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Saturated control design for linear differential inclusions subject to disturbance

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Abstract In this paper, saturated control design method is presented for robust stabilization of linear differential inclusions subject to disturbance. Convex hull quadratic Lyapunov functions are used to construct nonlinear state feedback laws. By the state feedbacks, stabilization, disturbance rejection with minimal reachable set and least L_2 gain are achieved simultaneously. Finally, the effectiveness of the proposed scheme is illustrated by a simulative example.

Keywords Linear differential inclusions · Saturated control · Lyapunov functions · Robust stability · Disturbance rejection

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1 Introduction

Because of the engineering significance and the theoretical challenges, the linear differential inclusion (LDI) has attracted tremendous attention in recent years. The reason is that the system described by an LDI can possess strong nonlinearity and time-varying uncertainty [1, 2]. Then it is natural that the control of LDIs has become a hot topic in the studies of control theory. Many authors have investigated LDIs and established numerous meaningful results, for instance [3–5]. For systems with time-varying uncertainties and those described by LDIs, it is now accepted that nonlinear control can work better than linear control [6]. In such an investigation, the convex hull quadratic Lyapunov function firstly presented in [7] is a powerful tool in the analysis of LDIs and saturated linear systems. A nonlinear control design method for LDIs via the convex hull quadratic Lyapunov function was presented in [8]. A pair of conjugate convex hull Lyapunov functions have demonstrated great potential in stability and performance analysis of LDIs [9]. Stability and performance for saturated systems via the convex hull quadratic Lyapunov function were researched in [10, 11]. In this paper, a saturated control design method is presented for robust stabilization of LDIs subject to disturbance.

This paper is organized as follows. Section 2 gives the preliminaries of the paper, which includes the description of the problem and several necessary lemmas. In Sect. 3, the convex hull quadratic Lyapunov functions are used for the construction of nonlinear state feedback laws. By the state feedbacks, stabilization, disturbance rejection with minimal reachable set and least L_2 gain are achieved simultaneously. Finally, the effectiveness of the proposed method is illustrated by a simulative example.

2 The system description and preliminaries

Consider the following linear differential inclusion (LDI):

$$\begin{bmatrix} \dot{x} \\ y \end{bmatrix} \in \mathrm{co}\left\{ \begin{bmatrix} A_i x + B_i \sigma(u) + T_i \omega \\ C_i x + D_i \omega \end{bmatrix}, \; i = 1, 2, \ldots, N \right\} \tag{1}$$

where $x \in R^n$, $u \in R^m$ are the state and the input, respectively, and $\sigma(.)$ is a standard saturation function with the saturation levels given by a vector $\bar{u} \in R^m$, $\bar{u}_l > 0, l = 1, 2, \ldots, m$. In particular,

$$\sigma(u_l) = \begin{cases} \bar{u}_l, & u_l \geq \bar{u}_l, \\ u_l, & u_l \in [-\bar{u}_l, \bar{u}_l], \\ -\bar{u}_l, & u_l \leq -\bar{u}_l, \end{cases}$$

$$\sigma(u) = \begin{bmatrix} \sigma(u_1) \\ \vdots \\ \sigma(u_m) \end{bmatrix}.$$

Here we have slightly abused the notation by using σ to denote both the scalar valued and the vector valued saturation functions. $\omega \in R^r$ is the disturbance and $y \in R^q$ is the output. A_i, B_i, T_i, C_i and D_i are the given real matrices of compatible dimensions.

For a positive-definite (semidefinite) matrix P, it is denoted as $P > 0$ ($P \geq 0$). When we say positive-definite (semidefinite), it is implied that the matrix is symmetric. Let $P \in R^{n\times n}$, $P > 0$, and a $\rho \in (0, \infty)$. Then denote a subset of R^n as follows:

$$\varepsilon(P, \rho) = \left\{ x \in R^n : x^T P x \leq \rho \right\}.$$

If $V(x) = x^T P x$, a level set of $V(.)$, denoted $L_V(\rho)$, is defined as

$$L_V(\rho) = \left\{ x \in R^n : V(x) \leq \rho \right\} = \varepsilon(P, \rho).$$

Let $H \in R^{m\times n}$, denote the lth row of H by h_l^T. Define

$$L(H) = \left\{ x \in R^n : |Hx|_\infty \leq 1 \right\}$$

where $|Hx|_\infty = \max_l |h_l^T x|$.

Let G be the set of $m \times m$ diagonal matrices whose diagonal elements are either 1 or 0. There are 2^m elements in G. Suppose that each element of G is labeled as G_η, i.e., $G = \{G_\eta, \eta = 1, 2, \ldots, 2^m\}$. If G_η belongs to G, then denote $G_\eta^- = I - G_\eta$. Obviously, G_η^- is also an element of G.

The convex hull quadratic function is constructed from a family of positive-definite matrices. Let $Q_j \in R^{n\times n}$, $Q_j = Q_j^T > 0$, $j = 1, 2, \ldots, J$, and

$$S^J = \left\{ s = [s_1, s_2, \ldots, s_J] : s_1 + s_2 + \cdots + s_J = 1, s_j \geq 0 \right\}.$$

Then the convex hull quadratic function is defined as

$$V_c(x) = \min_{s\in S^J} x^T \left(\sum_{j=1}^J s_j Q_j \right)^{-1} x. \tag{2}$$

It is obvious that $V_c(x)$ is a positive-definite function.

