized synchronization is derived. The proposed technique has been applied to synchronize prototype and Ikeda models by numerical simulation.

Keywords Generalized synchronization · Nonlinear observer · Active control · Modulated delay time · Krasovskii–Lyapunov theory

1 Introduction

Chaos synchronization has been a subject of intense study and is considered to be a fundamental mechanism behind a variety of behaviors in nature during the last decade [1]. Possible application areas of chaos synchronization are in secure communication, modeling brain activity and optimization of nonlinear system performance [2–5]. Different kinds of synchronization have been found: complete synchronization (CS) [6], phase synchronization (PS) [7–9], lag synchronization (LS) [10], anticipatory synchronization (AS) [11], generalized synchronization (GS) [12, 13], multiplexing synchronization (MS) [14], etc. CS means that the coupled systems remain in step with each other in the course of time. CS occurs only in coupled systems with identical elements. When the parameters of the coupled systems are mismatched from the point where oscillations are identical, the coupled systems can still remain synchronized in a generalized sense, i.e. the response system is a function of the driving system, which is called a generalized synchronization. There are several methods to detect the presence of GS between chaotic systems, such as the auxiliary systems approach [15], the method of nearest neighbors [16], nonlinear control [17], modified system approach [18], etc. Since the transformation between drive and response systems that embodies the GS can be very complicated, one needs special methods to detect the existence of the transformation and to study this kind of synchronization. In [12, 13, 16], some numerical methods are used for the detection of GS. GS is an extension of CS and GS has more applications than CS. Recently, Yang and Chua [19] realized generalized synchronization based on the linear transformed method. Juan and Xingyuan [17] investigated GS in autonomous chaotic systems using nonlinear control.

Due to finite signal transmission and memory effects, time-delayed systems are ubiquitous in nature,

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technology and society [20]. Recently, chaotic time-delayed system has been suggested as a good candidate for secure communication. In recent study it was discussed that time-delayed system is still vulnerable for communication because time delay τ can be exposed by several measures [21–25]. If the delay time τ is known, the time-delayed system becomes quite simple and the message encoded by the chaotic signal can be extracted by the common attack methods [26]. From the above point of view, we can see that the study of generalized synchronization in variable time-delayed system is of high practical importance. In recent study [27, 28], a scheme based on chaos synchronization is used for the identification of system's unknown parameters. Grassi and Mascolo [29] proposed a nonlinear observer design to synchronize hyperchaotic systems. An observer is a dynamic system designed to be driven by the output of another dynamic system (plant) and having the property that the state of the observer converges to the state of the plant.

While the concept of GS has been well established in low-dimensional systems, it has not yet studied in detail the coupled time-delayed systems and only a very few studies have been done in time-delayed systems [30, 31]. In particular, the mechanism of onset of generalized synchronization in modulated time-delayed system has not been yet clearly understood and requires attention.

In this paper, we present a new method for detection and characterization of generalized synchronization in modulated time-delayed systems. We call this technique the nonlinear active observer design. This method is a combination of nonlinear observer and active control. This method is different from the tools described in [15–18, 30, 31]. In addition, as we shall show below, the nonlinear active observer design can give analytical treatment for generalized synchronization between two different time-delayed systems with constant and modulated delay time.

The organization of the remaining part is as follows: In Sect. 2, the definition of nonlinear active observer design is presented. Some generalized synchronization condition using Krasovskii–Lyapunov theory is obtained. Results of simulation on prototype and Ikeda systems are given in Sect. 3. Finally, conclusions are drawn in Sect. 4.

2 Nonlinear active observer and condition of generalized synchronization

Let the general coupled time-delayed systems be in the form of

$$\dot{x} = F(x, x_{\tau_1}), \tag{1}$$

$$\dot{y} = G(y, y_{\tau_2}) + v(x, y), \tag{2}$$

where $x, y \in R^n$, $F, G : R^n \to R^n$ are the nonlinear vector fields, $v(x, y)$ is the control term and $x_{\tau_1} = x(t - \tau_1)$ and $\dot{x} = dx/dt$.

Definition Let the output of system (1) be $z = s(\phi(x), \phi(x_{\tau_1}))$. Then the dynamic system

$$\dot{y} = G(y, y_{\tau_2}) + h\big(z - t(y, y_{\tau_2})\big) \tag{3}$$

is said to be nonlinear generalized observer of system (1) if its state $y \to \phi(x)$ as $t \to \infty$ where $\phi : R^n \to R^n$ is the generalized synchronization function and $h : R^n \to R^n$ is a suitable chosen nonlinear function [32]. Moreover, system (3) is said to be global generalized observer of (1) if $y \to \phi(x)$ as $t \to \infty$ for any initial condition $x(0)$ and $y(0)$.

