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ORIGINAL PAPER

Exponential stability in the mean square for stochastic neural networks with mixed time-delays and Markovian jumping parameters

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Abstract In this paper, the stability analysis problem is considered for a class of stochastic neural networks with mixed time-delays and Markovian jumping parameters. The mixed delays include discrete and distributed time-delays, and the jumping parameters are generated from a continuous-time discrete-state homogeneous Markov process. The aim of this paper is to establish some criteria under which the delayed stochastic neural networks are exponentially stable in the mean square. By constructing suitable Lyapunov functionals, several stability conditions are derived on the basis of inequality techniques and the stochastic analysis. An example is also provided in the end of this paper to demonstrate the usefulness of the proposed criteria.

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1 Introduction

In the past two decades, neural networks have been successfully applied in many areas such as combinatorial optimization, signal processing and pattern recognition. In practice, time-delays are often encountered in various neural networks and many other engineering systems due to the finite speed of information transmission. The existence of time-delays may cause oscillation or instability in neural networks, which are harmful to the applications of neural networks, and so stability analysis problems for delayed neural networks have gained much research attention. Up till now, a great deal of results have been reported in the literature [1–6], where the time delays can be classified as constant, time-varying, and distributed delays.

In real nerve systems, the synaptic transmission is a noisy process brought on by random fluctuations from the release of neurotransmitters and other probabilistic causes [7]. Liao and Mao's research [8, 9] showed that a neural network could be stabilized or destabilized by certain stochastic inputs. Therefore, the stability analysis issue for stochastic neural networks becomes increasingly significant, and the relevant results can be seen in [10–14] and the references cited therein.

Hybrid systems driven by continuous-time Markov chain have been used to model many practical systems, where they may experience abrupt changes in their structure and parameters [15–18]. Stochastic neural network with Markovian jumping parameters is one of such hybrid systems, where the parameters are governed by a discrete-state homogeneous Markov process, and every state denotes a switching mode. For Markovian switching neural networks, there are some developments in the recent years. For example, Wang et al. [19] studied the exponential stability of neural networks with discrete time-invariant delays, Huang et al. [20] considered a stochastic neural network with discrete time-delays and parameter uncertainties. However, the stability analysis issue for Markovian stochastic neural networks with both discrete and distributed time-delays has not been fully investigated and there is still much room left for further investigation. This constitutes the motivation for the present research.

In this paper, we study the global exponential stability problem for a class of stochastic neural networks with mixed time-delays and Markovian jumping parameters. The delays include discrete and distributed time-delays, and the jumping parameters are generated from a finite state Markov chain. By applying Lyapunov method, the stochastic analysis and some inequality techniques, several sufficient conditions are obtained under which the delayed stochastic neural networks are exponentially stable in the mean square.

Notations: Throughout this paper, $\mathbb{R}^n$ and $\mathbb{R}^{n\times m}$ denote, respectively, the n-dimensional Euclidean space and the set of $n \times m$ matrices. The superscript "T" denotes matrix transposition and the notion $X > 0$ (respectively, $X \geq 0$) means that X is a real symmetric and positive definite (respectively, semi-definite) matrix. $|\cdot|$ is the Euclidean norm in $\mathbb{R}^n$. I_n is the $n \times n$ identity matrix and for a matrix A, $\lambda_{\max}(A)$ (respectively, $\lambda_{\min}(A)$) represents the largest (respectively, smallest) eigenvalue of A. Moreover, let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions. Denote by $L^2_{\mathcal{F}_0}([-\tau, 0]; \mathbb{R}^n)$ the family of all $\mathcal{F}_0$-measurable $C([-\tau, 0]; \mathbb{R}^n)$-valued random variables $\phi = \{\phi(s) : -\tau \leq s \leq 0\}$ with the norm $\|\phi\| = \sup_{-\tau\leq s\leq 0} |\phi(s)|^2$.

2 Model and analysis

The neural network with mixed time-varying delays has the following form:

$$du(t) = \Big[-Du(t) + A\tilde{f}\big(u(t)\big) + B\tilde{g}\big(u\big(t-\tau_1(t)\big)\big) + C\int_{t-\tau_2(t)}^{t} \tilde{h}\big(u(s)\big)ds + J\Big]dt, \tag{1}$$

where $u(t) = (u_1(t), u_2(t), \ldots, u_n(t))^{\mathrm{T}}$ is the state vector of the n neurons; $D = \mathrm{diag}\{d_1, d_2, \ldots, d_n\} > 0$, $A = (a_{ij})_{n\times n}$, $B = (b_{ij})_{n\times n}$, $C = (c_{ij})_{n\times n}$ represent the firing rate of the neurons and the connection weight matrices, respectively; $J = (J_1, J_2, \ldots, J_n)^{\mathrm{T}}$ is a constant external input vector, and $\tilde{f}(u) = (\tilde{f}_1(u_1), \tilde{f}_2(u_2), \ldots, \tilde{f}_n(u_n))$, $\tilde{g}(u) = (\tilde{g}_1(u_1), \tilde{g}_2(u_2), \ldots, \tilde{g}_n(u_n))$, $\tilde{h}(u) = (\tilde{h}_1(u_1), \tilde{h}_2(u_2), \ldots, \tilde{h}_n(u_n))$ are the neuron activation functions which satisfy the following assumption:

Assumption 1 There exist constant matrices K, L_1, $L_2 \in \mathbb{R}^{n\times n}$ such that

$$\begin{aligned} &\big|\tilde{f}(u) - \tilde{f}(v)\big| \leq \big|K(u-v)\big|,\\ &\big|\tilde{g}(u) - \tilde{g}(v)\big| \leq \big|L_1(u-v)\big|,\\ &\big|\tilde{h}(u) - \tilde{h}(v)\big| \leq \big|L_2(u-v)\big|,\\ &\quad \forall u, v \in \mathbb{R}^n. \end{aligned}$$

$\tau_1(t)$ and $\tau_2(t)$ are the time-varying delays satisfying:

Assumption 2 $0 \leq \tau_1(t) \leq \tau_1$, $0 \leq \tau_2(t) \leq \tau_2$, $\dot{\tau}_1(t) \leq \delta_1 < 1$, $\dot{\tau}_2(t) \leq \delta_2 < 1$.

