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# Lag synchronization of a class of chaotic systems with unknown parameters

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**Abstract** Research on chaos synchronization of dynamical systems has been largely reported in literature. However, synchronization of different structure—uncertain dynamical systems—has received less attention. This paper addresses synchronization of a class of time-delay chaotic systems containing uncertain parameters. A unified scheme is established for synchronization between two strictly different time-delay uncertain chaotic systems. The synchronization is successfully achieved by designing an adaptive controller with the estimates of the unknown parameters and the nonlinear feedback gain. The result is rigorously proved by the Lyapunov stability theorem. Moreover, we illustrate the application of the proposed scheme by numerical simulation, which demonstrates the effectiveness and feasibility of the proposed synchronization method.

**Keywords** Chaos synchronization · Time-delay · Adaptive control · Uncertain parameters

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## 1 Introduction

In the past three decades, there has been intensive interest in the research of synchronizing chaotic dynamical systems ([1–31], and references therein). In the 1990, Pecora and Carroll addressed the synchronization of chaotic systems using drive–response scheme [1]. Up to now, various types of synchronization phenomena have been found, such as complete synchronization [1, 3, 4], generalized synchronization [5–7], phase synchronization [8, 9], lag synchronization [10–15], projective synchronization [16, 17], and so on.

Notice that because of uncertainties in nature and artificial models, synchronization of uncertain dynamical systems is a subject of great interest. Many existing synchronization methods concern the synchronization of two chaotic systems with identical structure or slight parameter mismatch [18–20], also of dynamical systems with strictly different model structure [21–28]. Moreover, the system parameters are inevitably perturbed by external inartificial factors and cannot be exactly known a priori. Therefore, synchronization of two different chaotic systems with unknown parameters is more essential and useful in real-life applications [29, 30].

In engineering applications, time-delay always exists and affects the dynamical behaviors of chaotic systems. For example, in the telephone communication system, the voice one hears on the receiver side at time $t + \tau$ is the voice from the transmitter side at time $t$.

So, strictly speaking, it is not reasonable to require the response system to synchronize the drive system at exactly the same time. In [9–15, 31], the authors discuss the lag synchronization of chaotic systems.

However, less research has been done in synchronizing uncertain nonlinear dynamical systems with time-delay and strictly different structures, which is of great relevance in many fields, such as secure communication, signal engineering, biology, etc. In this paper, our aim is to discuss lag synchronization of uncertain nonlinear dynamical systems with different structure model. This scheme is presented based on parameter identification and Lyapunov functional method in Sect. 2. Section 3 examplifies the application of this procedure by the Genesio system and Lorenz system. Finally, the conclusion is drawn in Sect. 4.

## 2 The synchronization scheme

In this section, the main result for adaptive synchronization and lag synchronization of uncertain dynamical systems with time-delay is proposed.

Consider the drive chaotic system with time-delay in the form of

$$\dot{x}(t) = f\big(x(t-\tau)\big) + C\big(x(t-\tau)\big)\hat{\theta}, \tag{1}$$

where $x \in R^n$ is the state vector of the system, $f : R^n \to R^n$ is a continuous vector function, $C : R^n \to R^{n\times p}$ is a matrix function, $\hat{\theta} \in R^p$ denotes the parameter vector and $\tau > 0$ is a propagation delay.

*Remark 1* Nonlinear chaotic system (1) depends on the parameters, and many chaotic systems belong to (1), such as Lorenz system, Chen chaotic system when propagation delay $\tau = 0$.

The controlled response system with different structure is described as follows:

$$\dot{y}(t) = g\big(y(t)\big) + D\big(y(t)\big)\hat{\delta} + U(t), \tag{2}$$

where $y \in R^n$ is the state vector of the system, $g : R^n \to R^n$ is a continuous vector function, $D : R^n \to R^{n\times d}$ is a matrix function, $\hat{\delta} \in R^d$ is a parameter vector and $U(t)$ is the controller.

*Remark 2* When $g(y) = f(x), C(x(t))\hat{\theta} = D(y(t))\hat{\delta}$, the chaotic systems mentioned above are identical.

Now, the problem is to design a suitable controller to synchronize the two systems in spite of the difference in their structure. Denote the error between the two systems as $e(t) = y(t) - x(t-\tau)$. The problem is to design an adaptive synchronization algorithm

$$\begin{aligned} &U(t) = U\big(x, y, \tilde{\theta}, \tilde{\delta}, t\big), \\ &\tilde{\theta}(t) = \tilde{\theta}(x, y, t), \\ &\tilde{\delta}(t) = \tilde{\delta}(x, y, t), \end{aligned} \tag{3}$$

where $\tilde{\theta}$ and $\tilde{\delta}$ are the vectors of the parameter estimates of the original systems parameters. The object is to design the proper $U(t), \tilde{\theta}(t), \tilde{\delta}(t)$ and realize asymptotically synchronization between the states $\tilde{\theta}$ and $\hat{\theta}$, $\tilde{\delta}$ and $\hat{\delta}$, $y(t)$ of the response system (2) and $x(t)$ of the drive system (1), i.e., to achieve

$$\begin{aligned} &y(t) - x(t-\tau) \to 0, \quad t \to \infty, \\ &\tilde{\theta}(t) - \hat{\theta} \to 0, \quad t \to \infty, \\ &\tilde{\delta}(t) - \hat{\delta} \to 0, \quad t \to \infty. \end{aligned}$$

