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Synchronization for chaotic Lur'e systems with sector-restricted nonlinearities via delayed feedback control

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Abstract In this paper, the effects of time delay on chaotic master–slave synchronization scheme are considered. Using delayed feedback control scheme, a delay-dependent stability criterion is derived for the synchronization of chaotic systems that are represented by Lur'e system with sector-restricted nonlinearities. The derived criterion is a sufficient condition for absolute stability of error dynamics between the master and the slave system. Using a convex representation of the nonlinearity, the stability condition based on the Lyapunov–Krasovskii functional is obtained via LMI formulation. The proposed delay-dependent synchronization criterion is less conservative than the existing ones. The effectiveness of our work is verified through numerical examples.

Keywords Lur'e systems · Synchronization · Delay-dependent criterion · Absolute stability · LMIs

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1 Introduction

Synchronization problems are easily found in many physical and biological systems such as heart beat, walking, coordinated robot motion and so on. The synchronization between chaotic systems is a more interesting issue, because chaotic systems are hard to expect their behavior and very sensitive to initial conditions. Since the pioneer work of Pecora and Carroll [1] was presented for synchronization of two identical chaotic systems, chaotic synchronization has received much attention due to its theoretical and practical importance. For example, the chaotic synchronization scheme is extensively studied in various areas, including secure communication, modeling brain activity and information processing [2–4].

Recently, many researches for the chaotic synchronization of Lur'e systems were presented, because various chaotic systems, such as Chua's circuit, n-scroll attractors and hyperchaotic attractors, can be modeled as Lur'e systems [2–22, 27–29]. The Lur'e system is a continuum of a linear system and a feedback nonlinearity satisfying sector bound constraints. The stability of the Lur'e system is called absolute stability, which means global asymptotic stability. For this reason, many researchers have studied the chaotic synchronization for Lur'e systems and applied it to various applications. Recently, practical issues of the synchronization, such as propagation delay, noise and model uncertainty, have been considered. Since Chen and Liu [8] introduced the delay of the chaotic synchronization and showed that the delay may break the synchronization, especially many research efforts have been focused on the effect of the propagation delay for the chaotic synchronization.

Yalcin et al. [9] conducted the first research considering the effect of time delay in chaotic synchronization of Lur'e system and presented sufficient conditions for stability. Liao and Chen [10] proposed a synchronization scheme for Lur'e systems with time delay using a feedback controller. After the research of Liao and Chen, various synchronization schemes for the chaotic Lur'e systems with time delay were presented in [11–17]. In those synchronization schemes, a delayed state feedback controller[1] was designed and its gain matrix was derived from a sufficient condition for stability of error dynamics between a master system and a slave system. In [12], quadratic programming was used to obtain the gain matrix of the synchronization controller. Furthermore, in [11, 13] and [15], the gain matrix for a given time delay was obtained through LMI conditions that are derived from sufficient stability conditions and that can be easily solved by various numerical methods [25].

There are several studies that considered the delay effect on the chaotic synchronization [18–22]. In those researches, synchronization criteria were derived for given gain matrices of the synchronization controller and time delay, which are sufficient conditions for absolute stability of the synchronization. In [18], delay-independent and delay-dependent stability criteria for master–slave synchronization scheme of Lur'e systems with time delay were derived through LMI formulation. However, the model transformation technique used in [9] and [18] can lead to conservative conditions by inducing additional dynamics as addressed in [23]. In order to derive less conservative conditions for synchronization, synchronization criteria that does not use the model transformation were presented independently in [19, 21] and [20]. Xiang et al. [20] used the integral inequality and a free weighting matrix approach. In [19], a more general Lur'e–Postnikov–Lyapunov functional was presented to derive a less conservative criterion. Guo and Zhong [21] applied a free weighting matrix approach employed on the Leibniz–Newton formula and an equality constraint for the synchronization criterion. Furthermore, a synchronization method for the chaotic Lur'e system was extended for time-varying delay in [22].

In this paper, we consider the synchronization criterion for master–slave Lur'e systems with a delayed feedback controller. A delay-dependent synchronization criterion is presented for the time delayed state feedback controller. Convex representation of the nonlinearity of the Lur'e system is introduced, and then, a sector-bounded constraint of the nonlinearity is converted to an equality constraint. The Finsler lemma [24] is utilized for handling the equality constraint, so that a less conservative delay-dependent synchronization criterion is obtained. Furthermore, a new Lyapunov–Krasovskii function that employs redundant state of differential equations shifted in time by a fraction of the time delay is also applied to reduce conservatism in searching the maximum allowable delay, such that the error dynamics of synchronization are absolutely stable. This is a part of the implicit model transformation based approaches and it is called the delay discretization approach [26]. The derived criterion is formulated by LMIs that are easily solvable using various numerical methods [25]. For numerical examples, the maximum allowable delay for synchronization of chaotic Lur'e systems is found using the proposed criterion and is compared to the ones of the previous studies.

Notations $\mathbb{R}^n$ denotes the n-dimensional Euclidean space. $\mathbb{R}^{n\times m}$ is the set of all $n \times m$ real matrices. For a real matrix X, $X > 0$ or $X < 0$ means that X is a positive/negative definite symmetric matrix, respectively. I is an identity matrix with appropriate dimension and 0 is a null matrix with appropriate dimension. For a given matrix $A \in \mathbb{R}^{m\times n}$ such that $\operatorname{rank}(A) = r$, we define $A^{\perp} \in \mathbb{R}^{n\times(n-r)}$ as the right-orthogonal comple-

[1] In some studies, both state and delayed state are used to feedback.

ment of A by $AA^{\perp}=0$. diag$(\cdots)$ represents a block diagonal matrix.