Throughout this paper, denote $\tau = \max\{\tau_1, \tau_2\}$.

Let u^* be one equilibrium point of (1) and shift this equilibrium point to the origin by just letting $x = u - u^*$. Then one can have

$$dx(t) = \Big[-Dx(t) + Af\big(x(t)\big) + Bg\big(x\big(t-\tau_1(t)\big)\big) + C\int_{t-\tau_2(t)}^{t} h\big(x(s)\big)ds\Big]dt, \tag{2}$$

where $f(x(t)) = \tilde{f}(x(t) + u^*) - \tilde{f}(u^*)$, $g(x(t)) = \tilde{g}(x(t)+u^*) - \tilde{g}(u^*)$, $h(x(t)) = \tilde{h}(x(t)+u^*) - \tilde{h}(u^*)$.

Obviously, from Assumption 1, we have

$$|f(x)| \le |Kx|,$$
$$|g(x)| \le |L_1 x|,$$
$$|h(x)| \le |L_2 x|,$$
$$\forall x, y \in \mathbb{R}^n.$$

Let $\{r(t), t \ge 0\}$ be a right-continuous Markov process on the probability space, which takes values in the finite space $S = \{1, 2, \dots, N\}$ with generator $\Gamma = (\gamma_{ij})(i, j \in S)$ given by:

$$\mathcal{P}\big(r(t+\Delta) = j | r(t) = i\big) = \begin{cases} \gamma_{ij}\Delta + o(\Delta), & \text{if } j \ne i, \\ 1 + \gamma_{ii}\Delta + o(\Delta), & \text{if } j = i, \end{cases}$$

where $\gamma_{ij} \ge 0$ $(i \ne j)$ is the transition rate from i to j and $\gamma_{ii} = -\sum_{j \ne i} \gamma_{ij}$; $\Delta > 0$ and $\lim_{\Delta \to 0} o(\Delta)/\Delta = 0$. Based on the discussions in the Introduction section in this paper, we consider the following stochastic neural networks with Markovian jumping parameters:

$$\begin{aligned} dx(t) = \Big[& -D\big(r(t)\big)x(t) \\ & + A\big(r(t)\big) f\big(x(t)\big) + B\big(r(t)\big) g\big(x(t - \tau_1(t))\big) \\ & + C\big(r(t)\big) \int_{t-\tau_2(t)}^{t} h\big(x(s)\big) ds \Big] dt \\ & + \sigma\big(t, x(t), x\big(t - \tau_1(t)\big), \\ & \quad x\big(t - \tau_2(t)\big), r(t)\big) dw(t). \end{aligned} \tag{3}$$

where $r(t)$ is the Markov chain defined above, $w(t) = (w_1(t), w_2(t), \dots, w_n(t))^{\mathrm{T}}$ is an n-dimensional Brownian motion defined on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \ge 0}, \mathbb{P})$ and $\sigma : R^+ \times R^n \times R^n \times R^n \times S \to R^{n \times n}$ is a Borel measurable function.

Note that the Markov process $\{r(t), t \ge 0\}$ takes values in the finite space $S = \{1, 2, \dots, N\}$. For the sake of simplicity, denote

$$D(i) = D_i, \qquad A(i) = A_i,$$
$$B(i) = B_i, \qquad C(i) = C_i.$$

To conduct the stability analysis, two more assumptions are needed:

Assumption 3 There exist matrices $R_{0i} \ge 0$, $R_{1i} \ge 0$, $R_{2i} \ge 0$ $(i = 1, 2, \dots, N)$ such that

$$\begin{aligned} & \operatorname{trace}\big[\sigma^{\mathrm{T}}\big(t, x(t), x\big(t - \tau_1(t)\big), x\big(t - \tau_2(t)\big), i\big) \\ & \qquad \sigma\big(t, x(t), x\big(t - \tau_1(t)\big), x\big(t - \tau_2(t)\big), i\big)\big] \\ & \quad \le x^{\mathrm{T}}(t) R_{0i} x(t) + x^{\mathrm{T}}\big(t - \tau_1(t)\big) R_{1i} x\big(t - \tau_1(t)\big) \\ & \qquad + x^{\mathrm{T}}\big(t - \tau_2(t)\big) R_{2i} x\big(t - \tau_2(t)\big). \end{aligned}$$

Assumption 4 $\sigma(t, 0, 0, 0, r(t)) \equiv 0$.

Definition 1 For the stochastic neural network (3), the equilibrium point is exponentially stable in the mean square if, for all network modes, there exist scalars $\alpha > 0$ and $\beta > 0$ such that

$$\mathbb{E}\big|x(t, \phi)\big|^2 \le \beta e^{-\alpha t} \sup_{-\tau \le \theta \le 0} \mathbb{E}\big|\phi(\theta)\big|^2 \tag{4}$$

holds for every initial value $\phi \in L^2_{\mathcal{F}_0}([-\tau, 0]; \mathbb{R}^n)$.

