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ORIGINAL PAPER

Anti-synchronization of Liu system and Lorenz system with known or unknown parameters

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Abstract This work is concerned with anti-synchronization of Liu system and Lorenz system. Based on Lyapunov stability theory, different controllers are designed to anti-synchronize the two non-identical chaotic systems, active control is used when parameters are known, while the adaptive control law and the parameter update rule are derived via adaptive control when parameters are uncertain. Moreover, the convergence speeds of the scheme can be adjusted by changing the control coefficients. Finally, numerical simulations are also shown to verify the results.

Keywords Anti-synchronization · Liu system · Lorenz system · Lyapunov function

1 Introduction

Inspired by the pioneering work of Pecora and Carroll [1], the synchronization of chaotic systems has been a subject of active research field due to its potential applications for secure communications and control [2–4]. Up to now, many types of synchronization have been proposed in dynamical systems, such as complete synchronization [5], generalized synchronization [6], lag synchronization [7], phase synchronization [8], anti-phase synchronization [9], etc.

As a prevailing phenomenon in symmetrical oscillators, anti-synchronization implies that the state vectors of synchronized systems have the same absolute values but opposite signs. Namely, it is said that anti-synchronization of two systems, S_1 and S_2, is achieved if the following equation holds:

$$\lim_{t\to\infty} \|x_2 + x_1\| = 0, \tag{1}$$

where $x_1(t)$ and $x_2(t)$ are state vectors of the systems S_1 and S_2.

Some progress has been made in the research of anti-synchronization, and most of the works involved mainly with identical chaotic systems [10, 11]. In fact, in engineering, it is hardly the case that every component can be assumed to be identical. Therefore, how to realize anti-synchronization of two non-identical chaotic systems is an interesting and attractive question, while the anti-synchronization between two different chaotic systems is seldom reported in literatures.

In this paper, we investigate the anti-synchronization problem of Liu system and Lorenz system by two different methods. Active control is applied when system parameters are known and adaptive control is used when parameters are unknown. In both cases, sufficient conditions for the anti-synchronization are obtained analytically, based on Lyapunov stability theory. Finally, numerical simulations are employed to verify the effectiveness of the proposed scheme.

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2 Systems description

The Liu system [12] considered in this paper is given as follows:

$$\begin{aligned}\dot{x}_1 &= a_1(y_1 - x_1),\\ \dot{y}_1 &= b_1x_1 - x_1z_1,\\ \dot{z}_1 &= -c_1z_1 + 4x_1^2,\end{aligned} \tag{2}$$

where x_1, y_1, z_1 are state variables, and a_1, b_1, c_1 are real constants. When $a_1 = 10.0$, $b_1 = 40.0$, $c_1 = 2.5$, system (2) is chaotic. The chaotic attractors are shown in Fig. 1.

The Lorenz system is described by

$$\begin{aligned}\dot{x}_2 &= a_2(y_2 - x_2),\\ \dot{y}_2 &= b_2x_2 - y_2 - x_2z_2,\\ \dot{z}_2 &= -c_2z_2 + x_2y_2,\end{aligned} \tag{3}$$

where x_2, y_2, z_2 are state variables, and a_2, b_2, c_2 are real constants. When $a_2 = 10.0$, $b_2 = 28.0$, $c_2 = 8/3$, system (3) is chaotic [13].

In the next sections, we will study anti-synchronization between Liu system and Lorenz system with known or unknown parameters by two different methods.

3 Anti-synchronization between Liu system and Lorenz system with known parameters

In order to observe anti-synchronization behavior between Liu system and Lorenz system via active control, Liu system (2) is assumed as the drive system and Lorenz system is taken as the response system with controllers in the form

$$\begin{aligned}\dot{x}_2 &= a_2(y_2 - x_2) + u_1,\\ \dot{y}_2 &= b_2x_2 - y_2 - x_2z_2 + u_2,\\ \dot{z}_2 &= -c_2z_2 + x_2y_2 + u_3.\end{aligned} \tag{4}$$

The error dynamical system between the drive system (2) and the response system (4) is described