From the definition of $V_c(x)$, we have

$$V_c(x) = \min\left\{ \alpha : \alpha \geq x^T \left(\sum_{j=1}^J s_j Q_j \right)^{-1} x, s_j \in S^J \right\}.$$

By the Schur complement, $V_c(x)$ and the optimal value of s can be computed by solving a linear matrix inequality constraint

$$\begin{aligned} V_c(x) = &\min_{s_1,\ldots,s_j} \alpha \\ \text{s.t.} \quad &\begin{bmatrix} \alpha & x^T \\ x & \sum_{j=1}^N s_j Q_j \end{bmatrix} \geq 0, \\ &\sum_{j=1}^N s_j = 1, \quad s_j \geq 0 \end{aligned}$$

which is an optimization problem and can be easily solved with the techniques presented in [1].

Define a function $s^*(x)$ as follows:

$$s^*(x) = \arg\min_{s\in S^J} x^T \left(\sum_{j=1}^J s_j Q_j \right)^{-1} x. \tag{3}$$

We see that for a given x the optimal value of s is $s^*(x)$ such that $V_c(x) = x^T (\sum_{j=1}^J s_j^* Q_j)^{-1} x$. Generally,

s^* is uniquely determined by x and is a continuous function of x except for some degenerated cases. For example, this may happen if some Q_j can be expressed as the convex combination of other matrices in the set.

For a compact convex set L, a point x on the boundary of L (denoted as ∂L) is called an extreme point if it cannot be represented as a convex combination of any other points in L. A compact convex set is completely determined by its extreme points.

The following applies the definitions established in [7]. We begin with characterizing the set of extreme points of $L_{V_c}(\rho)$. It holds that

$$L_{V_c}(\rho) = \mathrm{co}\{\varepsilon(Q_j^{-1}, \rho), j = 1, 2, \ldots, J\},$$

an extreme point must be on the boundaries of both $L_{V_c}(\rho)$ and $\varepsilon(Q_j^{-1}, \rho)$ for some $j = 1, 2, \ldots, J$. Denote

$$\begin{aligned}\sqrt{\rho}E_k &= \partial L_{V_c}(\rho) \cap \partial\varepsilon(Q_k^{-1}, \rho) \\ &= \{x : V_c(x) = x^T Q_k^{-1} x = \rho\}.\end{aligned}$$

Then $\bigcup_{k=1}^{J} \sqrt{\rho}E_k$ contains all the extreme points of $L_{V_c}(\rho)$.

The following lemmas are verified in [7, 8].

Lemma 1 *For each* $\rho > 0, k = 1, 2, \ldots, J$, *then*

$$\begin{aligned}\sqrt{\rho}E_k = \{&x \in \partial L_{V_c}(\rho) : x^T Q_k^{-1}(Q_j - Q_k)Q_k^{-1}x \le 0, \\ &j = 1, 2, \ldots, J\}\end{aligned}$$

holds.

Lemma 2 *Let* $x \in R^n$. *For simplicity and without loss generality, assume that* $s_k^*(x) > 0$ *for* $k = 1, 2, \ldots, J_0$ *and* $s_k^*(x) = 0$ *for* $k = J_0 + 1, \ldots, J$. *Denote*

$$Q(s^*) = \sum_{k=1}^{J_0} s_k^* Q_k,$$

$$x_k = Q_k Q(s^*)^{-1} x, \quad k = 1, 2, \ldots, J_0.$$

Then

$$V_c(x_k) = V_c(x) = x_k^T Q_k^{-1} x_k$$

and

$$x_k \in (V_c(x))^{1/2} E_k,$$

for $k = 1, 2, \ldots, J_0$. *Moreover,* $x = \sum_{k=1}^{J_0} s_k^* x_k$, *and for* $k = 1, 2, \ldots, J_0$,

$$\nabla V_c(x) = \nabla V_c(x_k) = 2Q_k^{-1}x_k = 2Q(s^*)^{-1}x \tag{4}$$

where $\nabla V_c(x)$ *denotes the gradient of* V_c *at* x.

Lemma 3 *For a matrix* $H \in R^{m\times n}$, *and a matrix* $P > 0$, *and* $\rho > 0$, *then* $\varepsilon(P, \rho) \subset L(H)$, *if and only if*

$$\begin{bmatrix} 1 & \rho z_l^T \\ \rho z_l & \rho P^{-1} \end{bmatrix} \ge 0$$

where z_l^T *is the* l*th row of* HP^{-1}.

3 Main results

This section begins with the problem of stabilization. For this objective, we only consider the state inclusion without disturbance, i.e.,

$$\dot{x} \in \mathrm{co}\{A_i x + B_i\sigma(u), i = 1, 2, \ldots, N\}. \tag{5}$$

Let

$$\bar{U} = \mathrm{diag}\{\bar{u}_1, \ldots, \bar{u}_m\}$$

where $\bar{u}_l > 0$ for every $l = 1, 2, \ldots, m$ is the saturation level for the lth component of $\sigma(\cdot)$.

Theorem 1 *Let* $Q_k \in R^{n\times n}, k = 1, 2, \ldots, J$ *be* J *positive-definite matrices, and* $V_c(x)$ *be the function defined in* (2). *Denote* $P_k = Q_k^{-1}$. *For every* $\rho > 0$, *there is an ellipsoid* $\varepsilon(P_k, \rho)$. *Let there exist* $2J$ *matrices* $F_k, H_k \in R^{m\times n}, k = 1, 2, \ldots, J$, *and* $N \times J^2$ *nonnegative real numbers* $\lambda_{ijk} \ge 0$, $i = 1, 2, \ldots, N$; $j, k = 1, 2, \ldots, J$, *such that* $\varepsilon(P_k, \rho) \subset L(\bar{U}^{-1}H_k)$ *and*

$$\begin{aligned}&Q_k A_i^T + A_i Q_k + (G_\eta F_k + G_\eta^- H_k Q_k)^T B_i^T \\ &\quad + B_i(G_\eta F_k + G_\eta^- H_k Q_k) \\ &\le \sum_{j=1}^{J} \lambda_{ijk}(Q_j - Q_k), \quad \eta = 1, 2, \ldots, 2^m.\end{aligned} \tag{6}$$