Lemma 1 [21] *Given any real matrices X, Y and one matrix $P > 0$ of appropriate dimensions, the following inequality holds*:

$$X^{\mathrm{T}} Y + Y^{\mathrm{T}} X \le \frac{1}{\varepsilon} X^{\mathrm{T}} P X + \varepsilon Y^{\mathrm{T}} P^{-1} Y,$$

where ε is a positive scalar.

Lemma 2 [22] *For any matrix $M > 0$, scalar $\gamma > 0$, vector function $\psi : [0, \gamma] \to \mathbb{R}^n$ such that the integrations concerned are well defined, the following inequality holds*:

$$\left(\int_0^\gamma \psi(t)\, dt \right)^{\mathrm{T}} M \left(\int_0^\gamma \psi(t)\, dt \right) \le \gamma \left(\int_0^\gamma \psi^{\mathrm{T}}(t) M \psi(t)\, dt \right).$$

3 Main results

In this section, a set of conditions are given to guarantee the exponential stability of the Markovian stochastic neural network (3).

Theorem 1 *Under Assumptions 1–4, the system (3) is exponentially stable in the mean square if there exist matrices* $P_i > 0$, $Q_{2i} > 0$, $Q_{3i} > 0$, $Q_{4i} > 0$, *scalars* $\varepsilon_i > 0$, $\eta_i > 0$, $\rho_i > 0$ *and constants* β_1, β_2, β_3, β_4 *such that the following inequalities hold for all* $i \in S$:

$$\beta_2 + \beta_3 + \tau_2\beta_4 < \beta_1,$$

$$\lambda_{\max}(\Omega_i + Q_{2i} + Q_{3i} + \tau_2 Q_{4i}) < -\beta_1;$$

$$\tau_1\lambda_{\max}\left(\sum_{j=1}^{N}\gamma_{ij}Q_{2j}\right) \le \beta_2,$$

$$\tau_2\lambda_{\max}\left(\sum_{j=1}^{N}\gamma_{ij}Q_{3j}\right) \le \beta_3,$$

$$\tau_2\lambda_{\max}\left(\sum_{j=1}^{N}\gamma_{ij}Q_{4j}\right) \le \beta_4;$$

$$\lambda_{\max}(P_i)R_{1i} + \eta_i^{-1}L_1^{\mathrm{T}}L_1 - (1-\delta_1)Q_{2i} \le 0,$$

$$\lambda_{\max}(P_i)R_{2i} - (1-\delta_2)Q_{3i} \le 0,$$

$$\rho_i^{-1}\tau_2 L_2^{\mathrm{T}}L_2 - (1-\delta_2)Q_{4i} \le 0;$$

where

$$\begin{aligned}\Omega_i &= -P_iD_i - D_iP_i + \varepsilon_i P_iA_iA_i^{\mathrm{T}}P_i \\ &\quad + \eta_i P_iB_iB_i^{\mathrm{T}}P_i + \rho_i P_iC_iC_i^{\mathrm{T}}P_i + \varepsilon_i^{-1}K^{\mathrm{T}}K \\ &\quad + \lambda_{\max}(P_i)R_{0i} + \sum_{j=1}^{N}\gamma_{ij}P_j.\end{aligned}$$

Proof Let

$$\mu_1 = \max_{i\in S}\lambda_{\max}(P_i), \qquad \mu_2 = \max_{i\in S}\lambda_{\max}(Q_{2i}),$$
$$\mu_3 = \max_{i\in S}\lambda_{\max}(Q_{3i}), \qquad \mu_4 = \max_{i\in S}\lambda_{\max}(Q_{4i});$$

obviously, $\mu_i > 0$ $(i = 1, 2, 3, 4)$.

Firstly, we will prove that the following equation

$$\begin{aligned}\xi(\alpha) &= \alpha\mu_1 + \left(e^{\alpha\tau_1} - 1\right)\mu_2 + \left(e^{\alpha\tau_2} - 1\right)(\mu_3 + \tau_2\mu_4) \\ &\quad + e^{2\alpha\tau_1}\beta_2 + e^{2\alpha\tau_2}(\beta_3 + \tau_2\beta_4) - \beta_1 = 0 \qquad (5)\end{aligned}$$

has a unique positive root denoted by α. Choose i for $\lambda_{\min}(Q_{2i})$ to be the smallest of $\lambda_{\min}(Q_{2j})$ $(1 \le j \le N)$, i.e.,

$$\lambda_{\min}(Q_{2i}) = \min_{1\le j\le N}\lambda_{\min}(Q_{2j}),$$

and let $y \ne 0$ be the corresponding eigenvector of Q_{2i}. Then

$$y^{\mathrm{T}}Q_{2i}y = \lambda_{\min}(Q_{2i})|y|^2,$$

while

$$\begin{aligned}y^{\mathrm{T}}\left(\sum_{j=1}^{N}\gamma_{ij}Q_{2j}\right)y &= \sum_{j\ne i}^{N}\gamma_{ij}y^{\mathrm{T}}Q_{2j}y + \gamma_{ii}y^{\mathrm{T}}Q_{2i}y \\ &\ge \sum_{j\ne i}^{N}\gamma_{ij}\lambda_{\min}(Q_{2j})|y|^2 + \gamma_{ii}\lambda_{\min}(Q_{2i})|y|^2 \\ &\ge \lambda_{\min}(Q_{2i})|y|^2\sum_{j=1}^{N}\gamma_{ij} = 0;\end{aligned}$$

and so one has

$$\lambda_{\max}\left(\sum_{j=1}^{N}\gamma_{ij}Q_{2j}\right)|y|^2 \ge y^{\mathrm{T}}\left(\sum_{j=1}^{N}\gamma_{ij}Q_{2j}\right)y \ge 0.$$