2 Problem formulation

Consider the following master–slave synchronization scheme of chaotic Lur'e systems with a master system $\mathcal{M}$, a slave system $\mathcal{S}$ and time delayed output feedback $\mathcal{C}$:

$$\begin{aligned}
&\mathcal{M}: \quad \begin{cases} \dot{x}=Ax(t)+B\varphi(\mu(t)), \\ z_m(t)=Hx(t), \\ \mu(t)=Cx(t), \end{cases} \\
&\mathcal{S}: \quad \begin{cases} \dot{y}=Ay(t)+B\varphi(\kappa(t))+u(t), \\ z_s(t)=Hy(t), \\ \kappa(t)=Cy(t), \end{cases} \\
&\mathcal{C}: \quad u(t)=L(z_m(t-\tau)-z_s(t-\tau)),
\end{aligned} \tag{1}$$

where $\tau>0$ is the time delay; $x(t)$, $y(t)$ and $z_m(t)$, $z_s(t)$ are state vectors $\mathbb{R}^n$ and the output vectors $\mathbb{R}^l$ of the Lur'e systems, respectively; $A\in\mathbb{R}^{n\times n}$, $B\in\mathbb{R}^{n\times p}$, $C\in\mathbb{R}^{p\times n}$, $H\in\mathbb{R}^{l\times n}$ are constant matrices; the controller gain matrix $L\in\mathbb{R}^{n\times l}$ is a given constant matrix. $\mu(t)$ and $\kappa(t)$ are the input vectors of the nonlinearity function $\phi(\cdot)$. The nonlinearity of the Lur'e system $\varphi(\cdot):\mathbb{R}^m\to\mathbb{R}^p$ is a memoryless vector valued function of which ith element $\varphi_i(\cdot)$ is in a certain sector such that

$$\bar{\alpha}_i \le \frac{\varphi_i(\sigma_i(t))}{\sigma_i(t)} \le \bar{\beta}_i \tag{2}$$

where $\sigma_i(t)$ is the ith element of the $\sigma(\cdot)$, $\bar{\alpha}_i$ and $\bar{\beta}_i$ are the lower and upper bounds of the sector, respectively. We assume that the nonlinearity $\varphi(\cdot)$ also satisfies a slope constraint such that

$$\alpha_i \le \frac{d\varphi_i(\sigma_i(t))}{d\sigma_i(t)} \le \beta_i. \tag{3}$$

The synchronization scheme (1) achieves synchronization of states between two systems by utilizing a time delayed output feedback as an input to the slave system $\mathcal{S}$. Let us define an error of synchronization as

$$e(t)=x(t)-y(t), \tag{4}$$

then the following error dynamics of the synchronization can be obtained:

$$\begin{aligned}
&\dot{e}(t)=Ae(t)+Me(t-\tau)+B\big(\varphi\big(\mu(t)\big)-\varphi\big(\kappa(t)\big)\big), \\
&e(\theta)=\psi(\theta), \quad \forall\theta\in[-\tau,0]
\end{aligned} \tag{5}$$

where $M=-LH$ and $\psi(\cdot)$ is a continuous vector valued function for initial values.

Using the slope constraint of the nonlinear function in (3), we can derive new sector bounds for the error of the nonlinear functions $\varphi(\mu(t))$ and $\varphi(\kappa(t))$. By the mean value theorem, there exists a constant $\delta\in(\mu_i(t),\kappa_i(t))$ such that

$$\varphi_i\big(\mu_i(t)\big)-\varphi_i\big(\kappa_i(t)\big)=\frac{d\varphi_i(\delta)}{d\sigma_i}\big(\mu_i(t)-\kappa_i(t)\big) \tag{6}$$

where $\mu_i(t)$ and $\kappa_i(t)$ are the ith elements of $\mu(t)$ and $\kappa(t)$, respectively. From the slope bounds in (3), we have

$$\alpha_i \le \frac{d\varphi_i(\delta)}{d\sigma_i} \le \beta_i.$$

Since $\mu(t)-\kappa(t)=C(x(t)-y(t))=Ce(t)$, we also have

$$\alpha_i c_i e(t) \le \varphi_i\big(\mu_i(t)\big)-\varphi_i\big(\kappa_i(t)\big) \le \beta_i c_i e(t) \tag{7}$$

where c_i is the ith row vector of the matrix C. Let us denote $\nu_i(t)=c_ie(t)$, then we obtain a new nonlinear function $\phi_i(\nu_i(t))$ bounded by a sector that belongs to $[\alpha_i,\beta_i]$ such that

$$\alpha_i \le \frac{\phi_i(\nu_i(t))}{\nu_i(t)} \le \beta_i, \tag{8}$$

where $\phi_i(\nu_i(t)) \triangleq \varphi_i(\mu_i(t))-\varphi_i(\kappa_i(t)))$.

Therefore, the error dynamics (5) can be represented as a Lur'e system with the new sector-bounded nonlinear function $\phi(\nu(t))$ as follows:

$$\mathcal{E}: \quad \dot{e}(t)=Ae(t)+Me(t-\tau)+B\phi\big(\nu(t)\big) \tag{9}$$

where $\nu(t)=Ce(t)$.

Remark 1 In most researches for synchronization of Lur'e systems [9–22] it is assumed that the sector bounds of $\varphi(\mu(t))-\varphi(\kappa(t))$ and $\phi(\cdot)$ are identical. However, when the inequality of the sector is not satisfied for whole domain, we could not derive globally asymptotic stability condition but locally asymptotic stability condition. Even if the nonlinearity such as a saturation function of Chua's circuit is not differentiable at some points in the domain, the inequality (8) can be utilized to find the sector bound easily. The sector bound is obtained by considering both (8) and $\varphi_i(\mu_i(t)) - \varphi_i(\kappa_i(t))$ for non-differentiable points in the domain.

