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The study of a ratio-dependent predator–prey model with stage structure in the prey

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Abstract In this paper, a ratio-dependent predator–prey model with stage structure in the prey is constructed and investigated. In the first part of this paper, some sufficient conditions for the existence and stability of three equilibriums are obtained. In the second part, we consider the effect of impulsive release of predator on the original system. A sufficient condition for the global asymptotical stability of the prey-eradication periodic solution is obtained. We also get the condition, under which the prey would never be eradicated, i.e., the impulsive system is permanent. At last, we give a brief discussion.

Keywords Ratio-dependent · Stage structure · Predator–prey · Impulsive · Permanent · Prey-eradication periodic solution

1 Introduction

One of the most interesting topics in mathematical ecology concerns the survival of species in ecological system. Continuous models, usually in the form of differential equations, have formed a large part of the traditional mathematical ecological literature. Recently, there is a growing explicit biological physiological evidence [1–3] that in many situations, especially when the predators have to search for food (and therefore have to share or compete for food), a more suitable general predator–prey theory should be based on the so-called ratio-dependent theory, which can be roughly stated as that the per capita predator growth rate should be a function of the ratio of prey to predator abundance, and so should be the so-called predator functional response. This is strongly supported by numerous fields and laboratory experiments and observations [4, 5]. Specifically, the standard Lotka–Volterra type models, on which nearly all existing theory is built, assumes that the per capita rate of predation depends on the prey numbers only. An alternative assumption is that, as the numbers of predator change slowly (relative to prey change), there is often competition among the predators, and the per capita rate of predation depends on the numbers of both prey and predator, most likely and simply on their ratio. Generally, a ratio-dependent predator–prey model takes the form

$$\begin{cases} x' = xf(x) - yP(\frac{x}{y}), \\ y' = cyQ(\frac{x}{y}) - dy. \end{cases}$$

Here $P(x)$ is the so-called predator functional response, $P(x)$, $Q(x)$ satisfy the usual properties such

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as being nonnegative and increasing, and equal to zero at zero. In [6], a delayed Gause-type ratio-dependent predator–prey model was discussed. However, ratio-dependent predator–prey model with Michaelis–Menten type functional response received great attention among theoretical and mathematical biologists [7–12]. That is, the authors studied models with predator functional response $P(\frac{x}{y}) = \frac{cxy}{my+x}$.

Systems with impulsive effects describing evolution processes are characterized by the fact that at certain moments of time they abruptly experience a change of state. Impulsive effects are studied in almost every domain of applied science. Especially, impulsive differential equations can describe population dynamical models, since many life phenomena and human exploitation are almost impulsive in the natural world. Some recent and general references on the theory of impulsive equations are [13–15]. There are also many applications of impulsive equations [16–20].

Stage structure is an important notation in epidemiology and demography. In the natural world, there are many species whose individuals have a life history that can be divided into two stages, immature and mature. In [21–23], stage-structured models of single species growth consisting of immature and mature individuals was proposed and discussed. In recent years, stage-structured models have been received much attention. In [24, 25] predator–prey models with stage structure were investigated. In [26–28], stage-structured competitive models were studied. In [17, 19, 29–31] epidemic models with stage structure were investigated. In [7, 8, 10], the authors proposed and investigated ratio-dependent predator–prey models with stage structure. In [8, 10], the author considered the case that the predator had two stages, while in [7] the authors considered the case that the prey had two stages. However, these articles had not considered the effect of impulsive interruption. In [17, 19], the authors considered the effects of impulsive interruption on stage-structured models. Inspired by [7, 8, 17, 19, 21], in this paper, we shall consider ratio-dependent predator–prey model with stage structure in the prey and the effect of impulsive release of predator. The predator being released may be migrated from other areas.

The present paper constructs and analyzes an ODE (ordinary differential equations) model of ratio-dependent predator–prey system with stage structure in the prey and the effect of impulsive interruption on the original system. The paper is organized as follows. In Sect. 2, the main biological assumptions are formulated, and the model is constructed. In Sect. 3, by qualitative analysis, we get the conditions for the existence and stability of three equilibriums of the system. In Sect. 4, the effect of impulsive interruption is considered. We get the sufficient condition for the global asymptotic attractivity of the prey-eradication periodic solution and we also get the sufficient condition for the permanence of the impulsive system. An example is given to simulate our theoretical results. Finally, a brief discussion is provided in the last section.

2 Model formulation

The basic ratio-dependent predator–prey model with stage structure in the prey is

$$\begin{cases} x_1'(t) = rx_2(t) - d_1x_1(t) - re^{-d_1\tau}x_2(t-\tau), \\ x_2'(t) = re^{-d_1\tau}x_2(t-\tau) - d_2x_2(t) \\ \qquad -\beta x_2^2(t) - P(\frac{x_2(t)}{y(t)})y(t), \\ y'(t) = kP(\frac{x_2(t)}{y(t)})y(t) - d_3y(t), \end{cases} \tag{2.1}$$

where $x_1(t)$ and $x_2(t)$ denote the densities of immature prey and mature prey population, respectively. $y(t)$ denotes the density of predator population. All coefficients are positive constants. The models are derived with the following assumptions.