Since $|y| > 0$, we can conclude

$$\lambda_{\max}\left(\sum_{j=1}^{N}\gamma_{ij}Q_{2j}\right) \ge 0.$$

From the condition $\tau_1\lambda_{\max}(\sum_{j=1}^{N}\gamma_{ij}Q_{2j}) \le \beta_2$, we easily have $\beta_2 \ge 0$. Similarly, one has $\beta_3 \ge 0$ and $\beta_4 \ge 0$. Moreover, from the condition $\beta_2 + \beta_3 + \tau_2\beta_4 < \beta_1$, we can deduce that (5) has a unique root $\alpha > 0$ by the existence and uniqueness theorem of equations.

Next, we choose a Lyapunov functional candidate $V(t, x(t), i)$ as

$$\begin{aligned}V\big(t, x(t), i\big) &= V_1\big(t, x(t), i\big) + V_2\big(t, x(t), i\big) \\ &\quad + V_3\big(t, x(t), i\big) + V_4\big(t, x(t), i\big) \qquad (6)\end{aligned}$$

with

$$V_1\big(t,x(t),i\big)=e^{\alpha t}x^{\mathrm{T}}(t)P_ix(t),$$

$$V_2\big(t,x(t),i\big)=\int_{t-\tau_1(t)}^{t}e^{\alpha(s+\tau_1)}x^{\mathrm{T}}(s)Q_{2i}x(s)\,ds,$$

$$V_3\big(t,x(t),i\big)=\int_{t-\tau_2(t)}^{t}e^{\alpha(s+\tau_2)}x^{\mathrm{T}}(s)Q_{3i}x(s)\,ds,$$

$$V_4\big(t,x(t),i\big)=\int_{-\tau_2(t)}^{0}\int_{t+s}^{t}e^{\alpha(\theta+\tau_2)}x^{\mathrm{T}}(\theta)Q_{4i}x(\theta)\,ds\,d\theta.$$

For stochastic network (3), the weak infinitesimal operator can be computed as follows

$$\begin{aligned}
&\mathcal{L}V_1\big(t,x(t),i\big)\\
&=e^{\alpha t}\bigg\{\alpha x^{\mathrm{T}}(t)P_ix(t)+2x^{\mathrm{T}}(t)P_i\bigg[-D_ix(t)+A_if\big(x(t)\big)+B_ig\big(x\big(t-\tau_1(t)\big)\big)+C_i\int_{t-\tau_2(t)}^{t}h\big(x(s)\big)ds\bigg]\\
&\quad+\sum_{j=1}^{N}\gamma_{ij}x^{\mathrm{T}}(t)P_jx(t)\\
&\quad+\mathrm{trace}\big[\sigma^{\mathrm{T}}\big(t,x(t),x\big(t-\tau_1(t)\big),x\big(t-\tau_2(t)\big),i\big)P_i\sigma\big(t,x(t),x\big(t-\tau_1(t)\big),x\big(t-\tau_2(t)\big),i\big)\big]\bigg\}\\
&\le e^{\alpha t}\bigg\{\alpha x^{\mathrm{T}}(t)P_ix(t)+2x^{\mathrm{T}}(t)P_i\bigg[-D_ix(t)+A_if\big(x(t)\big)+B_ig\big(x\big(t-\tau_1(t)\big)\big)+C_i\int_{t-\tau_2(t)}^{t}h\big(x(s)\big)ds\bigg]\\
&\quad+\lambda_{\max}(P_i)\big[x^{\mathrm{T}}(t)R_{0i}x(t)+x^{\mathrm{T}}\big(t-\tau_1(t)\big)R_{1i}x\big(t-\tau_1(t)\big)\\
&\quad+x^{\mathrm{T}}\big(t-\tau_2(t)\big)R_{2i}x\big(t-\tau_2(t)\big)\big]+\sum_{j=1}^{N}\gamma_{ij}x^{\mathrm{T}}(t)P_jx(t)\bigg\}.
\end{aligned}$$

By employing Lemmas 1 and 2, the following inequalities hold:

$$\begin{aligned}
2x^{\mathrm{T}}(t)P_iA_if\big(x(t)\big)&\le\varepsilon_ix^{\mathrm{T}}(t)P_iA_iA_i^{\mathrm{T}}P_ix(t)+\varepsilon_i^{-1}f^{\mathrm{T}}\big(x(t)\big)f\big(x(t)\big)\\
&\le\varepsilon_ix^{\mathrm{T}}(t)P_iA_iA_i^{\mathrm{T}}P_ix(t)+\varepsilon_i^{-1}x^{\mathrm{T}}(t)K^{\mathrm{T}}Kx(t);\\
2x^{\mathrm{T}}(t)P_iB_ig\big(x\big(t-\tau_1(t)\big)\big)&\le\eta_ix^{\mathrm{T}}(t)P_iB_iB_i^{\mathrm{T}}P_ix(t)+\eta_i^{-1}x^{\mathrm{T}}\big(t-\tau_1(t)\big)L_1^{\mathrm{T}}L_1x\big(t-\tau_1(t)\big);\\
2x^{\mathrm{T}}(t)P_iC_i\int_{t-\tau_2(t)}^{t}h\big(x(s)\big)ds&\le\rho_ix^{\mathrm{T}}(t)P_iC_iC_i^{\mathrm{T}}P_ix(t)+\rho_i^{-1}\bigg(\int_{t-\tau_2(t)}^{t}h\big(x(s)\big)ds\bigg)^{\mathrm{T}}\bigg(\int_{t-\tau_2(t)}^{t}h\big(x(s)\big)ds\bigg)\\
&\le\rho_ix^{\mathrm{T}}(t)P_iC_iC_i^{\mathrm{T}}P_ix(t)+\rho_i^{-1}\tau_2\int_{t-\tau_2(t)}^{t}x^{\mathrm{T}}(s)L_2^{\mathrm{T}}L_2x(s)\,ds;
\end{aligned}$$