**Theorem** *The drive system* (1) *synchronizes with the response system* (2), *if we choose the controller* $U(t)$ *as follows*:

$$\begin{aligned} U(t) = &-g\big(y(t)\big) + f\big(x(t-\tau)\big) - D\big(y(t)\big)\tilde{\delta} \\ &+ C\big(x(t-\tau)\big)\tilde{\theta} + Ke(t), \end{aligned} \tag{4}$$

*where* $\tilde{\theta}$ *and* $\tilde{\delta}$ *are the estimate vectors of original system parameter vectors* $\hat{\theta}$ *and* $\hat{\delta}$, *respectively.* $K = \mathrm{diag}(k_1, k_2, \dots, k_n) \in R^{n\times n}$ *is the feedback control gain. Then, if the gain* $K$ *is updated according to the following law*:

$$\dot{k}_i = -\varepsilon_i e_i^2(t), \quad i = 1, 2, \dots, n, \tag{5}$$

*where* $\varepsilon_i > 0$ *is an arbitrary constant, that guarantees an effective feedback gain.*

Meanwhile, the update law of the unknown parameter $\tilde{\theta}$ is taken as

$$\dot{\tilde{\theta}} = -\big[C^{\mathrm{T}}\big(x(t-\tau)\big)\big]e(t), \tag{6}$$

also, the update law of the parameter $\tilde{\delta}$ is taken as

$$\dot{\tilde{\delta}} = D^{\mathrm{T}}\big(y(t)\big)e(t), \tag{7}$$

thus, the synchronization of the drive system (1) and the response system (2) can be achieved globally asymptotically.

*Proof* The dynamic equation of the error states can be obtained easily by system (1) and system (2), which is expressed as follows:

$$\dot{e}(t) = g\big(y(t)\big) + D\big(y(t)\big)\hat{\delta} - f\big(x(t-\tau)\big) - C\big(x(t-\tau)\big)\hat{\theta} + U(t). \tag{8}$$

Considering the controller (4), the new error system is

$$\dot{e}(t) = D\big(y(t)\big)\delta - C\big(x(t-\tau)\big)\theta + Ke(t), \tag{9}$$

where $\theta = \hat{\theta} - \tilde{\theta}$ and $\delta = \hat{\delta} - \tilde{\delta}$ are the parameter estimate errors.

Choose the following Lyapunov function

$$V = \frac{1}{2}\big(e^{\mathrm{T}}(t)e(t) + \theta^{\mathrm{T}}\theta + \delta^{\mathrm{T}}\delta\big) + \frac{1}{2}\sum_{i=1}^{n}\frac{1}{\varepsilon_i}(k_i + l)^2, \tag{10}$$

where $l > 0$ is a constant. The time derivative of $V$ along the trajectories in the error system (9) is

$$\begin{aligned}
&\dot{V}(e,\theta,\delta)\\
&= e^{\mathrm{T}}(t)\dot{e}(t) + \dot{\theta}^{\mathrm{T}}\theta + \dot{\delta}^{\mathrm{T}}\delta - \sum_{i=1}^{n}(k_i + l)e_i^2\\
&= e^{\mathrm{T}}(t)\big[D\big(y(t)\big)\delta - C\big(x(t-\tau)\big)\theta + Ke(t)\big]\\
&\quad + \big(C^{\mathrm{T}}\big(x(t-\tau)\big)e(t)\big)^{\mathrm{T}}\theta\\
&\quad + \big(-D^{\mathrm{T}}\big(y(t)\big)e(t)\big)^{\mathrm{T}}\delta - \sum_{i=1}^{n}(k_i + l)e_i^2\\
&= e^{\mathrm{T}}(t)D\big(y(t)\big)\delta - e^{\mathrm{T}}(t)C\big(x(t-\tau)\big)\theta\\
&\quad + e^{\mathrm{T}}(t)Ke(t) + e^{\mathrm{T}}(t)C\big(x(t-\tau)\big)\theta\\
&\quad - e^{\mathrm{T}}(t)D\big(y(t)\big)\delta - e^{\mathrm{T}}(t)Ke(t) - le^{\mathrm{T}}(t)e(t)\\
&= -le^{\mathrm{T}}(t)e(t) < 0.
\end{aligned} \tag{11}$$

Then, according to the well-known invariant principle and Lyapunov stability theorem, the error systems are asymptotically stable. This completes the proof. □

*Remark 3* In [17], the authors study the projective synchronization of chaotic systems. However, time-delay is not considered in their paper. In [26], the authors address the sequential synchronization of special chaotic systems. In this paper, the synchronization scheme is more general and effective in synchronizing the systems than in [17, 26].