The nonlinear function $\phi(\cdot)$ can be represented by a convex combination of the sector bounds such as α_i and β_i. We can rewrite the $\phi_i(\cdot)$ as follows:

$$\phi_i\big(\nu_i(t)\big) = \big(\lambda_i^l\big(\nu_i(t)\big)\alpha_i + \lambda_i^u\big(\nu_i(t)\big)\beta_i\big)\nu_i(t) \tag{10}$$

where

$$\begin{aligned}\lambda_i^l\big(\nu_i(t)\big) &= \frac{\phi_i(\nu_i(t)) - \alpha_i\nu_i(t)}{(\beta_i-\alpha_i)\nu_i(t)},\\ \lambda_i^u\big(\nu_i(t)\big) &= \frac{\beta_i\nu_i(t) - \phi_i(\nu_i(t))}{(\beta_i-\alpha_i)\nu_i(t)}.\end{aligned} \tag{11}$$

Since $\lambda_i^l(\nu_i) + \lambda_i^u(\nu_i) = 1$, $\lambda_i^l(\nu_i) \geq 0$ and $\lambda_i^u(\nu_i) \geq 0$, the $\phi_i(\cdot)$ can be represented using a convex hull:

$$\phi_i(\nu_i(t)) = \Delta_i\big(\nu_i(t)\big)\nu_i(t) \tag{12}$$

where $\Delta_i(\nu_i(t))$ is an element of a convex hull $\mathrm{Co}\{\alpha_i, \beta_i\}$.

Let us define some diagonal matrices as

$$\begin{aligned}\Delta(\nu) &\triangleq \mathrm{diag}\big(\Delta_1(\nu_1), \ldots, \Delta_p(\nu_p)\big),\\ \alpha &\triangleq \mathrm{diag}(\alpha_1, \ldots, \alpha_p),\\ \beta &\triangleq \mathrm{diag}(\beta_1, \ldots, \beta_p).\end{aligned} \tag{13}$$

Then, the nonlinear function $\phi(\cdot)$ can be represented by

$$\phi\big(\nu(t)\big) = \Delta\big(\nu(t)\big)\nu(t) \tag{14}$$

where $\Delta(\nu(t))$ belongs to the following set:

$$\Phi := \big\{\Delta(\nu) | \Delta(\nu) \in \mathrm{Co}\{\alpha, \beta\}\big\}. \tag{15}$$

Like many papers [9–22] for the synchronization criteria, we suppose that the gain matrix of the synchronization controller is given, because the purpose of this paper is to find the maximum allowable delay bound such that the error dynamics of the synchronization (9) is absolutely stable. The following lemmas are useful for deriving the synchronization criterion.

Lemma 1 [19] *For any constant matrix $W \in \mathbb{R}^{n\times n}$, $W > 0$, scalar $\tau > 0$ and a vector function $e(t) : \mathbb{R} \to \mathbb{R}^n$ such that the following integration is well defined*:

$$\begin{aligned}&-\tau\int_{-\tau}^{0} \dot{e}^T(t+\xi)W\dot{e}(t+\xi)\,d\xi\\ &\quad\leq \begin{bmatrix} e(t)^T & e(t-\tau)^T\end{bmatrix}\begin{bmatrix} -W & W\\ W & -W\end{bmatrix}\begin{bmatrix} e(t)\\ e(t-\tau)\end{bmatrix}.\end{aligned} \tag{16}$$

The following Finsler lemma is useful to convert an inequality subject to an equality constraint to an inequality.

Lemma 2 *Finsler lemma* [24] *Let $x \in \mathbb{R}^n$, $\Theta = \Theta^T \in \mathbb{R}^{n\times n}$, $\Gamma \in \mathbb{R}^{m\times n}$. The following statements are equivalent*:

i. $x^T\Theta x < 0 \;\; s.t. \; \Gamma x = 0, \; \forall x \neq 0$,
ii. $\Gamma^{\perp T}\Theta\Gamma^{\perp} < 0$.

3 Main results

In this section, we derive a synchronization criterion for absolute stability of the chaotic Lur'e system (9) via LMI formulation.

Considering the system (9), it is rewritten for any θ such that $-\tau \leq \theta \leq 0$ as follows:

$$\begin{aligned}\dot{e}(t+\theta) &= Ae(t+\theta) + Me(t-\tau+\theta)\\ &\quad + B\phi\big(\nu(t+\theta)\big),\\ e(t+\theta) &= \psi(t+\theta), \quad \forall t \in [-\tau, 0]\end{aligned} \tag{17}$$

where $\psi(\cdot)$ is an initial condition. Suppose that the error dynamics (9) of the Lur'e system is global asymptotically stable for a given time delay $\tau > 0$, then the system (17) should also be stable for any $\theta \in [-\tau, 0]$.

In [26], the Lyapunov–Krasovskii functional using this property was applied to obtain a less conservative delay-dependent stability criterion and was called *the discretized Lyapunov functional* approach. In this paper, we apply the discretization approach to derive the less conservative criteria.

For simplicity, let us choose $\theta = -\tau/2$ and consider an augmented system with states $e(t)$ and $e(t-\frac{\tau}{2})$:

$$\dot{e}\left(t-\frac{\tau}{2}\right) = Ae\left(t-\frac{\tau}{2}\right) + Me\left(t-\frac{3\tau}{2}\right) + B\phi\left(\nu\left(t-\frac{\tau}{2}\right)\right), \quad (18)$$

$$\dot{e}(t) = Ae(t) + Me(t-\tau) + B\phi\big(\nu(t)\big)$$

with appropriate initial conditions.