so we have

$$\begin{aligned}
\mathcal{L}V_1\big(t,x(t),i\big)\le e^{\alpha t}\bigg\{&x^{\mathrm{T}}(t)\big(\alpha P_i+\Omega_i\big)x(t)+x^{\mathrm{T}}(t-\tau_1(t))\big(\lambda_{\max}(P_i)R_{1i}+\eta_i^{-1}L_1^{\mathrm{T}}L_1\big)x\big(t-\tau_1(t)\big)\\
&+x^{\mathrm{T}}\big(t-\tau_2(t)\big)\lambda_{\max}(P_i)R_{2i}x\big(t-\tau_2(t)\big)+\rho_i^{-1}\tau_2\int_{t-\tau_2(t)}^{t}x^{\mathrm{T}}(s)L_2^{\mathrm{T}}L_2x(s)\,ds\bigg\}.
\end{aligned}\tag{7}$$

Similarly, for $V_2(t,x(t),i)$, $V_3(t,x(t),i)$, $V_4(t,x(t),i)$, we have

$$\begin{aligned}
&\mathcal{L}V_2\big(t,x(t),i\big)\\
&\quad= e^{\alpha(t+\tau_1)}x^{\mathrm{T}}(t)Q_{2i}x(t)-\big(1-\dot{\tau}_1(t)\big)e^{\alpha(t+\tau_1-\tau_1(t))}x^{\mathrm{T}}\big(t-\tau_1(t)\big)Q_{2i}x\big(t-\tau_1(t)\big)\\
&\qquad+\sum_{j=1}^{N}\gamma_{ij}\int_{t-\tau_1(t)}^{t}e^{\alpha(s+\tau_1)}x^{\mathrm{T}}(s)Q_{2j}x(s)\,ds\\
&\quad\le e^{\alpha t}\big\{e^{\alpha\tau_1}x^{\mathrm{T}}(t)Q_{2i}x(t)-(1-\delta_1)x^{\mathrm{T}}\big(t-\tau_1(t)\big)Q_{2i}x\big(t-\tau_1(t)\big)\big\}+e^{\alpha(t+\tau_1)}\beta_2\tau_1^{-1}\int_{t-\tau_1}^{t}\big|x(s)\big|^2\,ds;
\end{aligned}\tag{8}$$

$$\begin{aligned}
&\mathcal{L}V_3\big(t,x(t),i\big)\\
&\quad= e^{\alpha(t+\tau_2)}x^{\mathrm{T}}(t)Q_{3i}x(t)-\big(1-\dot{\tau}_2(t)\big)e^{\alpha(t+\tau_2-\tau_2(t))}x^{\mathrm{T}}\big(t-\tau_2(t)\big)Q_{3i}x\big(t-\tau_2(t)\big)\\
&\qquad+\sum_{j=1}^{N}\gamma_{ij}\int_{t-\tau_2(t)}^{t}e^{\alpha(s+\tau_2)}x^{\mathrm{T}}(s)Q_{3j}x(s)\,ds\\
&\quad\le e^{\alpha t}\big\{e^{\alpha\tau_2}x^{\mathrm{T}}(t)Q_{3i}x(t)-(1-\delta_2)x^{\mathrm{T}}\big(t-\tau_2(t)\big)Q_{3i}x\big(t-\tau_2(t)\big)\big\}+e^{\alpha(t+\tau_2)}\beta_3\tau_2^{-1}\int_{t-\tau_2}^{t}\big|x(s)\big|^2\,ds;
\end{aligned}\tag{9}$$

$$\begin{aligned}
&\mathcal{L}V_4\big(t,x(t),i\big)\\
&\quad= \tau_2(t)e^{\alpha(t+\tau_2)}x^{\mathrm{T}}(t)Q_{4i}x(t)-\big(1-\dot{\tau}_2(t)\big)\int_{t-\tau_2(t)}^{t}e^{\alpha(s+\tau_2)}x^{\mathrm{T}}(s)Q_{4i}x(s)\,ds\\
&\qquad+\sum_{j=1}^{N}\gamma_{ij}\int_{-\tau_2(t)}^{0}\int_{t+s}^{t}e^{\alpha(\theta+\tau_2)}x^{\mathrm{T}}(\theta)Q_{4j}x(\theta)\,ds\,d\theta\\
&\quad\le e^{\alpha t}\Big\{\tau_2e^{\alpha\tau_2}x^{\mathrm{T}}(t)Q_{4i}x(t)-(1-\delta_2)\int_{t-\tau_2(t)}^{t}x^{\mathrm{T}}(s)Q_{4i}x(s)\,ds\Big\}+e^{\alpha(t+\tau_2)}\beta_4\int_{t-\tau_2}^{t}\big|x(s)\big|^2ds.
\end{aligned}\tag{10}$$