- (H_1) At any time $t > 0$, birth into the immature prey population is proportional to the existing mature prey population, with proportionality r. The immature prey population will transfer to the mature prey class after its birth with a maturity period of τ. The term $re^{-d_1\tau}x_2(t-\tau)$ represents the immature prey who were born at time $t-\tau$ (i.e., $rx_2(t-\tau)$) and survive at the time t (with the immature prey death rate d_1), and therefore represent the transformation from immature prey to mature prey.
- (H_2) The immature prey population has the natural death rate d_1. The death rate of the mature prey population is proportional both to the existing mature population and to the square of it with proportionality constants d_2 and β. The predator population has the natural death rate d_3.
- (H_3) The predator population only capture the mature prey population, since the immature prey population is hidden in the sanctuary. $P(\frac{x_2(t)}{y(t)})$ rep-

resents that the predator functional response is ratio-dependent.

(H$_4$) For ecological reasons, we always assume that the initial values $\Phi = (\phi_1, \phi_2, \phi_3)$ for system (2.1) satisfy

$$\phi_i \in C\big([-\tau, 0], R^3_+\big), \quad \phi_i(0) > 0,\ i = 1, 2, 3, \tag{2.2}$$

where $R^3_+ = \{(x_1, x_2, y) : x_1 \geq 0, x_2 \geq 0, y \geq 0\}$. For continuity of initial conditions, we require

$$\phi_1(0) = \int_{-\tau}^{0} r\phi_2(s)e^{d_1 s}\, ds, \tag{2.3}$$

where $\phi_1(0)$ represents the accumulated survivors of those prey members who were born between time $-\tau$ and 0.

In this paper, for simplicity, we study the ratio-dependent model with Michaelis–Menten functional response (i.e., $P(\frac{x_2(t)}{y(t)}) = \frac{cx_2(t)}{my(t)+x_2(t)}$),

$$\begin{cases} x_1'(t) = rx_2(t) - d_1x_1(t) - re^{-d_1\tau}x_2(t-\tau), \\ x_2'(t) = re^{-d_1\tau}x_2(t-\tau) - d_2x_2(t) - \beta x_2^2(t) \\ \qquad - \frac{cx_2(t)y(t)}{my(t)+x_2(t)}, \\ y'(t) = \frac{fx_2(t)y(t)}{my(t)+x_2(t)} - d_3y(t), \end{cases} \tag{2.4}$$

where $f = kc$, and $0 < k < 1$ is the conversion rate for predation. Since the second and third equations of system (2.4) have no relation to the variable $x_1(t)$, we only need to investigate the following system (2.5):

$$\begin{cases} x_2'(t) = re^{-d_1\tau}x_2(t-\tau) - d_2x_2(t) - \beta x_2^2(t) \\ \qquad - \frac{cx_2(t)y(t)}{my(t)+x_2(t)}, \\ y'(t) = \frac{fx_2(t)y(t)}{my(t)+x_2(t)} - d_3y(t). \end{cases} \tag{2.5}$$

In the next section of this paper, we study model (2.5) with the following initial conditions:

$$\phi_i \in C\big([-\tau, 0], R^2_+\big), \quad \phi_i(0) > 0,\ i = 2, 3. \tag{2.6}$$

3 Qualitative analysis for system (2.5)

To prove our main results we give the following definition and lemmas.

Definition 3.1 System (2.5) is said to be permanent if there are constants $m, M > 0$ (independent of the initial values) and a finite time T_0 such that for all solutions $(x_2(t), y(t))$ with initial conditions (2.6), $m \leq x_2(t) \leq M, m \leq y(t) \leq M$ hold for all $t \geq T_0$. Here T_0 may depend on the initial values.

Lemma 3.2 (see [21]) *If equation $P(\lambda) = Q(\lambda)e^{-\lambda\tau}$ has at least one solution with negative real part, and there is no pure imaginary number solution for this equation, then all of the solution of this equation will have negative real part, where $P(\lambda)$ and $Q(\lambda)$ are continuous function.*

Lemma 3.3 (see [28]) *Consider the following equation*:

$$x'(t) = ax(t-\tau) - bx(t) - cx^2(t), \tag{3.1}$$

where a, b, c, τ are positive constants, and $x(t) > 0$ for $-\tau \leq t \leq 0$; then we have

(i) *If $a > b$, then* $\lim_{t\to\infty} x(t) = \frac{a-b}{c}$.
(ii) *If $a \leq b$, then* $\lim_{t\to\infty} x(t) = 0$.