Denote

$$F(s^*) = \sum_{k=1}^{J} s_k^* F_k, \qquad Q(s^*) = \sum_{k=1}^{J} s_k^* Q_k,$$
$$H(s^*) = \sum_{k=1}^{J} s_k^* H_k Q_k Q(s^*)^{-1} \tag{7}$$

where $s^*(x)$ is the function defined in (3). Then for each $x \in L_{V_c}(\rho)$,

$$\max\{\nabla V_c(x)^T (A_i x + B_i \sigma(F(s^*)Q(s^*)^{-1}x)) : i = 1, 2, \ldots, N\} \le 0. \tag{8}$$

It implies that for $x \in L_{V_c}(\rho)$, the saturation control

$$\sigma(u) = \sigma(F(s^*)Q(s^*)^{-1}x) \tag{9}$$

stabilizes the system (5). Moreover, if $s^*(x)$ is continuous then the function defined in (9) is continuous too.

Proof We first prove that (8) is satisfied for all extreme points of $L_{V_c}(\rho)$, in particular, for all $x \in \sqrt{\rho}E_k$, $k = 1, 2, \ldots, J$.

Let $x \in \sqrt{\rho}E_k$. Then $V_c(x) = x^T Q_k^{-1} x = \rho$ and $s^*(x)$ is a vector whose kth elements is 1 and the rest are zeros. Hence $F(s^*)Q(s^*)^{-1}x = F_k Q_k^{-1} x$, $H(s^*) = \sum_{k=1}^{J} s_k^* H_k Q_k Q(s^*)^{-1} = H_k$ and $\nabla V_c(x) = 2Q_k^{-1}x$.

For each $i = 1, 2, \ldots, N$, $x \in \sqrt{\rho}E_k$, then

$$\begin{aligned}\nabla V_c(x)^T (A_i x + B_i u) &= 2x^T P_k A_i x + 2x^T P_k B_i \sigma(F(s^*)Q(s^*)^{-1}x) \\ &= 2x^T P_k A_i x + 2x^T P_k B_i \sigma(F_k Q_k^{-1} x).\end{aligned} \tag{10}$$

Denote the lth column of B_i by b_{il} and the lth row of $F_k Q_k^{-1}$ by f_{lk}^T, respectively. From (10), we have

$$\nabla V_c(x)^T (A_i x + B_i u) = 2x^T P_k A_i x + \sum_{l=1}^{m} 2x^T P_k b_{il} \sigma(f_{lk}^T x). \tag{11}$$

Let h_{lk}^T be the lth row of H_k. For each $x \in \varepsilon(P_k, \rho)$, consider $2x^T P_k b_{il} \sigma(f_{lk}^T x)$ by the following four cases:

(1) If $x^T P_k b_{il} \ge 0$ and $f_{lk}^T x \le -\bar{u}_l$, then

$$\begin{aligned}2x^T P_k b_{il} \sigma(f_{lk}^T x) &= -2x^T P_k b_{il} \bar{u}_l \\ &\le 2x^T P_k b_{il} h_{lk}^T x.\end{aligned}$$

Here we note that $-u_l \le h_{lk}^T x$, for $\forall x \in \varepsilon(P_k, \rho) \subset L(\bar{U}^{-1} H_k)$.

(2) If $x^T P_k b_{il} \ge 0$ and $f_{lk}^T x \ge -\bar{u}_l$, let us consider it in two cases. If $x^T P_k b_{il} \ge 0$ and $-\bar{u}_l \le f_{lk}^T x \le \bar{u}_l$, then

$$2x^T P_k b_{il} \sigma(f_{lk}^T x) = 2x^T P_k b_{il} f_{lk}^T x;$$

and if $x^T P_k b_{il} \ge 0$ and $f_{lk}^T x \ge \bar{u}_l$, then

$$\begin{aligned}2x^T P_k b_{il} \sigma(f_{lk}^T x) &= 2x^T P_k b_{il} u_l \\ &\le 2x^T P_k b_{il} f_{lk}^T x.\end{aligned}$$

So, if $x^T P_k b_{il} \ge 0$ and $f_{lk}^T x \ge -\bar{u}_l$, then

$$2x^T P_k b_{il} \sigma(f_{lk}^T x) \le 2x^T P_k b_{il} f_{lk}^T x.$$

(3) If $x^T P_k b_{il} \le 0$ and $f_{lk}^T x \ge \bar{u}_l$, then

$$\begin{aligned}2x^T P_k b_{il} \sigma(f_{lk}^T x) &= 2x^T P_k b_{il} \bar{u}_l \\ &\le 2x^T P_k b_{il} h_{lk}^T x.\end{aligned}$$

Here we note that $h_{lk}^T x \le u_l$, for $\forall x \in \varepsilon(P_k, \rho) \subset L(\bar{U}^{-1} H_k)$.