Let us define a new augmented state $e_{\mathrm{a}}(t)$ and some block matrices as

$$e_{\mathrm{a}}(t) = \begin{bmatrix} e(t-\frac{\tau}{2})^T & e(t)^T \end{bmatrix}^T, \quad (19)$$

$$\begin{aligned} &A_{\mathrm{a}} = \mathrm{diag}(A, A), \qquad M_{\mathrm{a}} = \mathrm{diag}(M, M), \\ &B_{\mathrm{a}} = \mathrm{diag}(B, B), \\ &C_{\mathrm{a}} = \mathrm{diag}(C, C), \qquad \Delta_{\mathrm{a}} = \mathrm{diag}(\Delta, \Delta), \\ &\alpha_{\mathrm{a}} = \mathrm{diag}(\alpha, \alpha), \qquad \beta_{\mathrm{a}} = \mathrm{diag}(\beta, \beta). \end{aligned} \quad (20)$$

Then, the error dynamics (18) is rewritten as

$$\begin{aligned} &\dot{e}_{\mathrm{a}}(t) = A_{\mathrm{a}}e_{\mathrm{a}}(t) + M_{\mathrm{a}}e_{\mathrm{a}}(t-\tau) + B_{\mathrm{a}}\phi_{\mathrm{a}}\big(\nu_{\mathrm{a}}(t)\big), \\ &\nu_{\mathrm{a}}(t) = C_{\mathrm{a}}e_{\mathrm{a}}(t), \\ &\phi_{\mathrm{a}}\big(\nu_{\mathrm{a}}(t)\big) = \begin{bmatrix} \phi(\nu(t-\frac{\tau}{2})) \\ \phi(\nu(t)) \end{bmatrix}. \end{aligned} \quad (21)$$

The convex representation (14) of the nonlinearity can be used to establish equality constraints. From (14), we have the following equality constraint:

$$\phi\big(\nu(t)\big) - \Delta\big(\nu(t)\big)\nu(t) = \phi\big(\nu(t)\big) - \Delta Ce(t) = 0, \quad \forall \Delta \in \Phi. \quad (22)$$

Therefore, an equality constraint for $\phi_{\mathrm{a}}(\nu_{\mathrm{a}}(t))$ is obtained as

$$\phi_{\mathrm{a}}\big(\nu(t)\big) - \begin{bmatrix} \Delta & 0 \\ 0 & \Delta \end{bmatrix} \begin{bmatrix} C & 0 \\ 0 & C \end{bmatrix} e_{\mathrm{a}}(t) = 0, \quad \forall \Delta \in \Phi. \quad (23)$$

Furthermore, we can establish an additional equality constraint from the error dynamics (21) and the definition of $e_{\mathrm{a}}(t)$ as follows:

$$\dot{e}_{\mathrm{a}}(t) - A_{\mathrm{a}}e_{\mathrm{a}}(t) - M_{\mathrm{a}}e_{\mathrm{a}}(t-\tau) - B_{\mathrm{a}}\phi_{\mathrm{a}}\big(\nu_{\mathrm{a}}(t)\big) = 0. \quad (24)$$

In the following theorem, a synchronization criterion for the chaotic Lur'e system (21) under equality constraints described above is presented using the Finsler lemma.

Theorem 1 *The error system described as* (21) *is absolutely stable for any delay τ such that $0 \le \tau \le \bar{\tau}$ if there exist positive definite matrices P, Q_1, Q_2, R_1 and $R_2 \in \mathbb{R}^{2n\times 2n}$ and a positive definite diagonal matrix $S \in \mathbb{R}^{2p\times 2p}$ satisfying the following LMI*:

$$\Gamma^{\perp^T}(\Delta)Y(\tau)\Gamma^{\perp}(\Delta) < 0 \quad (25)$$

where $\Gamma^{\perp}(\Delta)$ is a right-orthogonal complement of

$$\Gamma(\Delta) = \begin{bmatrix} I & 0 & -A & 0 & 0 & 0 & -M & 0 & 0 & 0 & 0 & 0 & -B & 0 \\ 0 & I & 0 & -A & 0 & 0 & 0 & -M & 0 & 0 & 0 & 0 & 0 & -B \\ 0 & 0 & -I & 0 & I & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -I & 0 & I & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & -I & 0 & 0 & 0 & I & 0 & 0 & 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & -I & 0 & 0 & 0 & I & 0 & 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & I & -I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I & -I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \Delta(\nu)C & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -I & 0 \\ 0 & 0 & 0 & \Delta(\nu)C & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -I \end{bmatrix} \quad (26)$$

and $Y(\tau)$ *is*

$$Y(\tau)=\begin{bmatrix} \tau R_1+\frac{\tau}{2}R_2 & P & 0 & 0 & 0 & 0 & 0 \\ P & Q_1+Q_2-2C_{\mathrm{a}}^T\alpha_{\mathrm{a}}S\beta_{\mathrm{a}}C_{\mathrm{a}} & 0 & 0 & 0 & 0 & C_{\mathrm{a}}^T S(\alpha_{\mathrm{a}}+\beta_{\mathrm{a}}) \\ 0 & 0 & -Q_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -Q_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{2}{\tau}R_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{\tau}R_2 & 0 \\ 0 & (\alpha_{\mathrm{a}}+\beta_{\mathrm{a}})SC_{\mathrm{a}} & 0 & 0 & 0 & 0 & -2S \end{bmatrix}. \tag{27}$$

Proof Consider the following Lyapunov–Krasovskii functional:

$$V\big(e_{\mathrm{a}}(t)\big)=V_1\big(e_{\mathrm{a}}(t)\big)+V_2\big(e_{\mathrm{a}}(t)\big)+V_3\big(e_{\mathrm{a}}(t)\big) \tag{28}$$

where

$$V_1\big(e_{\mathrm{a}}(t)\big)=e_{\mathrm{a}}(t)^T Pe_{\mathrm{a}}(t),$$

$$V_2\big(e_{\mathrm{a}}(t)\big)=\int_{t-\tau/2}^{t} e_{\mathrm{a}}^T(\xi)Q_1e_{\mathrm{a}}(\xi)\,d\xi + \int_{t-\tau}^{t} e_{\mathrm{a}}^T(\xi)Q_2e_{\mathrm{a}}(\xi)\,d\xi,$$

$$V_3\big(e_{\mathrm{a}}(t)\big)=\int_{t-\tau/2}^{t}\left(\frac{\tau}{2}-t+\xi\right)\dot e_{\mathrm{a}}^T(\xi)R_1\dot e_{\mathrm{a}}(\xi)\,d\xi + \int_{t-\tau}^{t}(\tau-t+\xi)\dot e_{\mathrm{a}}^T(\xi)R_2\dot e_{\mathrm{a}}(\xi)\,d\xi.$$