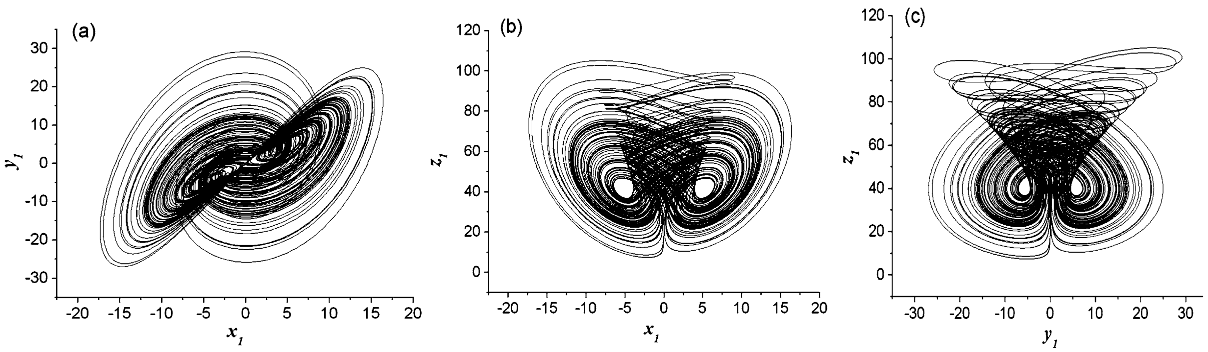


Fig. 1 Chaotic attractors of Liu system: (**a**) (x_1, y_1); (**b**) (x_1, z_1); (**c**) (y_1, z_1)

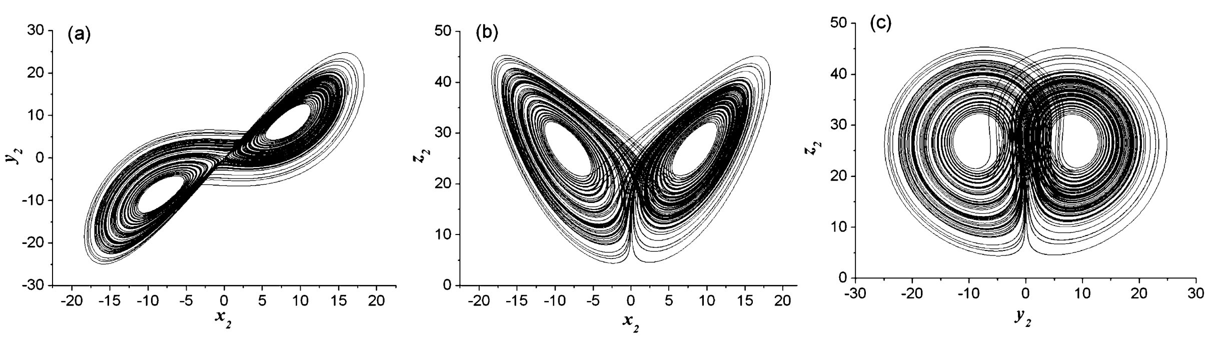


Fig. 2 Chaotic attractors of Lorenz system: (**a**) (x_2, y_2); (**b**) (x_2, z_2); (**c**) (y_2, z_2)