(4) If $x^T P_k b_{il} \le 0$ and $f_{lk}^T x \le \bar{u}_l$, then

$$2x^T P_k b_{il} \sigma(f_{lk}^T x) = 2x^T P_k b_{il} f_{lk}^T x.$$

Summing up the above discussion, we have

$$2x^T P_k b_{il} \sigma(f_{lk}^T x) \le \max\{2x^T P_k b_{il} h_{lk}^T x, 2x^T P_k b_{il} f_{lk}^T x\} \tag{12}$$

for every $x \in \varepsilon(P_k, \rho)$. Now we associate every $x \in \varepsilon(P_k, \rho)$ with a vector $v(x) \in R^m$ as follows: if $2x^T P_k b_{il} h_{lk}^T x < 2x^T P_k b_{il} f_{lk}^T x$, then $v_{l\eta} = 1$; otherwise $v_{l\eta} = 0$, $\eta = 1, 2, \ldots, 2^m$. Then

$$x^T P_k b_{il} \sigma(f_{lk}^T x) \le v_{l\eta} x^T P_k b_{il} f_{lk}^T x + (1 - v_{l\eta}) x^T P_k b_{il} h_{lk}^T x. \tag{13}$$

If $x \in \sqrt{\rho}E_k$, then we have $x \in \varepsilon(P_k, \rho)$. In view of (11) and (13), it follows that for every $x \in \sqrt{\rho}E_k$,

$$\nabla V_c(x)^T (A_i x + B_i u)$$

$$\le 2x^T P_k A_i x + 2\sum_{l=1}^{m}\big(v_{l\eta} x^T P_k b_{il} f_{lk}^T x + (1-v_{l\eta}) x^T P_k b_{il} h_{lk}^T x\big)$$
$$= 2x^T P_k A_i x + 2x^T P_k \sum_{l=1}^{m} b_{il}\big(v_{l\eta} f_{lk}^T + (1-v_{l\eta}) h_{lk}^T\big) x$$
$$= 2x^T P_k A_i x + 2x^T P_k B_i\big(G_\eta F_k Q_k^{-1} + G_\eta^- H_k\big)x. \tag{14}$$

Multiplying (6) from left and right sides by Q_k^{-1} respectively, we have

$$\big[A_i + B_i\big(G_\eta F_k Q_k^{-1} + G_\eta^- H_k\big)\big]^T Q_k^{-1} + Q_k^{-1}\big[A_i + B_i\big(G_\eta F_k Q_k^{-1} + G_\eta^- H_k\big)\big] \le \sum_{j=1}^{J} \lambda_{ijk} Q_k^{-1}(Q_j - Q_k) Q_k^{-1},$$
$$i = 1, 2, \dots, N, \eta = 1, 2, \dots, 2^m.$$

By Lemma 1, for every $x \in \sqrt{\rho}E_k$, we obtain

$$x^T\big[A_i + B_i\big(G_\eta F_k Q_k^{-1} + G_\eta^- H_k\big)\big]^T Q_k^{-1} + Q_k^{-1}\big[A_i + B_i\big(G_\eta F_k Q_k^{-1} + G_\eta^- H_k\big)\big]x$$
$$= 2x^T P_k A_i x + 2x^T P_k B_i\big(G_\eta F_k Q_k^{-1} + G_\eta^- H_k\big)x$$
$$\le \sum_{j=1}^{J} \lambda_{ijk} x^T Q_k^{-1}(Q_j - Q_k) Q_k^{-1} x$$
$$\le 0, \quad i = 1, 2, \dots, N, \eta = 1, 2, \dots, 2^m. \tag{15}$$

By (14) and (15), it can be deduced for any $x \in \sqrt{\rho}E_k$,

$$\nabla V_c(x)^T (A_i x + B_i u) \le 0. \tag{16}$$

This implies that (8) is satisfied for all $x \in \sqrt{\rho}E_k$.

Next, let $x_0 \in \partial L_{V_c}(\rho)$ be an arbitrary point. Then $V_c(x_0) = \rho$. By Lemma 2, x_0 is a convex combination of a set of $x_k' s$, each of which belongs to a certain $\sqrt{\rho}E_k$. For simplicity, assume that $s_k^*(x_0) > 0$ for $k = 1, 2, \dots, J_0$ and $s_k^*(x_0) = 0$ for $k > J_0$. Then $x_0 = \sum_{k=1}^{J_0} s_k^* x_k$. Recalling Lemma 2, we have $\nabla V_c(x_0) = 2Q(s^*)^{-1}x_0$ and $Q(s^*)^{-1}x_0 = Q_k^{-1}x_k, k = 1, 2, \dots, J_0$; furthermore,

$$F(s^*)Q(s^*)^{-1}x_0 = \sum_{k=1}^{J_0} s_k^* F_k Q_k^{-1} x_k,$$
$$H(s^*)x_0 = \sum_{k=1}^{J_0} s_k^* H_k Q_k Q(s^*)^{-1} x_0 = \sum_{k=1}^{J_0} s_k^* H_k x_k. \tag{17}$$

Note that $L_{V_c}(\rho) = \varepsilon(Q(s^*)^{-1}, \rho)$. Applying convex combination to the $\varepsilon(P_k, \rho) \subset L(\bar{U}^{-1}H_k)$, now we prove $\varepsilon(Q(s^*)^{-1}, \rho) \subset L(H(s^*))$. Let $Z_k = H_k Q_k, k = 1, 2, \dots, J_0$. By the condition $\varepsilon(P_k, \rho) \subset L(\bar{U}^{-1}H_k)$, and Lemma 3, we have

$$\begin{bmatrix} 1 & \rho \bar{u}_l^{-1} z_{lk}^T \\ \rho \bar{u}_l^{-1} z_{lk} & \rho Q_k \end{bmatrix} \ge 0,$$
$$k = 1, 2, \dots, J_0,\ l = 1, 2, \dots, m$$

where z_{lk}^T is the lth row of the matrix Z_k.