Time derivative of $V_1(e_{\mathrm{a}}(t))$ with respect to time along the trajectory of (21) is

$$\dot V_1\big(e_{\mathrm{a}}(t)\big)=\dot e_{\mathrm{a}}^T(t)Pe_{\mathrm{a}}(t)+e_{\mathrm{a}}^T(t)P\dot e_{\mathrm{a}}(t). \tag{29}$$

Similarly, time derivative of $V_2(e_{\mathrm{a}}(t))$ is found as

$$\dot V_2\big(e_{\mathrm{a}}(t)\big) = \begin{bmatrix} e_{\mathrm{a}}(t) \\ e_{\mathrm{a}}(t-\frac{\tau}{2})\end{bmatrix}^T \begin{bmatrix} Q_1 & 0 \\ 0 & -Q_1\end{bmatrix}\begin{bmatrix} e_{\mathrm{a}}(t) \\ e_{\mathrm{a}}(t-\frac{\tau}{2})\end{bmatrix} + \begin{bmatrix} e_{\mathrm{a}}(t) \\ e_{\mathrm{a}}(t-\tau)\end{bmatrix}^T \begin{bmatrix} Q_2 & 0 \\ 0 & -Q_2\end{bmatrix}\begin{bmatrix} e_{\mathrm{a}}(t) \\ e_{\mathrm{a}}(t-\tau)\end{bmatrix}. \tag{30}$$

The following inequality for time derivative of $V_3(e_{\mathrm{a}}(t))$ can be obtained by using Lemma 1:

$$\dot V_3\big(e_{\mathrm{a}}(t)\big) \le \frac{\tau}{2}\dot e_{\mathrm{a}}^T(t)R_1\dot e_{\mathrm{a}}(t)+\tau\dot e_{\mathrm{a}}^T(t)R_2\dot e_{\mathrm{a}}(t) + \frac{2}{\tau}\begin{bmatrix} e_{\mathrm{a}}(t) \\ e_{\mathrm{a}}(t-\frac{\tau}{2})\end{bmatrix}^T \begin{bmatrix} R_1 & -R_1 \\ -R_1 & R_1\end{bmatrix}\begin{bmatrix} e_{\mathrm{a}}(t) \\ e_{\mathrm{a}}(t-\frac{\tau}{2})\end{bmatrix} + \frac{1}{\tau}\begin{bmatrix} e_{\mathrm{a}}(t) \\ e_{\mathrm{a}}(t-\tau)\end{bmatrix}^T \begin{bmatrix} R_2 & -R_2 \\ -R_2 & R_2\end{bmatrix}\begin{bmatrix} e_{\mathrm{a}}(t) \\ e_{\mathrm{a}}(t-\tau)\end{bmatrix}. \tag{31}$$

From the sector constraint of the nonlinearity $\phi(\cdot)$, the following inequality is obtained:

$$-2\big(\phi_{\mathrm{a}}\big(\nu_{\mathrm{a}}(t)\big)-\alpha_{\mathrm{a}}\nu_{\mathrm{a}}(t)\big)^T S\big(\phi_{\mathrm{a}}\big(\nu_{\mathrm{a}}(t)\big)-\beta_{\mathrm{a}}\nu_{\mathrm{a}}(t)\big)\le 0. \tag{32}$$

By applying the well-known S-procedure [25] to (32) and utilizing (29)–(31), we have

$$\begin{aligned}\dot V\big(e_{\mathrm{a}}(t)\big) &\le \dot e_{\mathrm{a}}^T(t)Pe_{\mathrm{a}}(t)+e_{\mathrm{a}}^T(t)P\dot e(t)+\frac{\tau}{2}\dot e_{\mathrm{a}}^T(t)R_1\dot e_{\mathrm{a}}(t) \\ &\quad +\tau\dot e_{\mathrm{a}}^T(t)R_2\dot e_{\mathrm{a}}(t) \\ &\quad + \begin{bmatrix} e_{\mathrm{a}}(t) \\ e_{\mathrm{a}}(t-\frac{\tau}{2})\end{bmatrix}^T \begin{bmatrix} Q_1 & 0 \\ 0 & -Q_1\end{bmatrix}\begin{bmatrix} e_{\mathrm{a}}(t) \\ e_{\mathrm{a}}(t-\frac{\tau}{2})\end{bmatrix} \\ &\quad + \begin{bmatrix} e_{\mathrm{a}}(t) \\ e_{\mathrm{a}}(t-\tau)\end{bmatrix}^T \begin{bmatrix} Q_2 & 0 \\ 0 & -Q_2\end{bmatrix}\begin{bmatrix} e_{\mathrm{a}}(t) \\ e_{\mathrm{a}}(t-\tau)\end{bmatrix} \\ &\quad + \frac{2}{\tau}\begin{bmatrix} e_{\mathrm{a}}(t) \\ e_{\mathrm{a}}(t-\frac{\tau}{2})\end{bmatrix}^T \begin{bmatrix} R_1 & -R_1 \\ -R_1 & R_1\end{bmatrix}\begin{bmatrix} e_{\mathrm{a}}(t) \\ e_{\mathrm{a}}(t-\frac{\tau}{2})\end{bmatrix} \\ &\quad + \frac{1}{\tau}\begin{bmatrix} e_{\mathrm{a}}(t) \\ e_{\mathrm{a}}(t-\tau)\end{bmatrix}^T \begin{bmatrix} R_2 & -R_2 \\ -R_2 & R_2\end{bmatrix}\begin{bmatrix} e_{\mathrm{a}}(t) \\ e_{\mathrm{a}}(t-\tau)\end{bmatrix} \\ &\quad -2\big(\phi_{\mathrm{a}}\big(\nu_{\mathrm{a}}(t)\big)-\alpha_{\mathrm{a}}\nu_{\mathrm{a}}(t)\big)^T S\big(\phi_{\mathrm{a}}\big(\nu_{\mathrm{a}}(t)\big) \\ &\quad -\beta_{\mathrm{a}}\nu_{\mathrm{a}}(t)\big).\end{aligned} \tag{33}$$

