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Bursting synchronization of Hind–Rose system based on a single controller

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Abstract Bursting synchronization of Hind–Rose system is investigated. Two schemes with only a single controller are proposed to synchronize Hind–Rose chaotic system via the back-stepping method. Especially in the second scheme, only one state variable is contained in the controller. Based on Lyapunov stability theory, the sufficient conditions for synchronization are obtained analytically in both cases. Finally, numerical simulations are provided to show the effectiveness of the developed methods.

Keywords Synchronization · Hind–Rose system · Chaotic bursting · Lyapunov function

1 Introduction

Over the last three decades, chaos synchronization has become a popular research topic arousing interest of physical scientists and electrical engineers ([1–7], and the references therein). Synchronization strategies have great potential applications in several areas, such as secure communication, biological oscillations and animal gaits. Synchronization means that $\lim_{t\to\infty}\|x(t)-y(t)\|=0$, where $x(t)$ and $y(t)$ are the states of the master and slave systems, respectively. It has been shown that the synchronization behavior can be induced either by coupling the systems or by forcing them.

Recently, synchronous activity among neurons or neuronal ensembles has attracted considerable interest for it is an important phenomenon observed in many regions of the brain in sensory systems and in other neural networks [8–11]. It is known that bursts could provide a more reliable mode of information transfer [12]. Therefore, how to effectively synchronize two neuronal systems with chaotic bursting states may be an important problem for theoretical research and potential practical application in secure communications. As chaotic bursting is a characteristic of neurons, a lot of studies have been carried out on synchronization of neuron bursting ([13–17], and the references therein). In this letter, the synchronization of Hind–Rose neuronal systems is addressed by employing back-stepping procedure [18]. Based on Lyapunov stability theory, two schemes which need only a single controller are proposed and the sufficient conditions for the synchronization have been obtained analytically. Finally, numerical simulations are employed to verify the effectiveness of the proposed scheme.

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2 Systems description

The minimal model of bursting behavior in real neurons requires three variables and is of the form of the Hind–Rose system [8] given by

$$\begin{aligned}\dot{x}_1 &= ax_1^2 - bx_1^3 - x_2 - x_3 + I_{\mathrm{ext}},\\ \dot{x}_2 &= dx_1^2 - c - x_2,\\ \dot{x}_3 &= r\big(S(x_1+k) - x_3\big),\end{aligned} \tag{1}$$

where x_1, x_2, x_3 are state variables, and a, b, c, d, r, S, k, I_{ext} are real constants.

The Hind–Rose system is a slow-fast system. Slow oscillation of x_3 drives the fast subsystem (x_1, x_2) through periods of oscillatory and quiescent behavior. The model (1) may describe regular bursting or chaotic bursting for certain domains of the parameters. If the parameters are taken as $a = 3.0$, $b = 1.0$, $c = 1.0$, $d = 5.0$, $r = 0.006$, $S = 4.0$, $k = 1.6$, system (1) is regular bursting for $I_{\mathrm{ext}} = 2.0$ and chaotic bursting for $I_{\mathrm{ext}} = 3.0$, respectively (see Fig. 1). It may be useful because a chaotic bursting system may lead to more secure communications.

3 Bursting synchronization of Hind–Rose system

In this section, we propose a systematic design procedure to synchronize bursting system based on backstepping procedure. This method needs only a single controller to realize synchronization. The aim is to design a controller u_1 such that the controlled Hind–Rose chaotic system

$$\begin{aligned}\dot{y}_1 &= ay_1^2 - by_1^3 - y_2 - y_3 + I_{\mathrm{ext}} + u_1,\\ \dot{y}_2 &= dy_1^2 - c - y_2,\\ \dot{y}_3 &= r\big(S(y_1+k) - y_3\big)\end{aligned} \tag{2}$$

is synchronous with Hind–Rose chaotic system (1). To this end, we let the error variables be $e_1 = y_1 - x_1$, $e_2 = y_2 - x_2$, $e_3 = y_3 - x_3$, and choose Lyapunov function as

$$V = \frac{1}{2}\big(e_1^2 + e_2^2 + e_3^2\big). \tag{3}$$

The time derivative of Lyapunov function V along the solutions of (1) and (2) is

$$\begin{aligned}\dot{V} &= e_1\dot{e}_1 + e_2\dot{e}_2 + e_3\dot{e}_3\\ &= \big[a(y_1+x_1)e_1 - b\big(y_1^2 + y_1x_1 + x_1^2\big)e_1\\ &\quad - e_2 - e_3 + u_1\big]e_1\\ &\quad + \big[d(y_1+x_1)e_1 - e_2\big]e_2 + r(Se_1 - e_3)e_3.\end{aligned} \tag{4}$$

Therefore, we have the following theorem.