Let $Z = s_1^* Z_1 + s_2^* Z_2 + \cdots + s_{J_0}^* Z_{J_0}$, and z_l^T be the lth row of Z. Then by the convexity of $Q(s^*)$, we have

$$\begin{bmatrix} 1 & \rho \bar{u}_l^{-1} z_l^T \\ \rho \bar{u}_l^{-1} z_l & \rho Q(s^*) \end{bmatrix} \ge 0, \quad l = 1, 2, \dots, m. \tag{18}$$

In view of $H(s^*) = \sum_{k=1}^{J_0} s_k^* H_k Q_k Q(s^*)^{-1} = Z Q(s^*)^{-1}$, denote the lth row of $H(s^*)$ by $(h_l^*)^T$, then (18) can be rewritten as

$$\begin{bmatrix} 1 & \rho \bar{u}_l^{-1} (h_l^*)^T Q(s^*) \\ \rho \bar{u}_l^{-1} Q(s^*) h_l^* & \rho Q(s^*) \end{bmatrix} \ge 0,$$
$$l = 1, 2, \dots, m. \tag{19}$$

Thus $\varepsilon(Q(s^*)^{-1}, \rho) \subset L(\bar{U}^{-1}H(s^*))$.

Using Lemma 2 and the arguments as in (14), it can be deduced that for any $x_0 \in \partial L_{V_c}(\rho)$,

$$\nabla V_c(x_0)^T \big(A_i x_0 + B_i \sigma\big(F(s^*)Q(s^*)^{-1}x_0\big)\big)$$
$$= 2x_0^T Q(s^*)^{-1}\big(A_i x_0 + B_i \sigma\big(F(s^*)Q(s^*)^{-1}x_0\big)\big)$$
$$\le 2x_0^T Q(s^*)^{-1}\big(A_i + B_i\big(G_\eta F(s^*)Q(s^*)^{-1} + G_\eta^- H(s^*)\big)x_0. \tag{20}$$

In view of (17) and (20), we have

$$V_c(x_0)^T \big(A_i x_0 + B_i \sigma\big(F(s^*)Q(s^*)^{-1}x_0\big)\big)$$
$$\le 2x_0^T Q(s^*)^{-1}\big(A_i + B_i\big(G_\eta F(s^*)Q(s^*)^{-1}$$

$$
\begin{aligned}
&+ G_\eta^- H(s^*)))x_0 \\
&= 2x_0^T Q(s^*)^{-1} \sum_{k=1}^{J_0} s_k^* (A_i + B_i (G_\eta F_k Q_k^{-1} \\
&\quad + G_\eta^- H_k))x_k \\
&= 2 \sum_{k=1}^{J_0} s_k^* x_k^T Q_k^{-1} (A_i + B_i (G_\eta F_k Q_k^{-1} \\
&\quad + G_\eta^- H_k))x_k. \qquad (21)
\end{aligned}
$$

Since $x_k \in \sqrt{\rho} E_k$, by (15) and (21), we have

$$\nabla V_c(x_0)^T (A_i x_0 + B_i \sigma (F(s^*) Q(s^*)^{-1} x_0)) \leq 0.$$

Note that x_0 is an arbitrary point in $\partial L_{V_c}(\rho)$, thus (8) is proved. Since $F(s^*)$ and $Q(s^*)$ are continuous in s^*, and $Q(s^*) > 0$, in addition, the saturated function $\sigma(\cdot)$ is continuous, the continuity of $u = \sigma(F(s^*)Q(s^*)^{-1}x)$ follows from that of $s^*(x)$. □

Let us consider the LDI (1) with the disturbances

$$\dot{x} \in \text{co}\{A_i x + B_i \sigma(u) + T_i \omega, i = 1, 2, \ldots, N\} \qquad (22)$$

where the norm of ω is defined by

$$\|\omega\|_2 = \left(\int_0^\infty \omega^T(t) \omega(t)\, dt \right)^{1/2}.$$

Let $\sigma(u) = \sigma(F(s^*)Q(s^*)^{-1}x)$. The control design objective is disturbance rejection, i.e., to keep the state close to the origin. The disturbance rejection performance can be characterized by reachable set which can be estimated with a level set of a certain Lyapunov function.

Theorem 2 *Let $Q_k \in R^{n\times n}, k = 1, 2, \ldots, J$ be J positive-definite matrices, and $V_c(x)$ be the function defined in (2). Denote $P_k = Q_k^{-1}$. For every $\rho > 0$, there is an ellipsoid $\varepsilon(P_k, \rho)$. If there exist $2J$ matrices $F_k, H_k \in R^{m\times n}, k = 1, 2, \ldots, J$, and $N \times J^2$ nonnegative real numbers $\lambda_{ijk} \geq 0$, $i = 1, 2, \ldots, N$; $j, k = 1, 2, \ldots, J$, such that $\varepsilon(P_k, \rho) \subset L(\bar{U}^{-1} H_k)$ and*

$$\begin{bmatrix} M_{ik} & T_i \\ T_i^T & -I \end{bmatrix} \leq 0, \quad \forall i, k, \qquad (23)$$

where

$$
\begin{aligned}
M_{ik} &= Q_k A_i^T + A_i Q_k + (G_\eta F_k + G_\eta^- H_k Q_k)^T B_i^T \\
&\quad + B_i (G_\eta F_k + G_\eta^- H_k Q_k) \\
&\quad - \sum_{j=1}^{J} \lambda_{ijk} (Q_j - Q_k), \\
&\eta = 1, 2, \ldots, 2^m, \qquad (24)
\end{aligned}
$$

then for all ω bounded by $\|\omega\|_2 \leq \sqrt{\rho}$ and with $x(0) = 0$, the solutions of system (22) under the saturated control $\sigma(u) = \sigma(F(s^*)Q(s^*)^{-1}x)$ satisfy $x(t) \in L_{V_c}(\rho)$ for all $t > 0$.