Utilizing the conditions given in the theorem and combining with (6)–(10), we have

$$\begin{aligned}
\mathcal{L}V\big(t,x(t),i\big)&=\mathcal{L}V_1\big(t,x(t),i\big)+\mathcal{L}V_2\big(t,x(t),i\big)+\mathcal{L}V_3\big(t,x(t),i\big)+\mathcal{L}V_4\big(t,x(t),i\big)\\
&\le e^{\alpha t}x^{\mathrm{T}}(t)\big[\alpha P_i+\Omega_i+e^{\alpha\tau_1}Q_{2i}+e^{\alpha\tau_2}Q_{3i}+\tau_2e^{\alpha\tau_2}Q_{4i}\big]x(t)+e^{\alpha(t+\tau_1)}\beta_2\tau_1^{-1}\int_{t-\tau_1}^{t}\big|x(s)\big|^2\,ds\\
&\quad+e^{\alpha(t+\tau_2)}\beta_3\tau_2^{-1}\int_{t-\tau_2}^{t}\big|x(s)\big|^2\,ds+e^{\alpha(t+\tau_2)}\beta_4\int_{t-\tau_2}^{t}\big|x(s)\big|^2\,ds.
\end{aligned}$$

Since

$$\int_0^t e^{\alpha s}\bigg(\int_{s-\tau_1}^{s}\big|x(\theta)\big|^2\,d\theta\bigg)ds\le\int_{-\tau_1}^{t}\bigg(\int_{\theta}^{\theta+\tau_1}e^{\alpha s}\big|x(\theta)\big|^2\,ds\bigg)d\theta\le\tau_1e^{\alpha\tau_1}\int_{-\tau_1}^{t}e^{\alpha s}\big|x(s)\big|^2\,ds$$

and

$$\int_0^t e^{\alpha s}\bigg(\int_{s-\tau_2}^{s}\big|x(\theta)\big|^2\,d\theta\bigg)ds\le\tau_2e^{\alpha\tau_2}\int_{-\tau_2}^{t}e^{\alpha s}\big|x(s)\big|^2\,ds,$$

by using the generalized Itô's formula, one has

$$\begin{aligned}
\mathbb{E}V\big(t,x(t),r(t)\big) &= \mathbb{E}V\big(0,x(0),r(0)\big) + \int_0^t \mathbb{E}\mathcal{L}V\big(s,x(s),r(s)\big)ds \\
&\le \mathbb{E}V\big(0,x(0),r(0)\big) + \big[\alpha\mu_1 + \big(e^{\alpha\tau_1}-1\big)\mu_2 + \big(e^{\alpha\tau_2}-1\big)(\mu_3+\tau_2\mu_4) - \beta_1 + e^{2\alpha\tau_1}\beta_2 \\
&\quad + e^{2\alpha\tau_2}(\beta_3+\tau_2\beta_4)\big]\mathbb{E}\int_0^t e^{\alpha s}\big|x(s)\big|^2 ds + \big[e^{2\alpha\tau_1}\beta_2 + e^{2\alpha\tau_2}(\beta_3+\tau_2\beta_4)\big]\mathbb{E}\int_{-\tau}^0 \big|\phi(s)\big|^2 ds \\
&\le \mathbb{E}V\big(0,x(0),r(0)\big) + e^{2\alpha\tau}(\beta_2+\beta_3+\tau_2\beta_4)\tau\mathbb{E}\|\phi\|^2.
\end{aligned} \tag{11}$$

On the other hand, from

$$\mathbb{E}V\big(t,\phi,r(0)\big) \le \big(\mu_1 + \mu_2\tau_1 e^{\alpha\tau_1} + \mu_3\tau_2 e^{\alpha\tau_2} + \mu_4\tau_2^2 e^{\alpha\tau_2}\big)\mathbb{E}\|\phi\|^2,$$

$$\mathbb{E}V\big(t,x(t),r(t)\big) \ge e^{\alpha t}\min_{i\in S}\lambda_{\min}(P_i)\mathbb{E}\big|x(t)\big|^2;$$

it is easy to see that

$$\mathbb{E}\big|x(t)\big|^2 \le \beta e^{-\alpha t}\mathbb{E}\|\phi\|^2$$

holds for any initial value $\phi \in L^2_{\mathcal{F}_0}([-\tau,0];\mathbb{R}^n)$, where

$$\beta = \frac{\mu_1 + \mu_2\tau_1 e^{\alpha\tau_1} + \mu_3\tau_2 e^{\alpha\tau_2} + \mu_4\tau_2^2 e^{\alpha\tau_2} + \tau e^{2\alpha\tau}(\beta_2+\beta_3+\tau_2\beta_4)}{\min_{i\in S}\lambda_{\min}(P_i)}.$$

So the Markovian stochastic network (3) is globally exponentially stable in the mean square and this completes the proof. □

Remark 1 If we take $C(r(t)) = 0$ and remove the stochastic component, Theorem 1 is reduced to the result given in [19]. If we take $A(r(t)) = 0$ and $C(r(t)) = 0$, the criterion in Theorem 1 is just the one given in [20]. Therefore, the research in this paper is a generalization of the previous work and is therefore significant.