Theorem 1 *If we design the controller u_1 as*

$$\begin{aligned}u_1 &= b\big(x_1^2 + x_1y_1 + y_1^2\big)e_1 - (x_1+y_1)(ae_1 + de_2)\\ &\quad - e_1 + e_2 - (rS - 1)e_3,\end{aligned} \tag{5}$$

then the controlled Hind–Rose chaotic system (2) is globally asymptotically synchronous with Hind–Rose chaotic system (1).

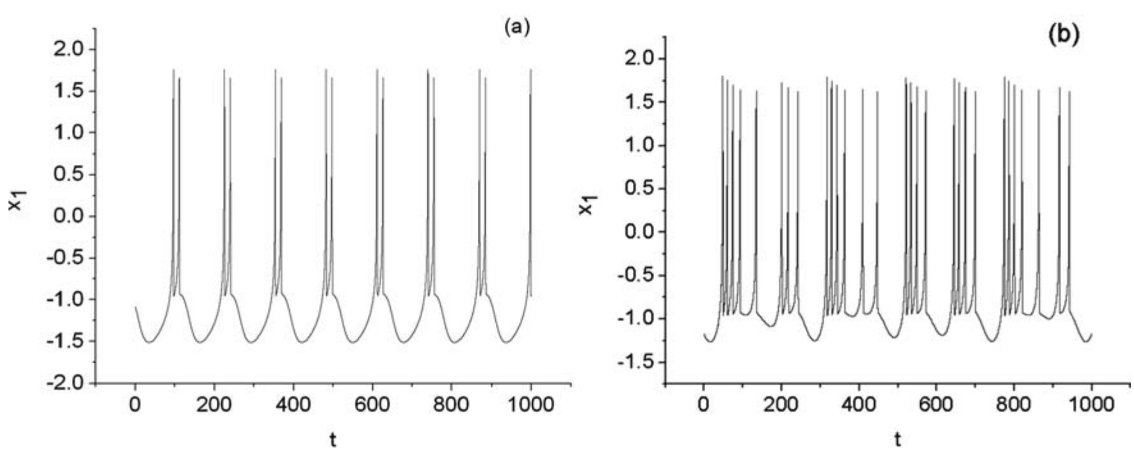

Fig. 1 **a** Regular bursting $I_{\mathrm{ext}} = 2.0$, **b** chaotic bursting $I_{\mathrm{ext}} = 3.0$

Proof Substituting (5) into (4), we get

$$\dot{V}(t) = -e_1^2 - e_2^2 - re_3^2 \leq 0. \tag{6}$$

Therefore, the states x_1, x_2, x_3 of the drive system (1) and the states y_1, y_2, y_3 of the response system (2) can be synchronized asymptotically and globally. This completes the proof. □

Obviously, the controller u_1 in (5) must have access to all the system state variables in order to achieve synchronization. But in practical problems, such as in using chaos synchronization for the communication purpose, only a subset of system state variables (usually only one system state variable) is known. The synchronization schemes based on only one system state variable are simple, efficient and easy to implement in practical applications. So we give the following synchronization scheme based on a single state variable.

Theorem 2 *If the controller is selected as*

$$U = -g(y_1 - x_1), \tag{7}$$

where g is a sufficiently large feedback gain, then the following controlled Hind–Rose chaotic system

$$\begin{aligned} \dot{y}_1 &= ay_1^2 - by_1^3 - y_2 - y_3 + I_{\text{ext}} + U, \\ \dot{y}_2 &= dy_1^2 - c - y_2, \\ \dot{y}_3 &= r\big(S(y_1 + k) - y_3\big) \end{aligned} \tag{8}$$

is synchronous with Hind–Rose chaotic system (1).