By simple calculations, we get that for system (2.5), trivial equilibrium $P_1(0, 0)$ always exists; boundary equilibrium $P_2(\overline{x}_2, 0) = (\frac{re^{-d_1\tau}-d_2}{\beta}, 0)$ exists if $re^{-d_1\tau} - d_2 > 0$. In addition, if $f > d_3$ and $re^{-d_1\tau} - d_2 > \frac{c(f-d_3)}{mf}$ hold, then a unique positive equilibrium $P_3(x_2^*, y^*)$ exists, where $x_2^* = \frac{(re^{-d_1\tau}-d_2)-\frac{c(f-d_3)}{mf}}{\beta}$, $y^* = \frac{(f-d_3)x_2^*}{md_3}$.

To the three equilibriums, we get the following results.

Theorem 3.4 *If $re^{-d_1\tau} - d_2 < 0$, then trivial equilibrium $P_1(0, 0)$ is locally asymptotically stable for system (2.5); if $re^{-d_1\tau} - d_2 > 0$, then it is unstable.*

Proof The characteristic equation of system (2.5) at equilibrium $P_1(0, 0)$ is

$$(\lambda + d_3)\big(\lambda - re^{-(d_1+\lambda)\tau} + d_2\big) = 0. \tag{3.2}$$

If $re^{-d_1\tau} - d_2 < 0$, then all of the eigenvalues of (3.2) have negative real part. Thus, equilibrium $P_1(0, 0)$ is locally asymptotically stable.

If $re^{-d_1\tau} - d_2 > 0$, then at least one of the eigenvalues of (3.2) has positive real part. Thus, equilibrium $P_1(0, 0)$ is unstable. The proof is complete. □

Remark 1 When $re^{-d_1\tau} - d_2 < 0$, the trivial equilibrium $P_1(0,0)$ is globally asymptotically stable for system (2.5).

In fact, from the first equation of system (2.5), we have

$$\begin{aligned} x_2'(t) &= re^{-d_1\tau}x_2(t-\tau) - d_2x_2(t) - \beta x_2^2(t) \\ &\quad - \frac{cx_2(t)y(t)}{my(t)+x_2(t)} \\ &\le re^{-d_1\tau}x_2(t-\tau) - d_2x_2(t) - \beta x_2^2(t). \end{aligned}$$

When $re^{-d_1\tau} - d_2 < 0$, by Lemma 3.3, we get $x_2(t) \to 0$ as $t \to \infty$. Then by the second equation of system (2.5), we easily get $y(t) \to 0$ as $t \to \infty$. Therefore, the trivial equilibrium $P_1(0,0)$ is globally asymptotically stable for system (2.5).

Theorem 3.5 *If $re^{-d_1\tau} - d_2 > 0$ and $f - d_3 < 0$, then boundary equilibrium $P_2(\overline{x}_2, 0)$ exists and it is locally asymptotically stable. If $re^{-d_1\tau} - d_2 > 0$ and $f - d_3 > 0$, then boundary equilibrium $P_2(\overline{x}_2, 0)$ exists and it is unstable.*

Proof The characteristic equation of system (2.5) at equilibrium $P_2(\overline{x}_2, 0)$ is

$$\begin{aligned} &\big[\lambda - (f-d_3)\big]\big[\lambda + re^{-d_1\tau}\big(1-e^{-\lambda\tau}\big) \\ &\quad + \big(re^{-d_1\tau} - d_2\big)\big] = 0. \end{aligned} \tag{3.3}$$

Obviously, one of the eigenvalues is $\lambda_1 = f - d_3$. If $f - d_3 < 0$, then $\lambda_1 < 0$; if $f - d_3 < 0$, then $\lambda_1 > 0$. All of the other eigenvalues are determined by equation

$$\lambda = -re^{-d_1\tau}\big(1-e^{-\lambda\tau}\big) - \big(re^{-d_1\tau} - d_2\big). \tag{3.4}$$

If $re^{-d_1\tau} - d_2 > 0$, then all of the eigenvalues of (3.4) have negative real part. In fact, if we let $G(\lambda) = -re^{-d_1\tau}(1-e^{-\lambda\tau}) - (re^{-d_1\tau} - d_2)$, then

$$G(0) = -\big(re^{-d_1\tau} - d_2\big) < 0,$$

and

$$G'(\lambda) = -r\tau e^{-(d_1+\lambda)\tau}.$$

It is easy to see that $G'(\lambda) < 0$ for all $\lambda > 0$. So there cannot exist an eigenvalue with nonnegative real part.

If $re^{-d_1\tau} - d_2 > 0$ and $f - d_3 < 0$, then all of the eigenvalues of (3.3) have negative real part. Therefore, equilibrium $P_1(0,0)$ is locally asymptotically stable.

If $re^{-d_1\tau} - d_2 > 0$ and $f - d_3 > 0$, then one of the eigenvalues of (3.3) has positive real part. Therefore, equilibrium $P_1(0,0)$ is unstable. The proof is complete. □

Remark 2 When $re^{-d_1\tau} - d_2 > 0$ and $f - d_3 < 0$, then boundary equilibrium $P_2(\overline{x}_2, 0)$ exists and it is globally asymptotically stable for system (2.5).