Proof Multiplying (23) on the left and the right sides by $\text{diag}\{P_k, I\}$, respectively, we have

$$\begin{bmatrix} P_k M_{ik} P_k & P_k T_i \\ T_i^T P_k & -I \end{bmatrix} \leq 0, \quad \forall i, k. \qquad (25)$$

This implies that for all $i = 1, 2, \ldots, N$; $k = 1, 2, \ldots, J$; $\eta = 1, 2, \ldots, 2^m$,

$$
\begin{aligned}
&2x^T P_k [A_i x + B_i (G_\eta F_k P_k + G_\eta^- H_k) x + T_i \omega] - \omega^T \omega \\
&\quad \leq \sum_{j=1}^{J} \lambda_{ijk} x^T P_k (Q_j - Q_k) P_k x, \\
&\quad \forall x \in R^n, \omega \in R^r. \qquad (26)
\end{aligned}
$$

Being similar to the proof of Theorem 1, we first verify

$$
\begin{aligned}
&\nabla V_c(x)^T (A_i x + B_i \sigma (F(s^*) Q(s^*)^{-1} x) + T_i \omega) \\
&\quad \leq \omega^T \omega \qquad (27)
\end{aligned}
$$

for every $x \in \sqrt{\rho} E_k$ by using (26). Then we extend the results to all other $x \in \partial L_{V_c}(\rho)$ by expressing it as a convex combination of $x_k \in \sqrt{\rho} E_k$, $k = 1, 2 \ldots, J_0$.

Since

$$
\begin{aligned}
&\dot{V}_c(x, \omega) \\
&\quad \leq \max\{(\nabla V_c(x))^T (A_i x + B_i \sigma (F(s^*) Q(s^*)^{-1} x) \\
&\qquad + T_i \omega) : i = 1, 2, \ldots, N\}, \qquad (28)
\end{aligned}
$$

where $\dot{V}_c(x, \omega)$ is the time derivative of V_c along with the trajectories of the closed-loop system, note that it depends on x and ω. By (27) and (28), we have

$$\dot{V}_c(x, \omega) \leq \omega^T \omega \qquad (29)$$

for all $x \in L_{V_c}(\rho)$ and $\omega \in R^r$. Now, suppose $x(0) = 0$ and $\|\omega\| \leq \sqrt{\rho}$. Then for any $t_0 > 0$, as long as

$x(t) \in L_{V_c}(\rho)$ for all $t \in (0, t_0)$, we have $V_c(x(t_0)) \leq \int_0^{t_0} \omega^T \omega \, dt \leq \rho$, i.e. $x(t_0) \in L_{V_c}(\rho)$. On the other hand, if there exists $t_0 > 0$ such that $V_c(x(t)) \leq \rho$ for all $t \in (0, t_0)$ and $V_c(x(t_0)) = \rho$ then we must have $\int_{t_0}^{\infty} \omega^T \omega \, dt = 0$ and $\dot{V}_c(x(t), \omega(t)) \leq 0$ for almost all $t \geq t_0$. Therefore, we conclude that $x(t) \in L_{V_c}(\rho)$ for all $t \geq 0$. □

We next address estimation of the L_2 gain.

Theorem 3 *Let $Q_k \in R^{n \times n}, k = 1, 2, \ldots, J$, be J positive-definite matrices, and $V_c(x)$ be the function defined in (2). Denote $P_k = Q_k^{-1}$. For every $\rho > 0$, there is an ellipsoid $\varepsilon(P_k, \rho)$. If there exist $2J$ matrices $F_k, H_k \in R^{m \times n}, k = 1, 2, \ldots, J$, and $N \times J^2$ nonnegative real numbers $\lambda_{ijk} \geq 0$, $i = 1, 2, \ldots, N$; $j, k = 1, 2, \ldots, J$, such that $\varepsilon(P_k, \rho) \subset L(\bar{U}^{-1} H_k)$ and*

$$\begin{bmatrix} M_{ik} & T_i & Q_k C_i^T \\ T_i^T & -I & D_i^T \\ C_i Q_k & D_i & -\delta^2 I \end{bmatrix} \leq 0, \quad \forall i, k, \tag{30}$$

where M_{ik} is given by (24), then for all ω bounded by $\|\omega\|_2 \leq \sqrt{\rho}$ and with $x(0) = 0$, the output of system (1) under the saturated control $\sigma(u) = \sigma(F(s^*) Q(s^*)^{-1} x)$, satisfies $\|y\|_2 \leq \delta \|\omega\|_2$.

Proof We will prove the theorem by showing that for all $x \in L_{V_c}(\rho)$, and $\omega \in R^r$, $\dot{V}_c(x, \omega) + (1/\delta^2) y^T y \leq \omega^T \omega$. Since (30) implies (23), by Theorem 2, we have $x(t) \in L_{V_c}(\rho)$ for all t and all $\|\omega\|_2 \leq \sqrt{\rho}$, and $x(0) = 0$. All the relationships established in the proof of Theorem 2 are true under the conditions of the current theorem. Multiplying (30) on the left and the right sides by $\operatorname{diag}\{P_k, I, I\}$, respectively, we have

$$\begin{bmatrix} P_k M_{ik} P_k & P_k T_i & C_i^T \\ T_i^T P_k & -I & D_i^T \\ C_i & D_i & -\delta^2 I \end{bmatrix} \leq 0, \quad \forall i, k. \tag{31}$$