Proof Let the error variables be $e_1 = y_1 - x_1$, $e_2 = y_2 - x_2$, $e_3 = y_3 - x_3$; we can get the dynamic error system from (1) and (8) as follows:

$$\begin{aligned} \dot{e}_1 &= a(y_1 + x_1)e_1 - b\big(y_1^2 + y_1x_1 + x_1^2\big)e_1 \\ &\quad - e_2 - e_3 - ge_1, \\ \dot{e}_2 &= d(y_1 + x_1)e_1 - e_2, \\ \dot{e}_3 &= r(Se_1 - e_3). \end{aligned} \tag{9}$$

Consider the following Lyapunov function:

$$V = \frac{1}{2}\big(e_1^2 + e_2^2 + \theta e_3^2\big), \tag{10}$$

where $\theta > 0$. The time derivative of V is

$$\begin{aligned} \dot{V} &= e_1\dot{e}_1 + e_2\dot{e}_2 + \theta e_3\dot{e}_3 \\ &= e_1\big[a(x_1 + y_1)e_1 - b\big(x_1^2 + x_1y_1 + y_1^2\big)e_1 \\ &\quad - e_2 - e_3 - ge_1\big] + e_2\big[d(x_1 + y_1)e_1 - e_2\big] \\ &\quad + e_3\theta r(Se_1 - e_3). \end{aligned} \tag{11}$$

Since a chaotic system has bounded trajectories, there exists a positive constant M, such that $|x_1| < M$ and $|y_1| < M$, thus

$$\begin{aligned} \dot{V} &\leq \big((2aM + 3bM^2) - g\big)e_1^2 + |e_1||e_2| + |e_1||e_3| \\ &\quad + 2dM|e_1||e_2| - e_2^2 + rS\theta|e_1||e_3| - r\theta e_3^2 \\ &= -\big(|e_1|, |e_2|, |e_3|\big)P\big(|e_1|, |e_2|, |e_3|\big)^{\mathrm{T}}, \end{aligned} \tag{12}$$

where

$$P = \begin{pmatrix} g - (2a + 3bM)M & -\frac{1}{2}(1 + 2dM) & -\frac{1}{2}(\theta rS + 1) \\ -\frac{1}{2}(1 + 2dM) & 1 & 0 \\ -\frac{1}{2}(\theta rS + 1) & 0 & \theta r \end{pmatrix}. \tag{13}$$

Obviously, to ensure that the origin of error system (9) is asymptotically stable, the matrix P should be positive definite. This is the case if the following two inequalities hold:

$$g - (2a + 3bM)M - \frac{1}{4}(1 + 2dM)^2 > 0, \tag{14}$$

$$\begin{aligned} &r\theta\left(g - (2a + 3bM)M - \frac{1}{4}(1 + 2dM)^2\right) \\ &\quad - \frac{1}{4}(r\theta S + 1)^2 > 0. \end{aligned} \tag{15}$$

Thus, if $g > (2a + 3bM)M + \frac{1}{4}(1 + 2dM)^2 + \frac{1}{4\theta r}(r\theta S + 1)^2$, then the matrix P is positive definite, and $\dot{V}$ is negative definite. Together with the LaSalle's invariant theorem [19], we know that the origin of error system (9) is asymptotically stable. Therefore, the controlled Hind–Rose chaotic system (8) is synchronous with Hind–Rose chaotic system (1). □

4 Numerical simulations

In order to demonstrate and verify the performance of the proposed methods, some numerical simulations are presented in this section. In the simulations, the system parameters are chosen to be $a = 3.0$, $b = 1.0$, $c = 1.0$, $d = 5.0$, $r = 0.006$, $S = 4.0$, $k = 1.6$ and $I_{\text{ext}} = 3.0$ with which Hind–Rose system is a chaotic bursting. The initial conditions of the master system and slave system are set to be $x_1(0) = 0.1$, $x_2(0) = 0.9$, $x_3(0) = 0.8$ and $y_1(0) = 0.3$, $y_2(0) = 0.2$, $y_3(0) = 0.9$, respectively. The simulation results are illustrated in Figs. 2–4. Figure 2a–c shows the dynamics errors of the drive system (1) and the response system (2) without controller. Figures 3 and 4 show the evolutions of the dynamics errors with controller (5) and controller (7) (where $g = 13.0$), respectively.