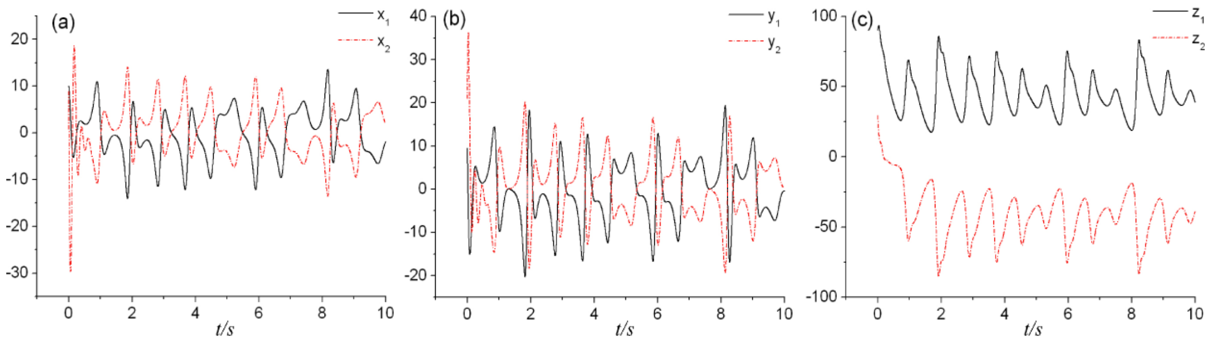


Fig. 3 Time evolutions of system (2) (*solid line*) and system (4) (*dash-dot line*): **(a)** x_1, x_2; **(b)** y_1, y_2; **(c)** z_1, z_2

by

$$\begin{aligned}
\dot{e}_1 &= a_2(e_2 - e_1) + (a_1 - a_2)(y_1 - x_1) + u_1,\\
\dot{e}_2 &= b_2 e_1 - e_2 - b_2 x_1 + b_1 x_1 + y_1\\
&\quad - x_2 z_2 - x_1 z_1 + u_2,\\
\dot{e}_3 &= -c_2 e_3 + c_2 z_1 - c_1 z_1 + 4x_1^2 + x_2 y_2 + u_3.
\end{aligned} \tag{5}$$

The controllers are chosen as follows:

$$\begin{aligned}
u_1 &= -(a_1 - a_2)(y_1 - x_1) - b_2 e_2,\\
u_2 &= (b_2 - b_1)x_1 - y_1 + x_2 z_2 + x_1 z_1 - a_2 e_1,\\
u_3 &= (c_1 - c_2)z_1 - 4x_1^2 - x_2 y_2.
\end{aligned} \tag{6}$$

Lyapunov function is chosen as

$$V = \frac{1}{2}\left(e_1^2 + e_2^2 + e_3^2\right). \tag{7}$$

With the choice of (6), the time derivative of Lyapunov function along the trajectories (5) is

$$\begin{aligned}
\dot{V} &= e_1\dot{e}_1 + e_2\dot{e}_2 + e_3\dot{e}_3\\
&= e_1\left[a_2(e_2 - e_1) - b_2 e_2\right]\\
&\quad + e_2(b_2 e_1 - e_2 - a_2 e_1) - c_2 e_3^2\\
&= -\left(a_2 e_1^2 + e_2^2 + c_2 e_3^2\right) < 0.
\end{aligned} \tag{8}$$

In light of Lyapunov stability theory, the error dynamical system can converge to the origin asymptotically. Consequently, the drive system (2) is anti-synchronous asymptotically with the response system (4) with the controllers (6).

Numerical simulations show the effectiveness of the above methods. In the simulations, the parameters are assumed as $a_1 = 10.0$, $b_1 = 40.0$, $c_1 = 2.5$, $a_2 = 10.0$, $b_2 = 28.0$ and $c_2 = 8.0/3.0$. The initial conditions of the drive system and the response system are taken as $(x_1(0), y_1(0), z_1(0)) = (10.0, 10.0, 90.0)$ and $(x_2(0), y_2(0), z_2(0)) = (10.0, 20.0, 30.0)$, respectively. Figure 3 shows the time evolutions of the drive system (2) and the response system (4) with the controllers (6). The time evolutions of the dynamics errors are plotted in Fig. 4.

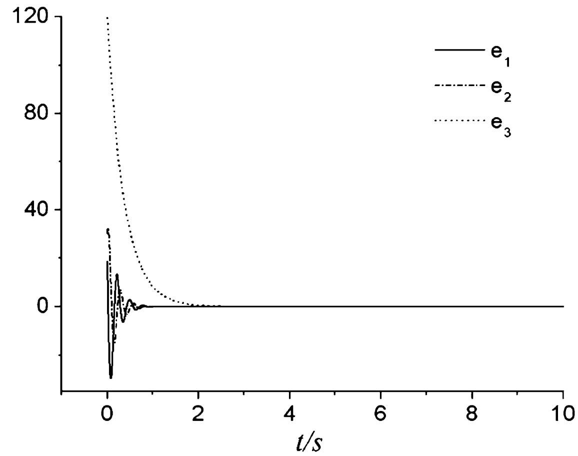


Fig. 4 Dynamics of synchronization errors states for systems (2) and (4) with time t