Let us define an extended vector $\zeta(t)$ as

$$\zeta(t)=\begin{bmatrix}\dot{e}_{\mathrm{a}}^{T}(t) & e_{\mathrm{a}}^{T}(t) & e_{\mathrm{a}}^{T}(t-\frac{\tau}{2}) & e_{\mathrm{a}}^{T}(t-\tau) & e_{\mathrm{a}}^{T}(t)-e_{\mathrm{a}}^{T}(t-\frac{\tau}{2}) & e_{\mathrm{a}}^{T}(t)-e_{\mathrm{a}}^{T}(t-\tau) & \phi_{a}^{T}(\nu_{a}(t))\end{bmatrix}^{T}. \tag{34}$$

By rewriting (33) for $\zeta(t)$, we have

$$\dot{V}\big(e_{\mathrm{a}}(t)\big)\le\zeta^{T}(t)Y\zeta(t). \tag{35}$$

Next, the equality constraints (22)–(23) for the augmented state (19) and the nonlinear function $\phi_{\mathrm{a}}(\cdot)$ are rewritten with the extended vector $\zeta(t)$ as follows:

$$\begin{bmatrix} I & -A_{\mathrm{a}} & 0 & -M_{\mathrm{a}} & 0 & 0 & -B_{\mathrm{a}}\\ 0 & \Delta_{\mathrm{a}}C_{\mathrm{a}} & 0 & 0 & 0 & 0 & -I\end{bmatrix}\zeta(t)=0. \tag{36}$$

Moreover, three additional equality constraints can be derived from $e_{\mathrm{a}}(t)-e_{\mathrm{a}}(t-\frac{\tau}{2})$ and $e_{\mathrm{a}}(t)-e_{\mathrm{a}}(t-\tau)$ as follows:

$$\begin{bmatrix} 0 & -I & I & 0 & I & 0 & 0\\ 0 & -I & 0 & I & 0 & I & 0\\ 0 & E_1 & E_2 & E_3 & 0 & 0 & 0\end{bmatrix}\zeta(t)=0, \tag{37}$$

where

$$E_1=\begin{bmatrix}0 & I\\ 0 & 0\end{bmatrix},\qquad E_2=\begin{bmatrix}-I & 0\\ 0 & I\end{bmatrix},$$

$$E_3=\begin{bmatrix}0 & 0\\ -I & 0\end{bmatrix}.$$

Combining (36) and (37) yields the following equality constraint:

$$\Gamma\zeta(t)=0. \tag{38}$$

Therefore, a sufficient condition for stability of the error dynamics (9) with a given time delay is that

$$\dot{V}\big(e_{\mathrm{a}}(t)\big)\le\zeta^{T}(t)Y(\tau)\zeta(t)<0,\quad \forall\zeta(t)\neq 0 \tag{39}$$

subject to

$$\Gamma\zeta(t)=0. \tag{40}$$

Applying the Finsler lemma to (39) and (40), we obtain the LMI (25). This completes the proof. □

4 Numerical examples

In this section, two examples, such as Chua's circuit and hyperchaotic attractor, are used to illustrate the effectiveness of the proposed synchronization criterion given in Theorem 1.

Example 1 Consider the following Chua's circuit shown in [9]:

$$\begin{cases}\dot{x}=a(y-h(x)),\\ \dot{y}=x-y+z,\\ \dot{z}=-by\end{cases} \tag{41}$$

with nonlinear characteristic

$$h(x)=m_1x+\frac{1}{2}(m_0-m_1)\big(|x+c|-|x-c|\big)$$

and parameters $m_0=-1/7$, $m_1=2/7$, $a=9$, $b=14.28$ and $c=1$. The system can be represented in Lur'e system with

$$A=\begin{bmatrix}-am_1 & a & 0\\ 1 & -1 & 1\\ 0 & -b & 0\end{bmatrix},$$

$$B=\begin{bmatrix}-a(m_0-m_1)\\ 0\\ 0\end{bmatrix}, \tag{42}$$

$$C=H=\begin{bmatrix}1 & 0 & 0\end{bmatrix}$$

and $\varphi(\mu)=\frac{1}{2}(|\mu+c|-|\mu-c|)$ belonging to sector [0, 1]. The gain matrices employed in the literature [9, 18–22] to illustrate the effectiveness of the criteria are shown in Table 1. We find the maximum allowable delay bound $\bar{\tau}$ and summarize in Table 2. Table 2 shows that the synchronization criterion of Theorem 1 is less conservative than the ones of previous studies [9, 18–22] Moreover, through numerical simulation, we find the delay bounds $\bar{\tau}^*$ by which the master–slave Chua's circuit can synchronize and the found $\bar{\tau}^*$ are as below:

$$\begin{aligned} &\text{For } L_1: \quad \bar{\tau}^* = 0.220, \\ &\text{For } L_2: \quad \bar{\tau}^* = 0.232, \\ &\text{For } L_3: \quad \bar{\tau}^* = 0.198. \end{aligned} \tag{43}$$

One of observations for L_1 is shown in Fig. 1. The output trajectory of master–slave system does not synchronize for the gain matrix L_1 at $\tau = 0.221$. During the simulation, we take the initial values $x(0) = [-0.2 \quad -0.33 \ 0.2]^T$ for the master system and $y(0) = [0.5 \quad -0.1 \ 0.66]^T$ for the slave system.