The generalized synchronization manifold of systems (1) and (2) is

$$y = \phi(x). \tag{4}$$

We consider the dynamics (1) in the form

$$\dot{u} = -Au + Bf(u_\tau), \tag{5}$$

where $f(u)$ is a nonlinear function of u, characterizing the systems, e.g. $f(u) = \sin u$ for Ikeda model [33, 34], $f(u) = \sin^2(u - u_0)$ for sine-square model [35], $f(u) = \frac{au}{1+u^c}$ for the Mackey–Glass model [36], etc.

Consider the two time delays τ_1 and τ_2 as a function of time [37–39], instead of constant delay, as

$$\begin{aligned} \tau_1(t) &= \tau_{10} + a_1 \sin(\omega_1 t), \\ \tau_2(t) &= \tau_{20} + a_2 e^{|\sin(\omega_2 t)|}, \end{aligned} \tag{6}$$

where τ_{10}, τ_{20} , $a_{1,2}$ and $\omega_{1,2}$ are non-zero constants.

Theorem *Let $s(x, x_{\tau_1}) = f(x_{\tau_1}) + kx$ be the synchronizing signal (k is the coupling strength) and*

$t(y, y_{\tau_2}) = ky + g(y_{\tau_2})$. *Also let*

$$h\big(z - t(y, y_{\tau_2})\big) = s\big(\phi(x), \phi(x_{\tau_1})\big) - t(y, y_{\tau_2}) + u \tag{7}$$

be a function in (3), *where* u *is the active control term. Then generalized synchronization between* (1) *and* (3) *occurs if*

$$k + b \geq \frac{|f'(\phi(x_{\tau_1}))|}{\sqrt{1-\tau_1'}} + \frac{|(\mu_2 - 1)g'(\phi(x_{\tau_2}))|}{\sqrt{1-\tau_2'}}, \tag{8}$$

where $\tau_1' = d\tau_1(t)/dt$.

Proof We consider the coupled systems as

$$\dot{x} = F(x, x_{\tau_1}) = -ax + \mu_1 f(x_{\tau_1}), \tag{9}$$

$$\begin{aligned}\dot{y} &= G(y, y_{\tau_2}) + v(x, y)\\ &= -by + \mu_2 g(y_{\tau_2}) + s\big(\phi(x), \phi(x_{\tau_1})\big)\\ &\quad - t(y, y_{\tau_2}) + u.\end{aligned} \tag{10}$$

Let $e = y - \phi(x)$ be the generalized synchronization error. Then the error dynamics is

$$\begin{aligned}\dot{e} &= -ke - by + (\mu_2 - 1)g(y_{\tau_2}) + f\big(\phi(x_{\tau_1})\big)\\ &\quad - u + ax\phi'(x) - \mu_1\phi'(x)f(x_{\tau_1}).\end{aligned}$$

If we choose the active control function u as

$$\begin{aligned}u &= -\big[ax\phi'(x) - \mu_1\phi'(x)f(x_{\tau_1}) - b\phi(x)\\ &\quad + (\mu_2 - 1)g\big(\phi(x_{\tau_2})\big) + f(y_{\tau_1})\big],\end{aligned} \tag{11}$$

the error dynamic becomes

$$\dot{e} = -r(t)e + s_1(t)e(t-\tau_1) + s_2(t)e(t-\tau_2), \tag{12}$$

where

$$r(t) = k + b, \qquad s_1(t) = -f'\big(\phi(x_{\tau_1})\big),$$
$$s_2(t) = (\mu_2 - 1)g'\big(\phi(x_{\tau_2})\big).$$

Consider a positive definite Krasovskii–Lyapunov functional [40, 41] of the form

$$V(t) = \frac{1}{2}e^2(t) + h_1(t)\int_{-\tau_1(t)}^{0} e^2(t+\theta_1)\,d\theta_1 + h_2(t)\int_{-\tau_2(t)}^{0} e^2(t+\theta_2)\,d\theta_2$$

where $h_1(t) > 0$, $h_2(t) > 0$ for any time.