4 Anti-synchronization between Liu system and Lorenz system with unknown parameters

In this section, we also assume Liu system with three unknown parameters as a drive system, and the controlled Lorenz system with three unknown parameters as a response system, which can be written as

$$\begin{aligned}\dot{x}_2 &= a_2(y_2 - x_2) + u_1,\\ \dot{y}_2 &= b_2x_2 - y_2 - x_2z_2 + u_2,\\ \dot{z}_2 &= -c_2z_2 + x_2y_2 + u_3,\end{aligned} \tag{9}$$

where $u_i = u_i(x_i, y_i, e_i)$ $(e_i = y_i + x_i, i = 1, 2, 3)$ are control functions. a_1, b_1, c_1 and a_2, b_2, c_2 are unknown parameters, respectively, which need to be estimated. In order to determine the control functions and to realize the anti-synchronization between drive system (2) and response system (9), the error dynamical system can be derived as following by adding (2) to (9):

$$\begin{aligned}\dot{e}_1 &= a_2(e_2 - e_1) + (a_1 - a_2)(y_1 - x_1) + u_1,\\ \dot{e}_2 &= b_2e_1 - e_2 - b_2x_1 + b_1x_1 + y_1 - x_2z_2\\ &\quad - x_1z_1 + u_2,\\ \dot{e}_3 &= -c_2e_3 + c_2z_1 - c_1z_1 + 4x_1^2 + x_2y_2 + u_3.\end{aligned} \tag{10}$$

We choose Lyapunov function as

$$V = \frac{1}{2}\big(e_1^2 + e_2^2 + e_3^2 + \tilde{a}_1^2 + \tilde{b}_1^2 + \tilde{c}_1^2 + \tilde{a}_2^2 + \tilde{b}_2^2 + \tilde{c}_2^2\big), \tag{11}$$

in which the variables $\tilde{a}_1 = a_1 - \bar{a}_1$, $\tilde{b}_1 = b_1 - \bar{b}_1$, $\tilde{c}_1 = c_1 - \bar{c}_1$, $\tilde{a}_2 = a_2 - \bar{a}_2$, $\tilde{b}_2 = b_2 - \bar{b}_2$, and $\tilde{c}_2 = c_2 - \bar{c}_2$. $\bar{a}_1$, $\bar{b}_1$, $\bar{c}_1$, $\bar{a}_2$, $\bar{b}_2$ and $\bar{c}_2$ are the estimate values of these unknown parameters, respectively.

The time derivative of Lyapunov function along the trajectories (10) is

$$\begin{aligned}\dot{V} &= e_1\dot{e}_1 + e_2\dot{e}_2 + e_3\dot{e}_3 + \tilde{a}_1\dot{\tilde{a}}_1 + \tilde{b}_1\dot{\tilde{b}}_1\\ &\quad + \tilde{c}_1\dot{\tilde{c}}_1 + \tilde{a}_2\dot{\tilde{a}}_2 + \tilde{b}_2\dot{\tilde{b}}_2 + \tilde{c}_2\dot{\tilde{c}}_2\\ &= e_1\big[a_2(e_2 - e_1) + (a_1 - a_2)(y_1 - x_1) + u_1\big]\\ &\quad + e_2(b_2e_1 - e_2 - b_2x_1 + b_1x_1 + y_1\\ &\quad - x_2z_2 - x_1z_1 + u_2)\\ &\quad + e_3\big(-c_2e_3 + c_2z_1 - c_1z_1 + 4x_1^2 + x_2y_2 + u_3\big)\\ &\quad + \tilde{a}_1\big(-\dot{\bar{a}}_1\big) + \tilde{b}_1\big(-\dot{\bar{b}}_1\big) + \tilde{c}_1\big(-\dot{\bar{c}}_1\big)\\ &\quad + \tilde{a}_2\big(-\dot{\bar{a}}_2\big) + \tilde{b}_2\big(-\dot{\bar{b}}_2\big) + \tilde{c}_2\big(-\dot{\bar{c}}_2\big).\end{aligned} \tag{12}$$