In fact, from the second equation of system (2.5), we have

$$y'(t) = \frac{fx_2(t)y(t)}{my(t)+x_2(t)} - d_3y(t) \le (f-d_3)y(t).$$

When $f - d_3 < 0$, we get $y(t) \to 0$ as $t \to \infty$. Thus, for an arbitrary positive number δ small enough, there exists a time t_1 such that $y(t) \le \delta$ for all $t \ge t_1$. Then by the first equation of system (2.5), we have

$$\begin{aligned} x_2'(t) &= re^{-d_1\tau}x_2(t-\tau) - d_2x_2(t) - \beta x_2^2(t) \\ &\quad - \frac{cx_2(t)y(t)}{my(t)+x_2(t)} \\ &\ge re^{-d_1\tau}x_2(t-\tau) - d_2x_2(t) - \beta x_2^2(t) - c\delta. \end{aligned}$$

Further, $x_2'(t) \le re^{-d_1\tau}x_2(t-\tau) - d_2x_2(t) - \beta x_2^2(t)$ is obvious. Taking into account that δ is small enough, by Lemma 3.3, we get that $x_2(t) \to \overline{x}_2$ as $t \to \infty$. Therefore boundary equilibrium $P_2(\overline{x}_2, 0)$ is globally asymptotically stable for system (2.5).

Theorem 3.6 *If $f > 2d_3$ and $re^{-d_1\tau} - d_2 > \frac{2c(f-d_3)}{mf}$, then equilibrium $P_3(x_2^*, y^*)$ is locally stable for system (2.5).*

Proof The characteristic equation of system (2.5) at equilibrium $P_3(x_2^*, y^*)$ is

$$\lambda^2 + a_1\lambda + a_2 = 0, \tag{3.5}$$

where

$$\begin{aligned} a_1 &= -re^{-(d_1+\lambda)\tau} + d_2 + 2\beta x_2^* \\ &\quad + \frac{mc(y^*)^2}{(my^*+x_2^*)^2} + d_3 - \frac{f(x_2^*)^2}{(my^*+x_2^*)^2}, \end{aligned}$$

$$
\begin{aligned}
a_2 &= \left(\frac{f(x_2^*)^2}{(my^* + x_2^*)^2} - d_3 \right) \\
&\quad \times \left(re^{-(d_1+\lambda)\tau} - d_2 - 2\beta x_2^* - \frac{mc(y^*)^2}{(my^* + x_2^*)^2} \right) \\
&\quad - \frac{mfc(x_2^* y^*)^2}{(my^* + x_2^*)^4}.
\end{aligned}
$$

In the next, we shall prove that all of the eigenvalues of (3.5) have negative real parts through two steps.

Step 1. There exists at least one eigenvalue which is negative real number.

Let $\lambda = u \in R$. Substituting it into (3.5), we get

$$
\begin{aligned}
&u^2 + \left[d_2 + 2\beta x_2^* + \frac{mc(y^*)^2}{(my^* + x_2^*)^2} + d_3 \right. \\
&\quad \left. - \frac{f(x_2^*)^2}{(my^* + x_2^*)^2} \right] u + \left[d_3 - \frac{f(x_2^*)^2}{(my^* + x_2^*)^2} \right] \\
&\quad \times \left[d_2 + 2\beta x_2^* + \frac{mc(y^*)^2}{(my^* + x_2^*)^2} \right] - \frac{mfc(x_2^* y^*)^2}{(my^* + x_2^*)^4} \\
&= re^{-(d_1+\lambda)\tau} \left[u + d_3 - \frac{f(x_2^*)^2}{(my^* + x_2^*)^2} \right]. \qquad (3.6)
\end{aligned}
$$

Denote the left-hand side and the right-hand side of (3.6) as $F(u)$ and $G(u)$, respectively. We have

$$
\begin{aligned}
F(0) &= \left[d_3 - \frac{f(x_2^*)^2}{(my^* + x_2^*)^2} \right] \\
&\quad \times \left[d_2 + 2\beta x_2^* + \frac{mc(y^*)^2}{(my^* + x_2^*)^2} \right] \\
&\quad - \frac{mfc(x_2^* y^*)^2}{(my^* + x_2^*)^4} \\
&= d_3 \frac{f - d_3}{f} [d_2 + 2\beta x_2^*] \\
&\quad + \frac{cd_3(f - d_3)^2 (f - 2d_3)}{mf^3}, \\
G(0) &= re^{-d_1\tau} \left[d_3 - \frac{f(x_2^*)^2}{(my^* + x_2^*)^2} \right] \\
&= re^{-d_1\tau} d_3 \frac{f - d_3}{f}.
\end{aligned}
$$

If $f > 2d_3$ and $re^{-d_1\tau} - d_2 > \frac{2c(f-d_3)}{mf}$, then $F(0) > G(0)$. On the other hand, we have

$$
\begin{aligned}
&G\left(-d_3 \frac{f - d_3}{f} \right) = 0 \\
&\quad > - \frac{mfc(x_2^* y^*)^2}{(my^* + x_2^*)^4} = F\left(-d_3 \frac{f - d_3}{f} \right).
\end{aligned}
$$

Thus, there exists a solution $u^* \in (-d_3 \frac{f-d_3}{f}, 0)$ for (3.6), since $F(u)$ and $G(u)$ are continuous. That is, (3.5) has at least one eigenvalue which is negative real number.