By Schur complements, (31) is equivalent to

$$\begin{bmatrix} P_k M_{ik} P_k & P_k T_i \\ T_i^T P_k & -I \end{bmatrix} + \frac{1}{\delta^2} \begin{bmatrix} C_i^T \\ D_i^T \end{bmatrix} \begin{bmatrix} C_i & D_i \end{bmatrix} \leq 0. \tag{32}$$

Then (32) implies that for all $x \in R^n, \omega \in R^r$,

$$\begin{aligned} &2x^T P_k \left[A_i x + B_i \left(G_\eta F_k P_k + G_\eta^- H_k \right) x + T_i \omega \right] \\ &\quad - \omega^T \omega + \frac{1}{\delta^2} (C_i x + D_i \omega)^T (C_i x + D_i \omega) \\ &\leq \sum_{j=1}^{J} \lambda_{ijk} x^T P_k (Q_j - Q_k) P_k x, \\ &\eta = 1, 2, \ldots, 2^m. \end{aligned} \tag{33}$$

Being similar to the proof of Theorem 1, we first verify

$$\begin{aligned} &\nabla V_c(x)^T \left(A_i x + B_i \sigma \left(F(s^*) Q(s^*)^{-1} x \right) + T_i \omega \right) \\ &\quad + \frac{1}{\delta^2} y^T y \leq \omega^T \omega \end{aligned} \tag{34}$$

for every $x \in \sqrt{\rho} E_k$ by using (33). Then we extend the results to all other $x \in \partial L_{V_c}(\rho)$ by expressing it as a convex combination of $x_k \in \sqrt{\rho} E_k, k = 1, 2, \ldots, J_0$. Thus

$$\begin{aligned} &\dot{V}_c(x, \omega) + \frac{1}{\delta^2} y^T y - \omega^T \omega \\ &\quad \leq \max \Big\{ \left(\nabla V_c(x) \right)^T \left(A_i x + B_i \sigma \left(F(s^*) Q(s^*)^{-1} x \right) \right. \\ &\qquad \left. + T_i \omega \right) + \frac{1}{\delta^2} y^T y - \omega^T \omega : i = 1, 2, \ldots, N \Big\}. \end{aligned} \tag{35}$$

By (34) and (35),

$$\dot{V}_c(x, \omega) + \frac{1}{\delta^2} y^T y - \omega^T \omega \leq 0. \tag{36}$$

Since $x(0) = 0$, $x(t) \in L_{V_c}(\rho)$ for all t and all $\|\omega\|_2 \leq \sqrt{\rho}$, integrating both sides of (36), we have $\|y\|_2 \leq \delta \|\omega\|_2$. □

4 Example

Consider a second-order LDI

$$\dot{x} \in \operatorname{co}\{A_1 x + B_1 u, A_2 x + B_2 u\} \tag{37}$$

where

$$A_1 = \begin{bmatrix} 0.1 & -0.8 \\ 1 & 1.5 \end{bmatrix}, \qquad B_1 = \begin{bmatrix} 0.1 \\ -1 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 0 & -0.5 \\ 1 & 1.5 \end{bmatrix}, \qquad B_2 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}.$$

V_c is composed from

$$Q_1 = \begin{bmatrix} 0.3194 & 0.0978 \\ 0.0978 & 0.1597 \end{bmatrix},$$

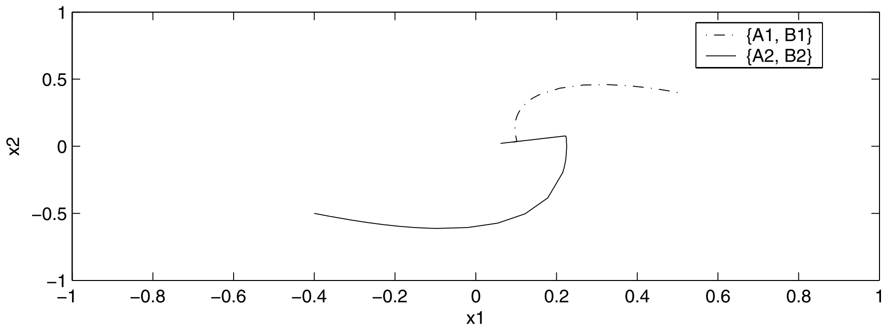

Fig. 1 State trajectories $x(t)$ of the two subsystems in (37) under state feedback control law (40) (the initial states are $[0.5, 0.4]^T, [-0.4, -0.5]^T$ respectively)

$$Q_2 = \begin{bmatrix} 0.2097 & 0.0825 \\ 0.0825 & 0.1650 \end{bmatrix}.$$

Let

$$P_1 = Q_1^{-1} = \begin{bmatrix} 3.8535 & -2.3598 \\ -2.3598 & 7.7069 \end{bmatrix},$$

$$P_2 = Q_2^{-1} = \begin{bmatrix} 5.9365 & -2.9682 \\ -2.9682 & 7.5447 \end{bmatrix}.$$