Let controllers are as follows:

$$\begin{aligned}u_1 &= \bar{a}_2(e_1 - e_2) + \big(\bar{a}_2 - \bar{a}_1\big)(y_1 - x_1) - k_1e_1,\\ u_2 &= -\big(\bar{b}_2e_1 - e_2 - \bar{b}_2x_1 + \bar{b}_1x_1\\ &\qquad + y_1 - x_2z_2 - x_1z_1\big) - k_2e_2,\\ u_3 &= -\big(-\bar{c}_2e_3 + \bar{c}_2z_1 - \bar{c}_1z_1 + 4x_1^2 + x_2y_2\big) - k_3e_3,\end{aligned} \tag{13}$$

and the parameter estimation update law as follows:

$$\begin{aligned}\dot{\bar{a}}_1 &= (y_1 - x_1)e_1 + \tilde{a}_1,\\ \dot{\bar{b}}_1 &= x_1e_2 + \tilde{b}_1,\\ \dot{\bar{c}}_1 &= z_1e_3 + \tilde{c}_1,\\ \dot{\bar{a}}_2 &= e_1(e_2 - e_1) - e_1(y_1 - x_1) + \tilde{a}_2,\\ \dot{\bar{b}}_2 &= e_2(e_1 - x_1) + \tilde{b}_2,\\ \dot{\bar{c}}_2 &= e_3(e_3 - z_1) + \tilde{c}_2,\end{aligned} \tag{14}$$

where k_1, k_2 and k_3 are three positive control coefficients, with which we can control the convergence speed of the scheme.

Then $\dot{V}$ becomes

$$\begin{aligned}\dot{V} &= -\big(k_1e_1^2 + k_2e_2^2 + k_3e_3^2 + \tilde{a}_1^2 + \tilde{b}_1^2\\ &\qquad + \tilde{c}_1^2 + \tilde{a}_2^2 + \tilde{b}_2^2 + \tilde{c}_2^2\big) < 0.\end{aligned} \tag{15}$$

According to Lyapunov stability theory, the error dynamical system can converge to the origin asymptotically. Consequently, the drive system (2) is anti-synchronous asymptotically with the response system (9) with the controllers (13) and the parameter estimation update law (14).

In what follows we would like to use numerical simulations to verify the effectiveness of the controllers (13) and the parameter estimation update law (14). In the simulations, the parameters are chosen as $a_1 = 10.0$, $b_1 = 40.0$, $c_1 = 2.5$, $a_2 = 10.0$, $b_2 = 28.0$ and $c_2 = 8.0/3.0$ enabling systems (2) and (9) to be chaotic without controls. The control coefficients for simplicity are chosen $k = k_1 = k_2 = k_3 = 1.0$. The initial conditions are $(x_1(0), y_1(0), z_1(0)) = (10.0, 10.0, 90.0)$ and $(x_2(0), y_2(0), z_2(0)) = (10.0, 20.0, 30.0)$, respectively. In addition, the initial condition of the parameter update law is (10.0, 10.0, 10.0, 10.0, 10.0, 10.0). Figure 5 displays the time evo-