We take derivative of the functional $V(t)$ along the trajectory of (12),

$$\begin{aligned}\frac{dV}{dt} &= e\dot{e} + \dot{h}_1(t)\int_{-\tau_1(t)}^{0} e^2(t+\theta_1)\,d\theta_1\\ &\quad + h_1(t)\big[e^2 - e^2(t-\tau_1) + \tau_1' e^2(t-\tau_1)\big]\\ &\quad + \dot{h}_2(t)\int_{-\tau_2(t)}^{0} e^2(t+\theta_2)\,d\theta_2\\ &\quad + h_2(t)\big[e^2 - e^2(t-\tau_2) + \tau_2' e^2(t-\tau_2)\big];\end{aligned}$$

if $\dot{h}_1(t) \leq 0$, $\dot{h}_2(t) \leq 0$ for arbitrary t, then

$$\begin{aligned}\dot{V}(t) &\leq -(r - h_1 - h_2)e^2 + s_1 e e_{\tau_1} + s_2 e e_{\tau_2}\\ &\quad - h_1(1-\tau_1')e_{\tau_1}^2 - h_2(1-\tau_2')e_{\tau_2}^2\\ &= -\left[r - h_1 - h_2 - \frac{s_1^2}{4h_1(1-\tau_1')} - \frac{s_2^2}{4h_2(1-\tau_2')}\right]e^2\\ &\quad - h_1(1-\tau_1')\left\{e_{\tau_1} - \frac{s_1 e}{2h_1(1-\tau_1')}\right\}^2\\ &\quad - h_2(1-\tau_2')\left\{e_{\tau_2} - \frac{s_2 e}{2h_2(1-\tau_2')}\right\}^2\\ &\leq -\Psi(h_1, h_2)e^2,\end{aligned}$$

where

$$\Psi(h_1, h_2) = r - h_1 - h_2 - \frac{s_1^2}{4h_1(1-\tau_1')} - \frac{s_2^2}{4h_2(1-\tau_2')}.$$

In order to show that $\frac{dV}{dt} < 0$ for all e, it is sufficient to show that $\Psi_{\min} > 0$. One can easily check that the absolute minimum of Ψ occurs at $h_1 = \frac{s_1}{2\sqrt{1-\tau_1'}}$, $h_2 = \frac{s_2}{2\sqrt{1-\tau_2'}}$ with $\Psi_{\min} = r - \frac{s_1}{\sqrt{1-\tau_1'}} - \frac{s_2}{\sqrt{1-\tau_2'}}$. Consequently, we have the condition for generalized synchronization as

$$r(t) \geq \frac{|s_1(t)|}{\sqrt{1-\tau_1'(t)}} + \frac{|s_2(t)|}{\sqrt{1-\tau_2'(t)}},$$

which gives the condition (8).

This completes the proof of the theorem. □

Remarks (1) In case of a constant time delay $\tau_1' = \frac{d\tau_1}{dt} = 0 = \tau_2'$, the above condition (8) is satisfied for a constant time delay.