Remark 2 The conditions in Theorem 1 can be converted into LMIs by making minor adjustment. The operation is as follows: replacing the conditions

$$\lambda_{\max}(\Omega_i + Q_{2i} + Q_{3i} + \tau_2 Q_{4i}) < -\beta_1,$$

$$\tau_1\lambda_{\max}\left(\sum_{j=1}^N \gamma_{ij}Q_{2j}\right) \le \beta_2,$$

$$\tau_2\lambda_{\max}\left(\sum_{j=1}^N \gamma_{ij}Q_{3j}\right) \le \beta_3 \quad \text{and}$$

$$\tau_2\lambda_{\max}\left(\sum_{j=1}^N \gamma_{ij}Q_{4j}\right) \le \beta_4$$

by

$$\Omega_i + Q_{2i} + Q_{3i} + \tau_2 Q_{4i} < -\beta_1 I,$$

$$\tau_1\sum_{j=1}^N \gamma_{ij}Q_{2j} \le \beta_2 I,$$

$$\tau_2\sum_{j=1}^N \gamma_{ij}Q_{3j} \le -\beta_3 I \quad \text{and}$$

$$\tau_2\sum_{j=1}^N \gamma_{ij}Q_{4j} \le -\beta_4 I,$$

respectively, and at the same time introducing positive scalars ν_i $(i = 1,2,\ldots,N)$ such that $P_i - \nu_i I < 0$, by using the method proposed in [19], it is easy to see that the conditions in Theorem 1 can be rewritten into LMIs.

Remark 3 The stability problem for the stochastic neural network (3) with parameter uncertainties can also be considered in a way similar to that in [20].

By the property of the Markov chain $r(\cdot)$, $\sum_{j=1}^{N}\gamma_{ij}=0$. If we take $Q_{2i}=Q_2$, $Q_{3i}=Q_3$, $Q_{4i}=Q_4$ $(i=1,2,\dots,N)$ in Theorem 1, we have the following corollary.

Corollary 1 *Under Assumptions 1–4, the system (3) is exponentially stable in the mean square if there exist matrices $P_i>0$, $Q_2>0$, $Q_3>0$, $Q_4>0$, scalars $\varepsilon_i>0$, $\eta_i>0$, $\rho_i>0$ and constant $\beta_1>0$ such that the following inequalities hold for all $i\in S$:*

$$\lambda_{\max}(\Omega_i+Q_2+Q_3+\tau_2Q_4)<-\beta_1;$$
$$\lambda_{\max}(P_i)R_{1i}+\eta_i^{-1}L_1^{\mathrm T}L_1-(1-\delta_1)Q_2\le 0,$$
$$\lambda_{\max}(P_i)R_{2i}-(1-\delta_2)Q_3\le 0,$$
$$\rho_i^{-1}\tau_2L_2^{\mathrm T}L_2-(1-\delta_2)Q_4\le 0;$$

where

$$\begin{aligned}\Omega_i={}&-P_iD_i-D_iP_i+\varepsilon_iP_iA_iA_i^{\mathrm T}P_i\\&+\eta_iP_iB_iB_i^{\mathrm T}P_i+\rho_iP_iC_iC_i^{\mathrm T}P_i\\&+\varepsilon_i^{-1}K^{\mathrm T}K+\lambda_{\max}(P_i)R_{0i}+\sum_{j=1}^{N}\gamma_{ij}P_j.\end{aligned}$$

For a deterministic system with Markovian switching parameters

$$\begin{aligned}dx(t)=\Big[&-D\big(r(t)\big)x(t)+A\big(r(t)\big)f\big(x(t)\big)\\&+B\big(r(t)\big)g\big(x\big(t-\tau_1(t)\big)\big)\\&+C\big(r(t)\big)\int_{t-\tau_2(t)}^{t}h\big(x(s)\big)ds\Big]dt,\end{aligned}\tag{12}$$

from Theorem 1 we can obtain its stability criterion as follows.

Corollary 2 *Under Assumptions 1–2, the system (12) is exponentially stable in the mean square if there exist matrices $P_i>0$, $Q_{2i}>0$, $Q_{3i}>0$, scalars $\varepsilon_i>0$, $\eta_i>0$, $\rho_i>0$ and constants β_1, β_2, β_3 such that the following inequalities hold for all $i\in S$:*

$$\beta_2+\tau_2\beta_3<\beta_1,$$
$$\lambda_{\max}(\Omega_i+Q_{2i}+\tau_2Q_{3i})<-\beta_1;$$
$$\tau_1\lambda_{\max}\left(\sum_{j=1}^{N}\gamma_{ij}Q_{2j}\right)\le\beta_2,$$
$$\tau_2\lambda_{\max}\left(\sum_{j=1}^{N}\gamma_{ij}Q_{3j}\right)\le\beta_3;$$
$$\eta_i^{-1}L_1^{\mathrm T}L_1-(1-\delta_1)Q_{2i}\le 0,$$
$$\rho_i^{-1}\tau_2L_2^{\mathrm T}L_2-(1-\delta_2)Q_{4i}\le 0;$$

where

$$\begin{aligned}\Omega_i={}&-P_iD_i-D_iP_i+\varepsilon_iP_iA_iA_i^{\mathrm T}P_i\\&+\eta_iP_iB_iB_i^{\mathrm T}P_i+\rho_iP_iC_iC_i^{\mathrm T}P_i\\&+\varepsilon_i^{-1}K^{\mathrm T}K+\sum_{j=1}^{N}\gamma_{ij}P_j.\end{aligned}$$

Proof Denote

$$\mu_1=\max_{i\in S}\lambda_{\max}(P_i),$$
$$\mu_2=\max_{i\in S}\lambda_{\max}(Q_{2i}),$$
$$\mu_3=\max_{i\in S}\lambda_{\max}(Q_{3i}).$$

Let α be the unique positive root of equation:

$$\begin{aligned}\xi(\alpha)={}&\alpha\mu_1+\big(e^{\alpha\tau_1}-1\big)\mu_2+\big(e^{\alpha\tau_2}-1\big)\tau_2\mu_3\\&+e^{2\alpha\tau_1}\beta_2+e^{2\alpha\tau_2}\tau_2\beta_3-\beta_1=0.\end{aligned}\tag{13}$$