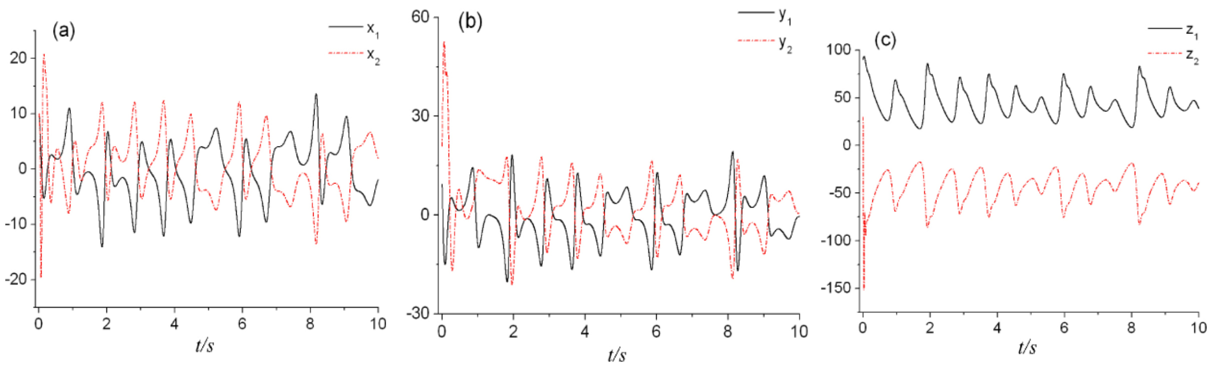
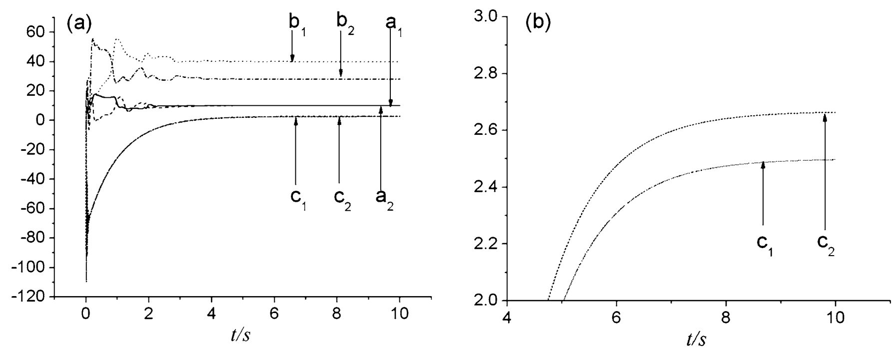


Fig. 5 Time evolutions of system (2) (*solid line*) and system (9) (*dash-dot line*): (**a**) x_1, x_2; (**b**) y_1, y_2; (**c**) z_1, z_2

Fig. 6 (**a**) Changing parameters a_1, b_1, c_1, a_2, b_2 and c_2 with time t; (**b**) Partial enlargement of c_1 and c_2 in (**a**)



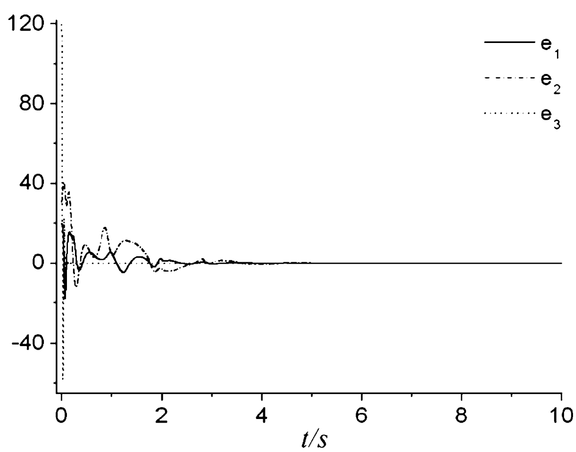


Fig. 7 Dynamics of synchronization error states for systems (2) and (9) with time t

lutions of the drive system (2) and the response system (4) with the controllers (13) and the parameter estimation update law (14). Figure 6 depicts the dynamics of the parameter estimation. The time evolutions of the dynamics errors are shown in Fig. 7.

We take e_1 and a_1 as an example, to show the fact that the convergence speed of the errors and the parameter estimate can be controlled by choosing different positive control coefficients k_1, k_2 and k_3 of the scheme. For simplicity, let $k = k_1 = k_2 = k_3$, too. The numerical results are plotted in Fig. 8 for $k = 1.0$, 10.0, 30.0, respectively. From that it is easy to see that the convergence speed becomes faster and faster with the increase of k.

5 Conclusion

This letter is concerned with anti-synchronization of two non-identical chaotic systems. Based on Lyapunov stability theory, anti-synchronizations between Liu system and Lorenz system with known or unknown parameters are realized. Moreover, via adaptive approaches, the parameters estimation rule is presented and the convergence speeds of the scheme can be adjusted. Numerical simulations are given to verify the results.