5 Conclusion

We have studied the synchronization of Hind–Rose system when it is a chaotic bursting, and proposed two schemes to synchronize two identical Hind–Rose chaotic systems. Both schemes need only one single controller to realize synchronization via back-stepping

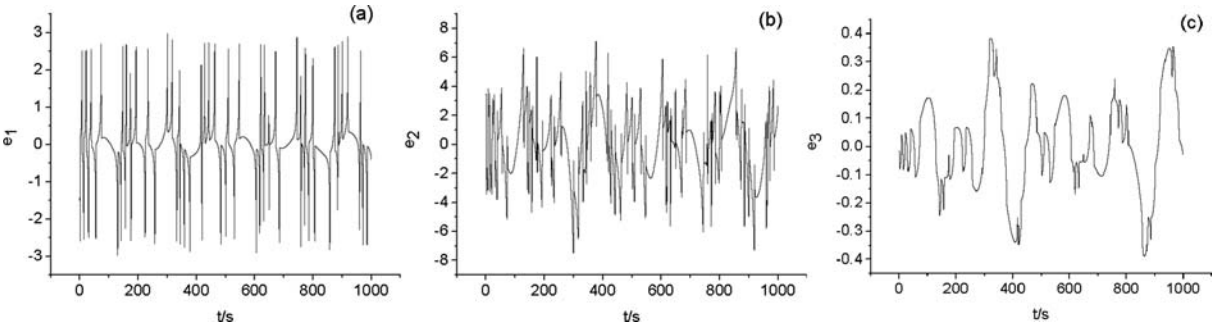

Fig. 2 Error in two Hind–Rose chaotic systems without controller

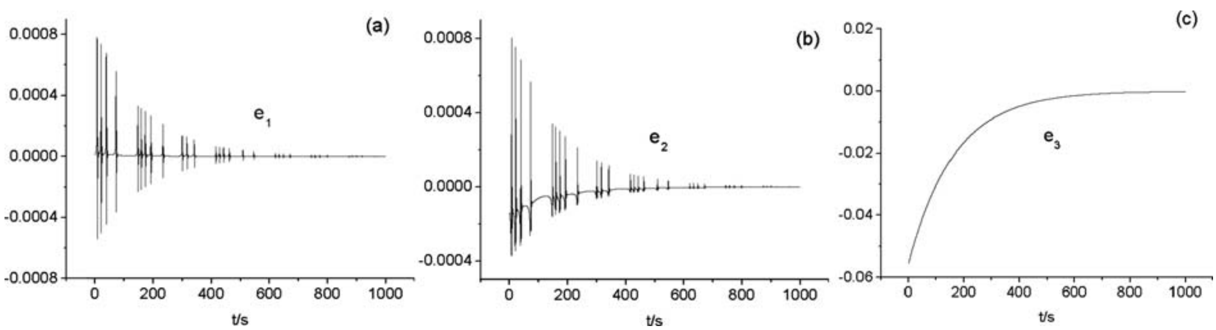

Fig. 3 Synchronization error in two Hind–Rose chaotic systems with the first scheme

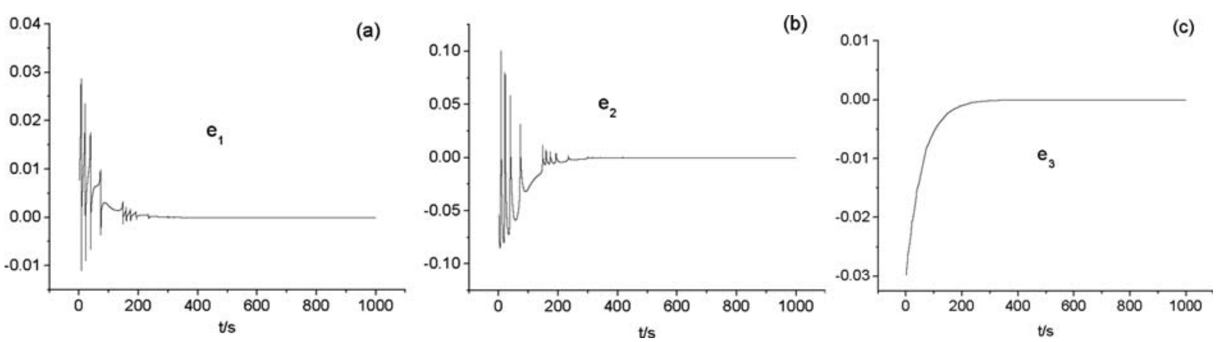

Fig. 4 Synchronization error in two Hind–Rose chaotic systems with the second scheme, the feedback gain $g = 13.0$

method. Especially in the second scheme, only one state variable is contained in controller, which is of important significance in synchronization. Furthermore, numerical simulations are provided to show the effectiveness of the developed methods.

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