Choose a Lyapunov functional candidate $V(t,x(t),i)$ as

$$\begin{aligned}&V\big(t,x(t),i\big)\\&\quad=V_1\big(t,x(t),i\big)+V_2\big(t,x(t),i\big)+V_3\big(t,x(t),i\big)\end{aligned}$$

with

$$V_1\big(t,x(t),i\big)=e^{\alpha t}x^{\mathrm T}(t)P_ix(t),$$
$$V_2\big(t,x(t),i\big)=\int_{t-\tau_1(t)}^{t}e^{\alpha(s+\tau_1)}x^{\mathrm T}(s)Q_{2i}x(s)\,ds,$$
$$V_3\big(t,x(t),i\big)=\int_{-\tau_2(t)}^{0}\int_{t+s}^{t}e^{\alpha(\theta+\tau_2)}x^{\mathrm T}(\theta)Q_{3i}x(\theta)\,ds\,d\theta.$$

Similarly to the proof of Theorem 1, we can prove that the system (12) is exponentially stable in the mean square. □

4 Illustrative example

Consider the following stochastic neural network with Markovian switching:

$$\begin{aligned}dx(t) = \Big[&-D\big(r(t)\big)x(t) + A\big(r(t)\big)f\big(x(t)\big)\\&+B\big(r(t)\big)g\big(x(t-\tau_1(t))\big)\\&+C\big(r(t)\big)\int_{t-\tau_2(t)}^{t} h\big(x(s)\big)ds\Big]dt\\&+\sigma\big(t, x(t), x\big(t-\tau_1(t)\big),\\&\quad x\big(t-\tau_2(t)\big), r(t)\big)dw(t),\end{aligned}\tag{14}$$

where $x(t) = (x_1(t), x_2(t))^{\mathrm{T}}$; $w(t)$ is a 2-dimensional Brownian motion; $r(t)$ is a right-continuous Markov chain taking value in $S = \{1, 2\}$ with generator $\Gamma = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$; $f_i(x_i) = 0.4[|x_i + 1| - |x_i - 1|]$, $g_i(x_i) = h_i(x_i) = 0.3[|x_i + 1| - |x_i - 1|]$ $(i = 1, 2)$; $\tau_1(t) = 0.5\cos t + 0.5$, $\tau_2(t) = 0.5\sin t + 0.5$. Then Assumptions 1 and 2 are satisfied with $K = 0.8I$, $L_1 = L_2 = 0.6I$, $\tau_1 = \tau_2 = 1$, $\delta_1 = \delta_2 = 0.5$. Take

$$\begin{aligned}&\sigma\big(t, x(t), x\big(t-\tau_1(t)\big), x\big(t-\tau_2(t)\big), 1\big)\\&\quad = \begin{pmatrix} 0.22(x_1(t) + x_2(t)) & 0.42x_2(t-\tau_1(t)) \\ 0.42x_1(t-\tau_1(t)) & 0.42x_2(t-\tau_2(t)) \end{pmatrix},\\&\sigma\big(t, x(t), x\big(t-\tau_1(t)\big), x\big(t-\tau_2(t)\big), 2\big)\\&\quad = \begin{pmatrix} 0.3x_1(t) + 0.2x_1(t-\tau_1(t))) & 0.25x_1(t-\tau_2(t)) \\ 0.5x_2(t-\tau_1(t)) & 0.3x_2(t) + 0.35x_2(t-\tau_2(t)) \end{pmatrix};\end{aligned}$$

then Assumption 3 is satisfied with $R_{01} = 0.1I$, $R_{11} = R_{21} = 0.2I$; $R_{02} = R_{12} = 0.2I$, $R_{22} = 0.3I$. Further let the parameters be as follows:

$$D_1 = \begin{pmatrix} 3 & 0 \\ 0 & 2.5 \end{pmatrix},\quad A_1 = \begin{pmatrix} 0.5 & 0.1 \\ 0 & 0.4 \end{pmatrix},$$

$$B_1 = \begin{pmatrix} 0.3 & 0 \\ 0.1 & 0.2 \end{pmatrix},\quad C_1 = \begin{pmatrix} 0.2 & 0 \\ 0 & 0.1 \end{pmatrix},$$

$$D_2 = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix},\quad A_2 = \begin{pmatrix} 0.5 & 0.1 \\ 0.1 & 0.5 \end{pmatrix},$$

$$B_2 = \begin{pmatrix} 0.3 & 0.1 \\ 0 & 0.3 \end{pmatrix},\quad C_2 = \begin{pmatrix} 0.2 & 0 \\ 0 & 0.2 \end{pmatrix}.$$

By simple computation, one can see that there exist $P_1 = I$, $P_2 = 1.2I$, $Q_2 = 0.8I$, $Q_3 = 0.6I$, $Q_4 = 0.4I$ and $\varepsilon_i = 1$, $\eta_i = 2$, $\rho_i = 2$ $(i = 1, 2)$, $\beta_1 = 2$ such that the conditions in Corollary 1 are satisfied for $i = 1, 2$. Therefore, one can conclude that the system (14) is globally exponentially stable in the mean square.

5 Conclusions

In this paper, the stochastic neural network with Markovian jumping parameters has been studied. Both discrete and distributed time-delays have been taken into account. The parameters in the neural system are determined by a finite-state homogeneous Markov process; that is, the modes among which neural network changes are finite. By using the Lyapunov method and some stochastic analysis techniques, sufficient conditions have been derived to guarantee the exponential stability in the mean square of the neural networks. An example is also given to show the effectiveness of our criteria.

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