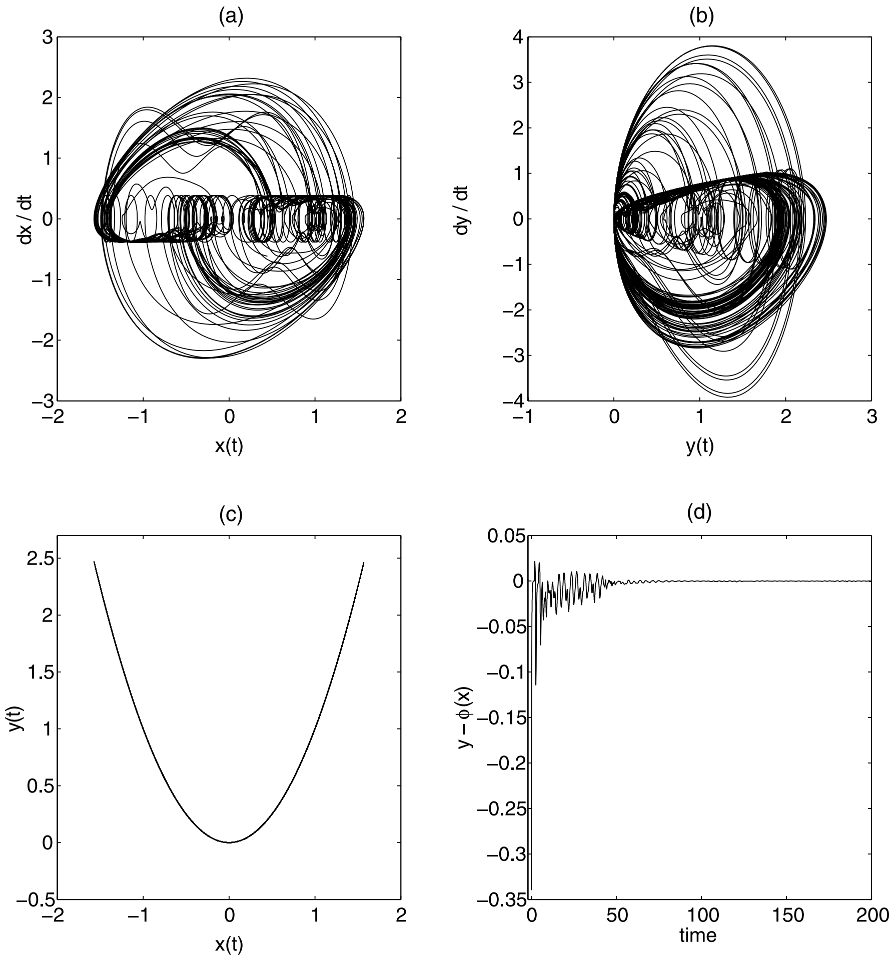

Fig. 1 (**a**) Chaotic attractor for the drive system (13), (**b**) response system's attractor, (**c**) generalized synchronization manifold for the coupling strength $k = 3.5$, (**d**) corresponding synchronization error

(2) The condition in the above theorem is successfully applied to a wide class of time-delayed systems with constant and variable time delay.

(3) One cannot obtain the generalized synchronization without the condition (8).

3 Numerical results

An example will be used to illustrate the effectiveness of the obtained results. We consider the two well-known chaotic time-delayed systems: prototype and Ikeda time-delayed systems, and their numerical simulations are performed. We consider prototype system as the drive and Ikeda system as the response system. Consider unidirectional coupled system as

$$\dot{x} = \delta x_{\tau_1} - \epsilon x_{\tau_1}^3, \tag{13}$$

$$\dot{y} = -by + \mu_2 \sin(y_{\tau_2}) + v, \tag{14}$$

where $v = s(\phi(x), \phi(x_{\tau_1})) - t(y, y_{\tau_2}) + u$ and u is given by (11). In prototype model, δ and ϵ are positive parameters and this system is proposed as a chaos generator and is studied in [42]. Physically y is the phase lag of the electric field across the resonator, b is the relaxation coefficient for the dynamical variable, and μ_2 is the laser intensity injected into the system. τ_2 is the round-trip time of the light in the resonator or feedback delay time in the coupled systems [33, 34]. The Ikeda model was introduced to describe the dynamics of an optical bistable resonator and is well known for delay-induced chaotic behavior [33, 34]. We take time delays τ_1 and τ_2 as a modulated time delay in the form of (6).

The systems (13) and (14) are chaotic for the set of parameter values $\delta = 1.0$, $\epsilon = 1.0$, $\tau_{10} = 1.6$, $a_1 = 0.26$, $\omega_1 = 0.8$ [37–39] and $b = 1.8$, $\mu_2 = 6.0$, $\tau_{20} = 2.0$, $a_2 = 0.5$, $\omega_2 = 0.02$, respectively. At first the function $\phi(x)$ is defined as $\phi(x) = x^2$, then active con-