Step 2. There cannot exist an eigenvalue which is a pure imaginary number.

Let $\lambda = iv$, where i is the unit of pure imaginary number and $v \in R$. Substituting it into (3.5), we get

$$
\begin{aligned}
&v^2 - \left[d_3 - \frac{f(x_2^*)^2}{(my^* + x_2^*)^2} \right] \\
&\quad \times \left[d_2 + 2\beta x_2^* + \frac{mc(y^*)^2}{(my^* + x_2^*)^2} \right] + \frac{mfc(x_2^* y^*)^2}{(my^* + x_2^*)^4} \\
&= -re^{-d_1\tau} \left[d_3 - \frac{f(x_2^*)^2}{(my^* + x_2^*)^2} \right] \cos(v\tau) \\
&\quad - \left(re^{-d_1\tau} v \right) \sin(v\tau), \qquad (3.7)
\end{aligned}
$$

and

$$
\begin{aligned}
&v \left[d_2 + 2\beta x_2^* + \frac{mc(y^*)^2}{(my^* + x_2^*)^2} + d_3 - \frac{f(x_2^*)^2}{(my^* + x_2^*)^2} \right] \\
&= -re^{-d_1\tau} \left[d_3 - \frac{f(x_2^*)^2}{(my^* + x_2^*)^2} \right] \sin(v\tau) \\
&\quad + \left(re^{-d_1\tau} v \right) \cos(v\tau). \qquad (3.8)
\end{aligned}
$$

Adding the square of (3.7) and (3.8), we get

$$
v^4 + Av^2 + B = 0, \qquad (3.9)
$$

where

$$
\begin{aligned}
A &= \left[d_2 + 2\beta x_2^* + \frac{mc(y^*)^2}{(my^* + x_2^*)^2} + d_3 \right. \\
&\quad \left. - \frac{f(x_2^*)^2}{(my^* + x_2^*)^2} \right]^2 - 2\left(d_3 - \frac{f(x_2^*)^2}{(my^* + x_2^*)^2} \right) \\
&\quad \times \left(d_2 + 2\beta x_2^* + \frac{mc(y^*)^2}{(my^* + x_2^*)^2} \right)
\end{aligned}
$$

$$
\begin{aligned}
&-2\frac{mfc(x_2^*y^*)^2}{(my^*+x_2^*)^4}-\left(re^{-d_1\tau}\right)^2\\
&=(d_2+2\beta x_2^*)^2-\left(re^{-d_1\tau}\right)^2+\left(\frac{mc(y^*)^2}{(my^*+x_2^*)^2}\right)^2\\
&\quad+2(d_2+2\beta x_2^*)\frac{mc(y^*)^2}{(my^*+x_2^*)^2}\\
&\quad+\left(d_3-\frac{f(x_2^*)^2}{(my^*+x_2^*)^2}\right)^2-2\frac{mfc(x_2^*y^*)^2}{(my^*+x_2^*)^4},
\end{aligned}
$$

$$
\begin{aligned}
B&=\left(d_3-\frac{f(x_2^*)^2}{(my^*+x_2^*)^2}\right)^2\\
&\quad\times\left[d_2+2\beta x_2^*+\frac{mc(y^*)^2}{(my^*+x_2^*)^2}\right]^2\\
&\quad+\frac{(mfc)^2(x_2^*y^*)^4}{(my^*+x_2^*)^4}\\
&\quad-2\frac{mfc(x_2^*y^*)^2}{(my^*+x_2^*)^4}\left(d_3-\frac{f(x_2^*)^2}{(my^*+x_2^*)^2}\right)\\
&\quad\times\left(d_2+2\beta x_2^*+\frac{mc(y^*)^2}{(my^*+x_2^*)^2}\right)\\
&\quad-\left(d_3-\frac{f(x_2^*)^2}{(my^*+x_2^*)^2}\right)^2\left(re^{-d_1\tau}\right)^2\\
&=\left\{\left(d_3-\frac{f(x_2^*)^2}{(my^*+x_2^*)^2}\right)\right.\\
&\quad\times\left[d_2+2\beta x_2^*+\frac{mc(y^*)^2}{(my^*+x_2^*)^2}\right]\\
&\quad\left.-\frac{mfc(x_2^*y^*)^2}{(my^*+x_2^*)^4}\right\}^2\\
&\quad-\left(d_3-\frac{f(x_2^*)^2}{(my^*+x_2^*)^2}\right)^2\left(re^{-d_1\tau}\right)^2\\
&=\left\{\left(d_3-\frac{f(x_2^*)^2}{(my^*+x_2^*)^2}\right)\left[\left(d_2+2\beta x_2^*-re^{-d_1\tau}\right)\right.\right.\\
&\quad\left.\left.+\frac{mc(y^*)^2}{(my^*+x_2^*)^2}\right]-\frac{mfc(x_2^*y^*)^2}{(my^*+x_2^*)^4}\right\}\\
&\quad\times\left\{\left(d_3-\frac{f(x_2^*)^2}{(my^*+x_2^*)^2}\right)\left[\left(d_2+2\beta x_2^*+re^{-d_1\tau}\right)\right.\right.\\
&\quad\left.\left.+\frac{mc(y^*)^2}{(my^*+x_2^*)^2}\right]-\frac{mfc(x_2^*y^*)^2}{(my^*+x_2^*)^4}\right\}.
\end{aligned}
$$