We demonstrate the effectiveness of the proposed synchronization criterion to compare with other synchronization schemes that paid attention to design a feedback controller for synchronization. We find the maximum allowable time delay $\bar{\tau}$ using gain matrices of a time delayed state feedback controller which are presented in [13, 14] and [16]. Table 3 shows the founded $\bar{\tau}$ using Theorem 1 and the ones presented in [14, 16] and [13]. We can find that the less conservative time delay bound is obtained by the proposed synchronization criterion. Synchronization error of master–slave Chua's circuit for different gain matrices with the maximum allowable time delay found by Theorem 1 is shown in Fig. 2.

Table 1 Gain matrices of the synchronization scheme used in delay effect analysis for Chua's circuit

Gain matrix L	Used in
$L_1 = [6.0229\ 1.3367\ -2.2164]^T$	[9, 19, 22]
$L_2 = [4.0229\ 1.3367\ -2.2164]^T$	[18]
$L_3 = [6.2121\ 1.0868\ -6.0356]^T$	[20, 21]

Remark 2 In spite of two points $x = 1$ and $x = -1$ at which $\varphi(\cdot)$ is not differentiable, we find the slope bounds as $\alpha = 0$, $\beta = 1$ except these two points. Using (8), bounds for the sector of the nonlinear function $\phi(\cdot)$ are obtained except the two points. However, we easily find that $\phi(\cdot)$ also satisfies (8) for the two points. Therefore, sector bounds of the $\phi(\nu(t)) = \varphi(\mu(t)) - \varphi(\kappa(t))$ are represented as $[\alpha, \beta] = [0, 1]$.

Example 2 Next, let us consider the following hyperchaotic system which consists of two unidirectionally coupled Chua's circuits [6, 9]:

$$\begin{cases} \dot{x}_1 &= a(x_2 - h(x_1)), \\ \dot{x}_2 &= x_1 - x_2 + x_3, \\ \dot{x}_3 &= -bx_2, \\ \dot{x}_4 &= a(x_5 - h(x_4)) + K(x_4 - x_1), \\ \dot{x}_5 &= x_4 - x_5 + x_6, \\ \dot{x}_6 &= -bx_5 \end{cases} \tag{44}$$

with nonlinear characteristic

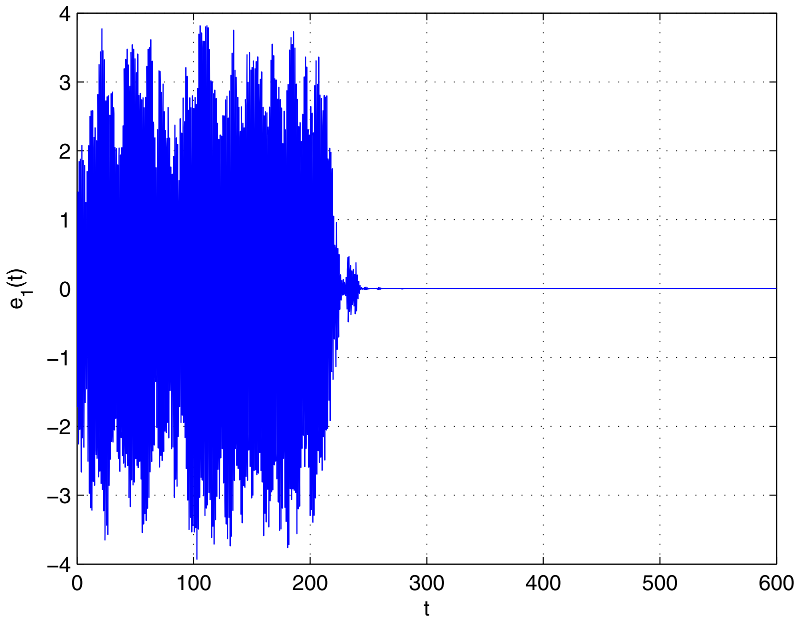
$$h(x) = m_1 x + \frac{1}{2}(m_0 - m_1)\big(|x + c| - |x - c|\big)$$

and parameters $m_0 = -1/7$, $m_1 = 2/7$, $a = 9$, $b = 14.28$, $c = 1$ and $K = 0.01$.

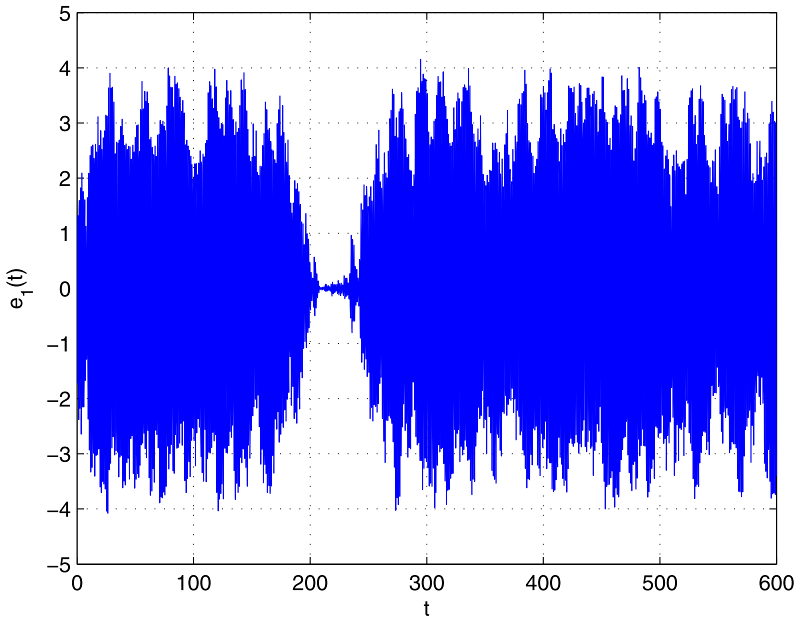
Table 2 The maximum allowable time delays $\bar{\tau}$ for each gain matrix