For $\rho = 1$, it yields ellipsoids $\varepsilon(P_1, 1)$ and $\varepsilon(P_2, 1)$. There exist $F_1 = [-0.14866.3489]$, $F_2 = [-0.29549.5012]$, $H_1 = [0.61671.8657]$, $H_2 = [-0.05372.3324]$ such that $\varepsilon(P_k, 1) \subset L(H_k), k = 1, 2$, and

$$Q_1 A_1^T + A_1 Q_1 + F_1^T B_1^T + B_1 F_1 \le 0.2123(Q_2 - Q_1),$$

$$Q_1 A_2^T + A_2 Q_1 + F_1^T B_2^T + B_2 F_1 \le 0.0148(Q_2 - Q_1),$$

$$Q_2 A_1^T + A_1 Q_2 + F_2^T B_1^T + B_1 F_2 \le 9.0349(Q_1 - Q_2),$$

$$Q_2 A_2^T + A_2 Q_2 + F_2^T B_2^T + B_2 F_2 \le 9.2147(Q_1 - Q_2),$$

$$Q_1 A_1^T + A_1 Q_1 + (H_1 Q_1)^T B_1^T + B_1 (H_1 Q_1) \le 0.2123(Q_2 - Q_1),$$

$$Q_1 A_2^T + A_2 Q_1 + (H_1 Q_1)^T B_2^T + B_2 (H_1 Q_1) \le 0.0148(Q_2 - Q_1),$$

$$Q_2 A_1^T + A_1 Q_2 + (H_2 Q_2)^T B_1^T + B_1 (H_2 Q_2) \le 9.0349(Q_1 - Q_2),$$

$$Q_2 A_2^T + A_2 Q_2 + (H_2 Q_2)^T B_2^T + B_2 (H_2 Q_2) \le 9.2147(Q_1 - Q_2). \tag{38}$$

So the condition (6) in Theorem 1 holds. For each $x \in L_{V_c}(1)$, let

$$F(s^*) = s_1^* F_1 + (1 - s_1^*) F_2, \; Q(s^*) = s_1^* Q_1 + (1 - s_1^*) Q_2 \tag{39}$$

where s_1^* is defined as

$$s^*(x) = \arg\min_{s_1 \ge 0} x^T \big(s_1 Q_1 + (1 - s_1) Q_2\big)^{-1} x.$$

Then the closed-loop system under

$$\sigma(u) = \sigma\big(F(s^*) Q(s^*)^{-1} x\big) \tag{40}$$

is stable when $x \in L_{V_c}(1)$.

Figures 1–3 show the time response of the state trajectories $x(t)$ under state feedback control law (40) and the control law $u(t)$, and the Lyapunov function $V_c(x(t))$ respectively. Figure 4 shows the time response of $s_1^*(x(t))$. For any $x(0) \in L_{V_c}(1)$, it has a similar simulative result.

5 Conclusions

In this paper, we present a saturated control design for robust stabilization of LDIs subject to disturbance. Convex hull quadratic Lyapunov functions are used to design the state feedback laws. We achieve the design objectives including stabilization, disturbance rejection with minimal reachable set and least L_2 gain

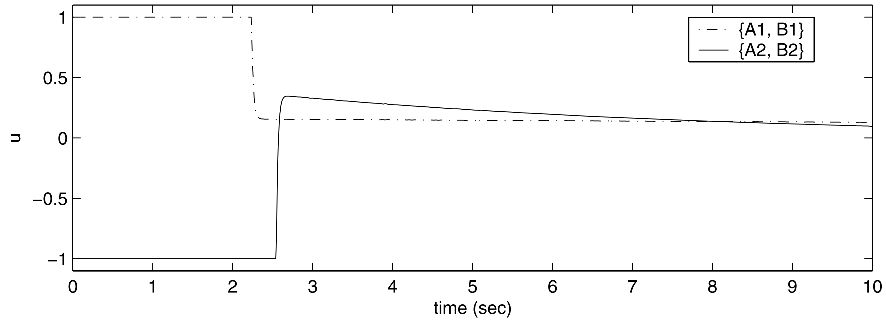

Fig. 2 Control law $u(t)$ of the two subsystems in (37)

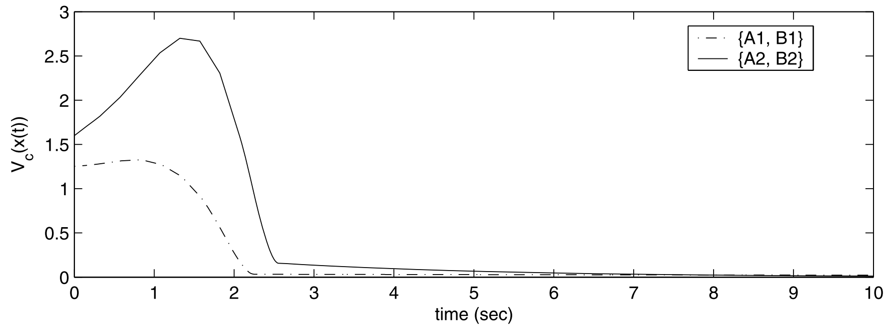

Fig. 3 Lyapunov function $V_c(x(t))$ of the two subsystems in (37)

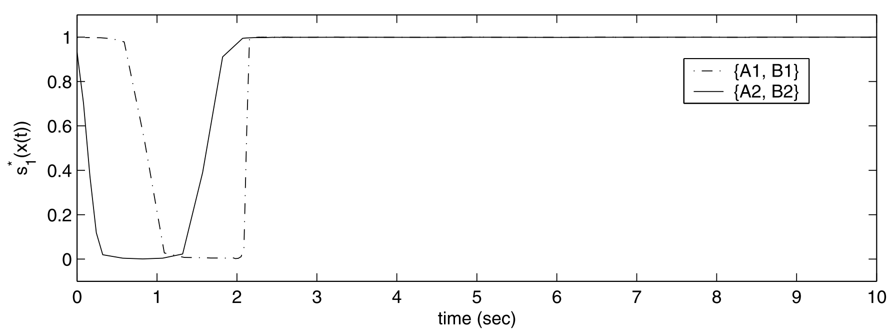

Fig. 4 $s_1^*(x(t))$ of the two subsystems in (37)

simultaneously. Finally, the simulation shows the effectiveness of the method.

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