If $f>2d_3$ and $re^{-d_1\tau}-d_2>\frac{2c(f-d_3)}{mf}$, then

$$(d_2+2\beta x_2^*)^2-\left(re^{-d_1\tau}\right)^2>0,$$

and

$$
\begin{aligned}
&\left(\frac{mc(y^*)^2}{(my^*+x_2^*)^2}\right)^2+\left(d_3-\frac{f(x_2^*)^2}{(my^*+x_2^*)^2}\right)^2\\
&\quad-2\frac{mfc(x_2^*y^*)^2}{(my^*+x_2^*)^4}\\
&\geq\left(\frac{mc(y^*)^2}{(my^*+x_2^*)^2}\right)^2+\frac{d_3^4}{f^2}-2\frac{mfc(x_2^*y^*)^2}{(my^*+x_2^*)^4}\\
&=\left(\frac{mc(y^*)^2}{(my^*+x_2^*)^2}\right)^2+\left[\frac{f(x_2^*)^2}{(my^*+x_2^*)^2}\right]^2\\
&\quad-2\frac{mfc(x_2^*y^*)^2}{(my^*+x_2^*)^4}\\
&=\left[\frac{mc(y^*)^2}{(my^*+x_2^*)^2}-\frac{f(x_2^*)^2}{(my^*+x_2^*)^2}\right]^2\geq 0.
\end{aligned}
$$

Thus, $A>0$, $B>0$, and there is no real number solution for (3.9). That is to say, the characteristic equation (3.5) has no pure imaginary eigenvalue.

By step 1, step 2 and Lemma 3.2, we get the desired result. This completes the proof. □

4 The effect of impulsive release of predator

In this section, we shall study the case that some predators are impulsively released with period T. What will the dynamical behaviors be? Will the system be permanent or will the prey be distinct? We shall study the following system with impulsive release of predator:

$$
\begin{cases}
x_1'(t)=rx_2(t)-d_1x_1(t)-re^{-d_1\tau}x_2(t-\tau),\\
x_2'(t)=re^{-d_1\tau}x_2(t-\tau)-d_2x_2(t)-\beta x_2^2(t)\\
\qquad-\frac{cx_2(t)y(t)}{my(t)+x_2(t)},\\
y'(t)=\frac{fx_2(t)y(t)}{my(t)+x_2(t)}-d_3y(t),\\
\quad t\neq nT,\\
x_1(t^+)=x_1(t),\\
x_2(t^+)=x_2(t), \qquad t=nT, n=1,2,3,\ldots,\\
y(t^+)=y(t)+\mu,
\end{cases}
\tag{4.1}
$$

where the parameters have similar biological meanings with those in system (2.5), T is the period of impulsive release and μ represents the impulsive release amount of the predator. In fact, we only need to study the following two-dimensional subsystem with initial values (2.6):

$$\begin{cases} x_2'(t) = re^{-d_1\tau}x_2(t-\tau) - d_2x_2(t) - \beta x_2^2(t) \\ \qquad -\frac{cx_2(t)y(t)}{my(t)+x_2(t)}, \\ y'(t) = \frac{fx_2(t)y(t)}{my(t)+x_2(t)} - d_3y(t), \\ \quad t \neq nT, \\ x_2(t^+) = x_2(t), \\ y(t^+) = y(t)+\mu, \qquad t = nT, n = 1, 2, 3, \ldots. \end{cases} \tag{4.2}$$

If $x_2(t) = 0$ for all $t \geq 0$, we get the following subsystem of system (4.2):

$$\begin{cases} y'(t) = -d_3y(t), & t \neq nT,\ n = 1, 2, 3, \ldots, \\ y(nT^+) = y(nT)+\mu, & t = nT,\ n = 1, 2, 3, \ldots. \end{cases} \tag{4.3}$$