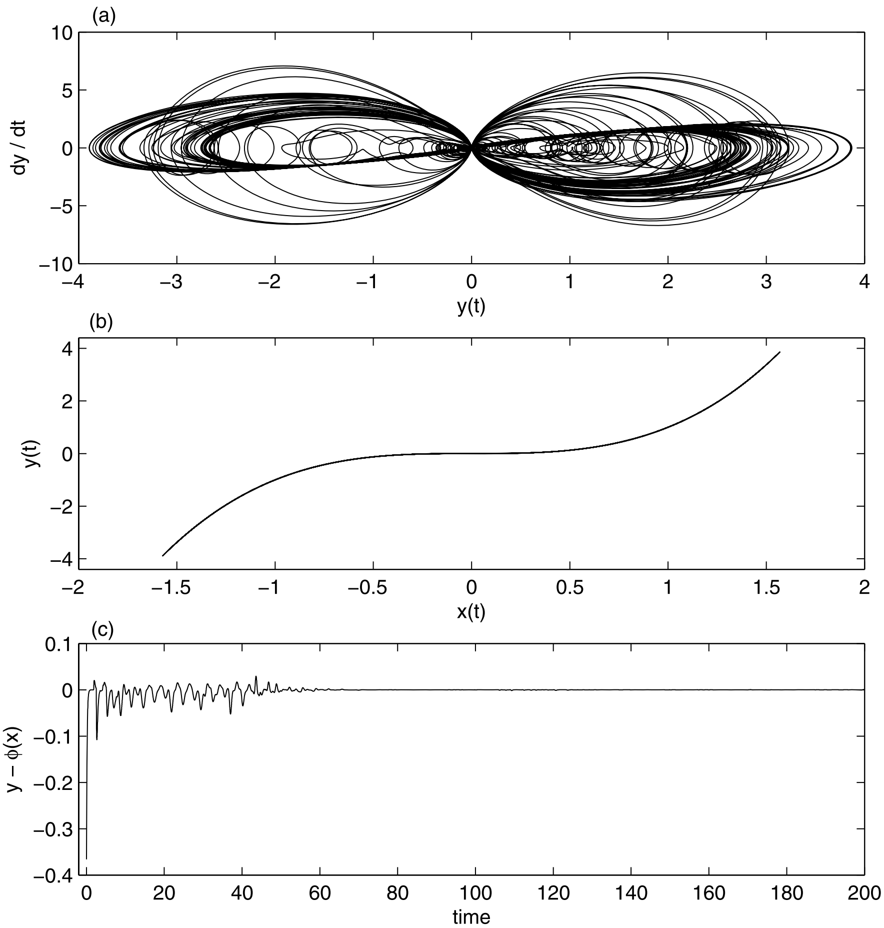

Fig. 2 (**a**) Chaotic attractor of the response system, (**b**) generalized synchronization manifold for $\phi(x) = x^3$, (**c**) corresponding generalized synchronization error

trol function is $u = 2ax^2 - 2\mu_1 x(\delta x_{\tau_1} - \epsilon x_{\tau_1}^3) - bx^2 + (\mu_2 - 1)\sin(x_{\tau_2}^2) + \delta y_{\tau_1} - \epsilon y_{\tau_1}^3$ and $s = \delta x_{\tau_1}^2 - \epsilon x_{\tau_1}^6 + kx^2$, $t = \sin(y_{\tau_2}) + ky$. Then the above condition (8) for generalized synchronization becomes

$$k + b \geq \frac{\frac{2\delta}{3}\sqrt{\frac{\delta}{3\epsilon}}}{\sqrt{1-\tau_1'}} + \frac{|\mu_2 - 1|}{\sqrt{1-\tau_2'}}. \tag{15}$$

Now

$$1 - \tau_1' = 1 - a_1\omega_1 \cos(\omega_1 t) \leq |1 - a_1\omega_1 \cos(\omega_1 t)|$$
$$\leq 1 + |a_1\omega_1 \cos(\omega_1 t)| \leq 1 + a_1\omega_1,$$

so $\sqrt{1-\tau_2'} \leq \sqrt{1 + a_1\omega_1}$ implies that

$$\frac{1}{\sqrt{1-\tau_1'}} \geq \frac{1}{\sqrt{1+a_1\omega_1}}.$$

Similarly,

$$\frac{1}{\sqrt{1-\tau_2'}} > \frac{1}{\sqrt{1+3a_2\omega_2}}.$$

For the above set of parameter values, generalized synchronization between (13) and (14) occurs if $k \geq 3.4766$. For our simulation, we choose $k = 3.5$. At this position, the chaotic attractors of the drive and response systems are shown in Fig. 1(a) and (b), respectively. In Fig. 1(c) the generalized synchronization manifold is shown. The corresponding synchronization manifold is shown in Fig. 1(d).

For $\phi(x) = x^3$, one can observe the generalized synchronization for $k = 4.0$ and the other parameters as before. At this position, the chaotic attractor of the response system is shown in Fig. 2(a). In this figure, the manifold is shown and corresponding synchronization error is given in Fig. 2(c).