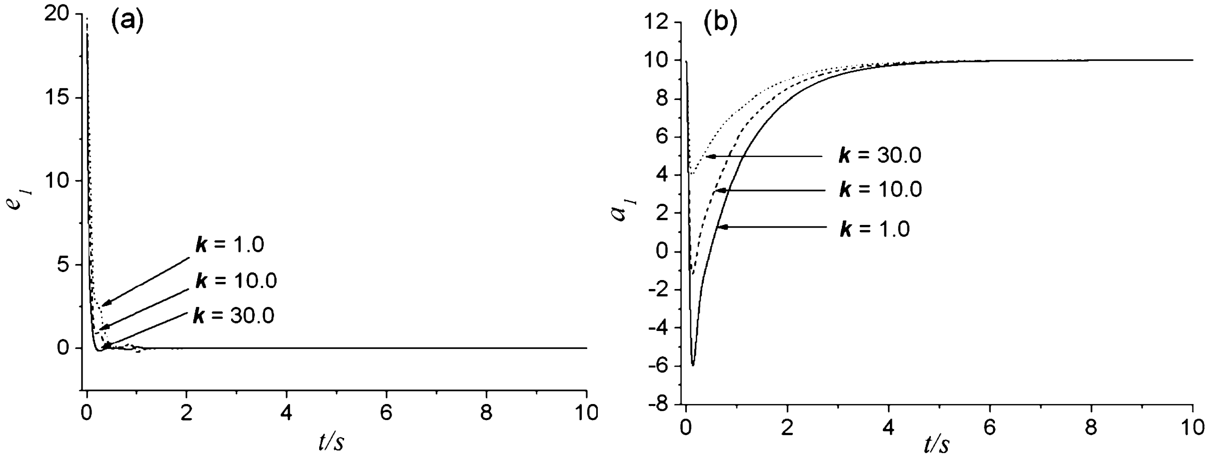


Fig. 8 The convergence speeds of the scheme for different k: (**a**) e_1; (**b**) a_1

References

1. Pecora, L.M., Carroll, T.L.: Synchronization in chaotic systems. Phys. Rev. Lett. **64**, 821–824 (1990)
2. Kocarev, L., Parlitz, U.: General approach for chaotic synchronization with applications to communication. Phys. Rev. Lett. **74**, 5028–5031 (1995)
3. Ma, J., Ying, H.P., Pu, Z.S.: An anti-control scheme for spiral under Lorenz chaotic signal. Chin. Phys. Lett. **22**(5), 1065–1068 (2005)
4. Corron, N.J., Hahs, D.W.: A new approach to communications using chaotic signals. IEEE Trans. Circ. Syst. **44**, 373–382 (1997)
5. Yu, H.J., Liu, Y.Z.: Chaotic synchronization based on stability criterion of linear systems. Phys. Lett. A **314**, 292–298 (2003)
6. Yang, S.S., Juan, C.K.: Generalized synchronization in chaotic systems. Chaos Solitons Fractals **9**, 1703–1707 (1998)
7. Rosenblum, M.G., Pikovsky, A.S., Kurths, J.: From phase to lag synchronization in coupled chaotic oscillators. Phys. Rev. Lett. **78**, 4193–4196 (1997)
8. Park, E.H., Zaks, M.A., Kurths, J.: Phase synchronization in the forced Lorenz system. Phys. Rev. E **60**, 6627–638 (1999)
9. Liu, W.Q.: Anti-phase synchronization in coupled chaotic oscillators. Phys. Rev. E **73**, 057203–057204 (2006)
10. Zhang, Y.P., Sun, J.T.: Chaotic synchronization and anti-synchronization based on suitable separation. Phys. Lett. A **330**, 442–447 (2004)
11. Hu, J., Chen, S.H., Chen, L.: Adaptive control for anti-synchronization of Chua's chaotic system. Phys. Lett. A **339**, 455–460 (2005)
12. Liu, C., Liu, T., Liu, L., Liu, K.: A new chaotic attractor. Chaos Solitons Fractals **22**, 1031–1038 (2004)
13. Lorenz, E.N.: Deterministic non-periodic flows. J. Atmos. Sci. **20**(1), 130–141 (1963)