Obviously $\widetilde{y}(t) = \frac{\mu e^{-d_3(t-nT)}}{1-e^{-d_3T}}$, $t \in (nT, (n+1)T]$, $n \in Z_+ = \{1, 2, 3, \ldots\}$, $\widetilde{y}(0^+) = \frac{\mu}{1-e^{-d_3T}}$ is a positive periodic solution of system (4.3). Therefore, the complete expression for the prey-eradication periodic solution of system (4.2) is obtained as $(0, \widetilde{y}(t)) = (0, \frac{\mu e^{-d_3(t-nT)}}{1-e^{-d_3T}})$, $t \in (nT, (n+1)T]$, $n \in Z_+$. Since the solution of system (4.3) is $y(t) = (y(0^+) - \frac{\mu}{1-e^{-d_3T}})e^{-d_3t} + \widetilde{y}(t)$, $t \in (n\tau, (n+1)\tau]$, $n \in Z_+$, so we derive

Lemma 4.1 (see [14]) *For every solution of system* (4.3) *with initial value* $y(0^+) > 0$, *it follows that* $y(t) \to \widetilde{y}(t)$ *as* $t \to \infty$.

Lemma 4.2 (see [13]) *Let* $V : R_+ \times R^2 \to R$ *and* $V \in V_0$. *Assume that*

$$\begin{cases} D^+V(t,X) \leq g(t, V(t,X)), \\ \quad t \neq nT,\ n = 1, 2, \ldots, \\ V(t, X(t^+)) \leq \Psi_n(V(t, X(t))), \\ \quad t = nT,\ n = 1, 2, \ldots, \end{cases} \tag{4.4}$$

where $g : R_+ \times R_+ \to R$ *is continuous in* $(nT, (n+1)T] \times R_+$ *and for each* $v \in R_+^2$, $n \in Z_+$

$$\lim_{(t,y)\to(nT^+,v)} g(t,y) = g\big(nT^+, v\big)$$

exists and is finite, $\Psi_n : R_+ \to R_+$ *is nondecreasing. Let* $R(t)$ *be the maximal solution of the scalar impulsive differential equation*

$$\begin{cases} U'(t) = g(t,U), & t \neq nT,\ n = 1, 2, \ldots, \\ U(t^+) = \Psi_n(U(t)), & t = nT,\ n = 1, 2, \ldots, \\ U(0^+) = U_0, \end{cases}$$

defined on $[0, \infty)$. *Then* $V(0^+, X_0) \leq U_0$ *implies that* $V(t, X(t)) \leq R(t)$, $t \geq 0$, *where* $X(t)$ *is any solution of* (4.4).

We have the following results about system (4.2).

Theorem 4.3 *If* $re^{-d_1\tau} - d_2 > \frac{c}{m}$, *then the prey will not be eradicated, i.e., system* (4.2) *is permanent.*

Proof If $re^{-d_1\tau} - d_2 > \frac{c}{m}$, then by the first equation of system (4.2), we have

$$\begin{aligned} x_2'(t) &= re^{-d_1\tau}x_2(t-\tau) - d_2x_2(t) - \beta x_2^2(t) \\ &\quad - \frac{cx_2(t)y(t)}{my(t)+x_2(t)} \\ &\geq re^{-d_1\tau}x_2(t-\tau) - d_2x_2(t) - \beta x_2^2(t) \\ &\quad - \frac{c}{m}x_2(t). \end{aligned}$$

By Lemma 3.3, we get $x_2(t) \geq \frac{re^{-d_1\tau}-d_2-\frac{c}{m}}{\beta} > 0$ as $t \to \infty$. Thus, the prey will not be eradicated and the system (4.2) is permanent. □

Theorem 4.4 *If* $re^{-d_1\tau} - d_2 < \frac{c}{m}$ *and* $\mu > \frac{(e^{d_3T}-1)(re^{-d_1\tau}-d_2)^2}{c\beta[1-\frac{m}{c}(re^{-d_1\tau}-d_2)]}$, *then the prey-eradication periodic solution* $(0, \widetilde{y}(t))$ *is globally attractive for system* (4.2) *and the prey will be eradicated.*

Proof If $re^{-d_1\tau} - d_2 < \frac{c}{m}$ and $\mu > \frac{(e^{d_3T}-1)(re^{-d_1\tau}-d_2)^2}{c\beta[1-\frac{m}{c}(re^{-d_1\tau}-d_2)]}$, then we can select positive constants δ_1, δ_2 small enough, such that $\sigma = re^{-d_1\tau} - d_2 - \frac{cy_{\min}}{my_{\min}+x_{2\max}} < 0$, where $x_{2\max} = \frac{re^{-d_1\tau}-d_2}{\beta} + \delta_1$ and $y_{\min} = \frac{\mu}{e^{d_3T}-1} - \delta_2$.