Gain matrix L		The maximum allowable delay $\bar{\tau}$
L_1	Yalcin et al. [9]	0.039
	Han [19]	0.1418
	Souza et al. [22]	0.141
	Guo et al. [21]	0.1418
	Theorem 1	**0.1637**
L_2	Han [19]	0.1418
	Guo et al. [21]	0.1418
	Theorem 1	**0.1542**
L_3	Han [19]	0.1544
	Guo et al. [21]	0.1544
	Theorem 1	**0.1763**

Fig. 1 Simulation results for master–slave synchronization: error signal $e_1(t)$ $(= x_1(t) - y_1(t))$ for different time delays τ



(a) $\tau = 0.220$



(b) $\tau = 0.221$

Table 3 The maximum allowable time delays $\bar{\tau}$ for each gain matrix

Presented gain matrix L	The maximum allowable delay $\bar{\tau}$	
$[\,4.1455\ 0.9280\ -4.2596\,]^T$	[16]	0.183
	Theorem 1	**0.1963**
$[\,3.9125\ 0.9545\ -3.8273\,]^T$	[14]	0.180
	Theorem 1	**0.1965**
$[\,4.0564\ 0.9285\ -4.1634\,]^T$	[13]	0.18403
	Theorem 1	**0.1942**

This system was represented in the Lur'e system with the following matrices:

$$A = \begin{bmatrix} -am_1 & a & 0 & 0 & 0 & 0 \\ 1 & -1 & 1 & 0 & 0 & 0 \\ 0 & -b & 0 & 0 & 0 & 0 \\ -K & 0 & 0 & -am_1+K & a & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & -b & 0 \end{bmatrix},$$

$$B = \begin{bmatrix} -a(m_0-m_1) & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & -a(m_0-m_1) \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \tag{45}$$

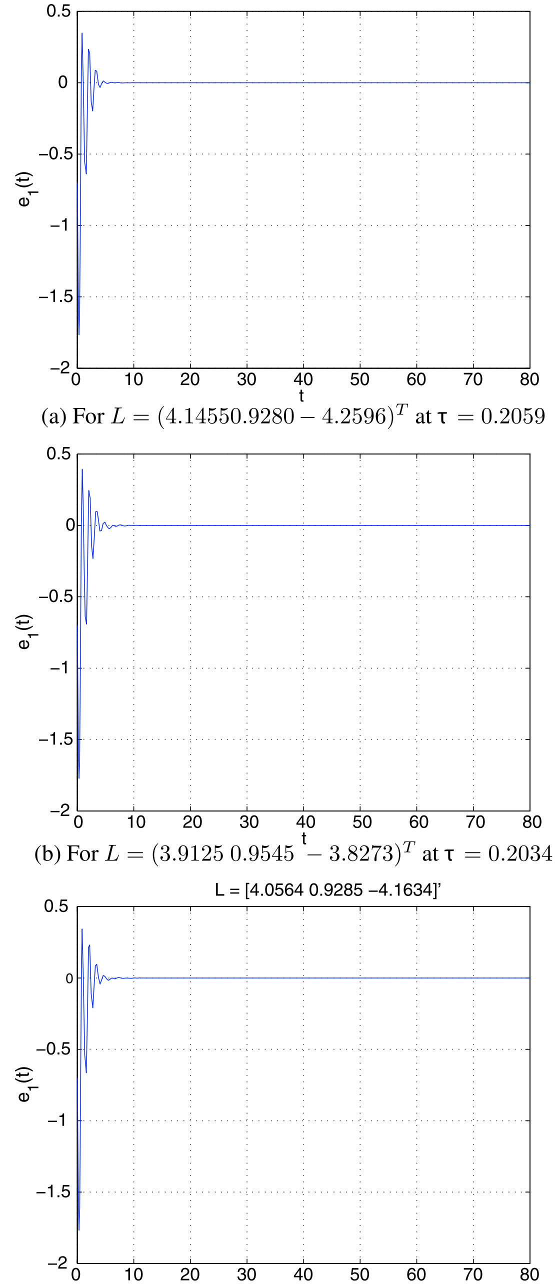
$$C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix},$$

$$\varphi(\mu) = \frac{1}{2}\big(|\mu + c| - |\mu - c|\big)$$

where $\varphi(\mu)$ belongs to sector [0, 1]. The gain matrix of the feedback controller is shown in [9]:

$$L = \begin{bmatrix} 7.6909 & 2.1313 & -3.9865 & -0.3491 & 0.1811 & -0.5180 \\ -1.0520 & 0.0835 & 0.3455 & 8.0879 & 1.8021 & -4.8256 \end{bmatrix}. \tag{46}$$

Table 4 contains a comparison of the maximum allowable time delay $\bar{\tau}$ and shows that the proposed synchronization criterion given in Theorem 1 is less conservative than the ones in [9] and [19].



(a) For $L = (4.14550.9280 - 4.2596)^T$ at $\tau = 0.2059$

(b) For $L = (3.9125\ 0.9545\ - 3.8273)^T$ at $\tau = 0.2034$

(c) For $L = (4.0564\ 0.9285\ - 4.1634)^T$ at $\tau = 0.2064$

Fig. 2 Synchronization error trajectory for different gain matrices at the time delay found using Theorem 1

5 Conclusion

In this paper, the problem of analyzing effect on time delay for master–slave synchronization represented by Lur'e systems with sector and slope re-

Table 4 The maximum allowable time delay $\bar{\tau}$ for the hyperchaotic system

	$\bar{\tau}$
Yalcin et al. [9]	0.038
Han [19]	0.1208
Theorem 1	**0.1358**

stricted nonlinearity have been addressed. A new synchronization criterion was presented, which is a sufficient condition of the error dynamic system for the given time delay. A convex representation for the nonlinear function of the Lur'e system was derived, and then, employed to equality constraints so that a less conservative criterion was obtained by utilizing the Finsler lemma. The Lyapunov–Krasovskii functional based on the delay discretization approach has been also used for the criterion and it can be easily extended to multiple discretized time delays. By two numerical examples, we have demonstrated the effectiveness of the proposed synchronization criterion.

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