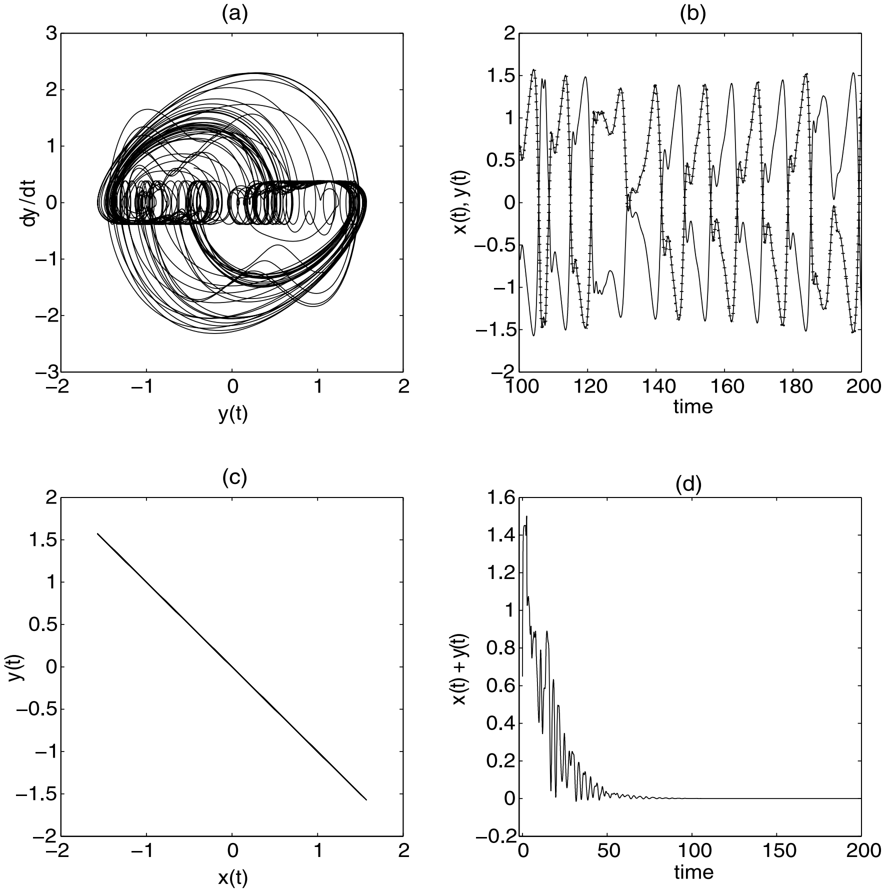

Fig. 3 (**a**) Chaotic attractor of the response system when $\phi(x) = -x$, (**b**) time series of $x(t)$ (*solid line*) and $y(t)$ (*dotted line*) for $k = 4.0$, (**c**) corresponding antisynchronization manifold, (**d**) corresponding antisynchronization error

If $\phi(x) = -x$, one can observe the antisynchronization for $k = 4.0$. The chaotic attractor of the response system is shown in Fig. 3(a). The time series of $x(t)$ and $y(t)$ are shown in Fig. 3(b). In this figure it is observed that the drive and response systems are in anti-phase pattern, i.e. the difference of the phase angles of the synchronized trajectories is π. Antisynchronization manifold is shown in Fig. 3(c) and corresponding manifold is shown in Fig. 3(d). The relation between the states of drive and response is $y = -x$.

4 Conclusions

We have addressed a method for generalized synchronization in time-delayed systems, using combination of nonlinear observer and active control. Chaos synchronization is often understood as a regime in which two coupled chaotic systems exhibit identical behavior. But practically chaos synchronization between two identical coupled systems is much more complicated. When the parameters of the coupled systems are slightly mismatched, then complete synchronization cannot be observed but the coupled systems can still remain synchronized in a generalized sense. Generalized synchronization is an extension of complete synchronization. Secure communication based on generalized synchronization can reduce the complexities of the transmitters and receivers. Up to now, the study on generalized synchronization in time-delayed system is still at the stage of beginning. The study of generalized synchronization in modulated time-delayed systems is of highly practical importance. As an important implication for practical applications it is demonstrated that generalized synchronization is robust with respect to parameter changes. This method also offers new possibilities for communication schemes using chaos synchronization.

Our work leads to several advantages over existing work [12, 13, 15–19, 30, 31]. (i) It is capable of realizing generalized synchronization of a general class of time-delayed chaotic systems, while previous work [12, 13, 15–19] only mentioned generalized synchronization on two coupled low-dimensional chaotic systems and only very few recent studies have dealt with generalized synchronization in time-delayed systems [30, 31]. So it can be considered as an extension of the dynamics from finite-dimensional to infinite-dimensional chaotic systems. (ii) It enables generalized synchronization be achieved in a systematic way and it does not require the initial conditions belonging to the same basin of attraction. It gives a uniform framework for construction of generalized synchronization in time-delayed systems. According to Krasovskii–Lyapunov theory, we have achieved the sufficient stability condition for generalized synchronization manifold. The validity and feasibility of our method have been verified by computer simulations of prototype and Ikeda systems.

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