By the first equation of system (4.2), we have

$$\begin{aligned} x_2'(t) &= re^{-d_1\tau}x_2(t-\tau) - d_2x_2(t) - \beta x_2^2(t) \\ &\quad - \frac{cx_2(t)y(t)}{my(t)+x_2(t)} \\ &\leq re^{-d_1\tau}x_2(t-\tau) - d_2x_2(t) - \beta x_2^2(t). \end{aligned}$$

By Lemma 3.3, we get $x_2(t) \leq \frac{re^{-d_1\tau}-d_2}{\beta}$ as $t \to \infty$. Thus, for the selected δ_1 small enough, there exists a time t_1 such that $x_2(t) \le x_{2\max}$ for all $t \ge t_1$. Further, we have

$$y'(t) = \frac{f x_2(t) y(t)}{m y(t) + x_2(t)} - d_3 y(t) \ge -d_3 y(t),$$

and by Lemma 4.1 and Lemma 4.2, we get that for the selected δ_2 small enough, there exists a $N_1 \in Z_+$ such that

$$\begin{aligned} y(t) \ge \widetilde{y}(t) - \delta_2 \ge \frac{\mu e^{-d_3 T}}{1 - e^{-d_3 T}} - \delta_2 \\ = y_{\min}, \quad \text{for } t \ge N_1 T. \end{aligned}$$

Thus, when $t > \max\{t_1, N_1 T\}$, we have

$$\frac{c x_2(t) y(t)}{m y(t) + x_2(t)} \ge \frac{c x_2(t) y(t)}{m y(t) + x_{2\max}} \ge \frac{c x_2(t) y_{\min}}{m y_{\min} + x_{2\max}}$$

and

$$\begin{aligned} x_2'(t) &= r e^{-d_1 \tau} x_2(t-\tau) - d_2 x_2(t) - \beta x_2^2(t) \\ &\quad - \frac{c x_2(t) y(t)}{m y(t) + x_2(t)} \\ &\le r e^{-d_1 \tau} x_2(t-\tau) - d_2 x_2(t) - \beta x_2^2(t) \\ &\quad - \frac{c x_2(t) y_{\min}}{m y_{\min} + x_{2\max}}. \end{aligned} \tag{4.5}$$

Since $\sigma = r e^{-d_1 \tau} - d_2 - \frac{c y_{\min}}{m y_{\min} + x_{2\max}} < 0$, from (4.5) and Lemma 3.3, we know that $x_2(t) \to 0$ as $t \to \infty$. Thus,

$$\begin{aligned} y'(t) &= \frac{f x_2(t) y(t)}{m y(t) + x_2(t)} - d_3 y(t) \\ &\to -d_3 y(t) \quad \text{as } t \to \infty. \end{aligned}$$

By Lemma 4.1, we get $y(t) \to \widetilde{y}(t)$ as $t \to \infty$. Therefore, the prey-eradication periodic solution is globally attractive for system (4.2) and the prey will be eradicated. □

Remark 3 From the above two theorems, we can see that under some conditions, we can control the prey population to extinction by proper release of predator. However, under some special conditions (i.e., $r e^{-d_1 \tau} - d_2 > \frac{c}{m}$), the prey cannot be eradicated, no matter how much predator populations are released. This result is very different from those obtained in [17, 19, 20]. This result also explains why the prey is very difficult to be eradicated under some situations. When the prey population density is very low, the predators have to work hard to prey the prey. Considering that the predation response function is ratio-dependent, the predators may die out before they can capture the prey. Our results show that under some conditions, a ratio-dependent predator–prey model may be more accurate than the traditional non-ratio-dependent model.

Example 1 In model (4.1), if let $r = 2.5$, $d_1 = 0.2$, $d_2 = 0.1$, $d_3 = 0.1$, $\beta = 0.01$, $\tau = 1.5$, $c = 3$, $m = 2$, $T = 0.1$, then we get $r e^{-d_1 \tau} - d_2 = 1.752045552$ and $\frac{c}{m} = 1.5$. By Theorem 4.3, we know that the prey will not be eradicated.

If let $r = 2$, $d_1 = 0.2$, $d_2 = 0.1$, $d_3 = 0.1$, $\beta = 0.01$, $\tau = 1.5$, $c = 3$, $m = 2$, $T = 0.1$, then we get $r e^{-d_1 \tau} - d_2 = 1.381636441$, $\frac{c}{m} = 1.5$ and $\frac{(e^{d_3 T} - 1)(r e^{-d_1 \tau} - d_2)^2}{c\beta[1 - \frac{m}{c}(r e^{-d_1 \tau} - d_2)]} = 8.104249932$. Thus, by Theorem 4.4, we know that the prey will be eradicated when $\mu > 8.104249932$.

5 Discussion

In this paper, a ratio-dependent predator–prey model with stage structure in the prey is constructed and investigated. In the first part of this article, we get the conditions for the existence and stability of three equilibriums of the continuous system (2.5). But, we have not get the global stability of the positive equilibrium. We leave it as a future work. In the second part of this article, we consider the effect of impulsive release of predator on the original system. We find that under some proper conditions, by impulsively releasing predator population, the prey will be eradicated. However, if the parameters satisfy $r e^{-d_1 \tau} - d_2 > \frac{c}{m}$, then no matter how much predators we release, the prey will never die out. We also provide an example to simulate our results. Obviously, our results are in accordance with some phenomena happening in real world, which in turn shows that our ratio-dependent predator–prey model is suitable and our results are interesting.

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