## 3 Application of synchronization

In this section, we give an example to illustrate the effectiveness of the proposed method. The Genesio system is taken as the drive system with time-delay. The drive system can be described by

$$\begin{aligned}
\dot{x}_1(t) &= x_2(t-\tau),\\
\dot{x}_2(t) &= x_3(t-\tau),\\
\dot{x}_3(t) &= -\hat{\alpha}x_1(t-\tau) - \hat{\beta}x_2(t-\tau)\\
&\quad - \hat{\gamma}x_3(t-\tau) + x_1^2(t-\tau).
\end{aligned} \tag{12}$$

The Lorenz system is chosen as the response system, and can be written as

$$\begin{aligned}
\dot{y}_1(t) &= -\hat{a}y_1(t) + \hat{a}y_2(t) + u_1(t),\\
\dot{y}_2(t) &= -y_1(t)y_3(t) + \hat{c}y_1(t) - y_2(t) + u_2(t),\\
\dot{y}_3(t) &= y_1(t)y_2(t) - \hat{b}y_3(t) + u_3(t),
\end{aligned} \tag{13}$$

where $\hat{\alpha}, \hat{\beta}, \hat{\gamma}, \hat{a}, \hat{b}, \hat{c}$ are the original system parameters. Systems are chaotic if the parameter values are as follows:

$$\hat{\alpha} = 6, \quad \hat{\beta} = 2.92, \quad \hat{\gamma} = 1.2,$$
$$\hat{a} = 10, \quad \hat{b} = 8/3, \quad \hat{c} = 28.$$

Then, the drive system and the response system can be rewritten as

$$\begin{pmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{pmatrix} = \begin{pmatrix}x_2\\ x_3\\ x_1^2\end{pmatrix} + \begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ -x_1 & -x_2 & -x_3\end{pmatrix}\begin{pmatrix}\hat{\alpha}\\ \hat{\beta}\\ \hat{\gamma}\end{pmatrix}, \tag{14}$$

and

$$\begin{pmatrix}\dot{y}_1\\ \dot{y}_2\\ \dot{y}_3\end{pmatrix} = \begin{pmatrix}0\\ -y_1y_3 - y_2\\ y_1y_2\end{pmatrix} + \begin{pmatrix}u_1\\ u_2\\ u_3\end{pmatrix}$$

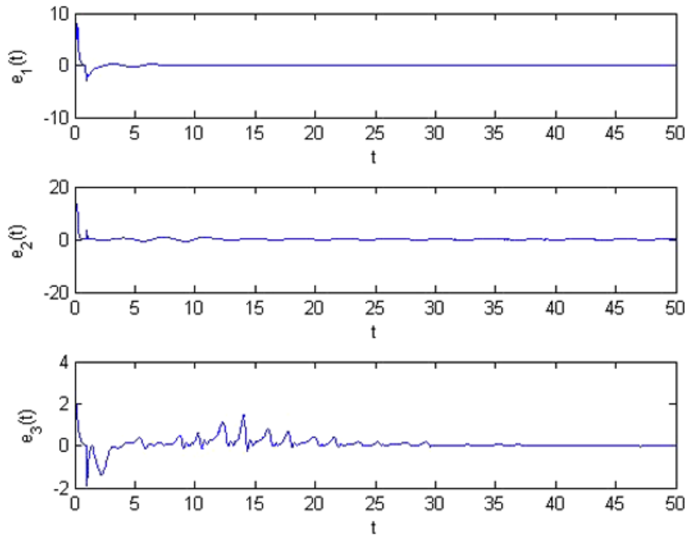


**Fig. 1** Time evolution of synchronization errors

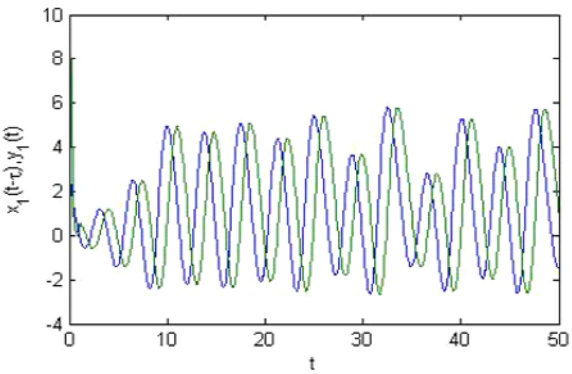


**Fig. 2** Dynamics of synchronization between Genesio system and Lorenz system

$$+\begin{pmatrix} -y_1+y_2 & 0 & 0 \\ 0 & 0 & y_1 \\ 0 & -y_3 & 0 \end{pmatrix}\begin{pmatrix} \hat{a} \\ \hat{b} \\ \hat{c} \end{pmatrix}. \tag{15}$$

According to (3), the controller is

$$\begin{cases} u_1 = x_{2\tau} - \tilde{a}(-y_1+y_2) + k_1 e_1, \\ u_2 = y_2 + y_1 y_3 + x_{3\tau} - \tilde{c} y_1 + k_2 e_2, \\ u_3 = -y_1 y_2 + x_{1\tau}^2 + \tilde{b} y_3 - \tilde{\alpha} x_{1\tau} - \tilde{\beta} x_{2\tau} \\ \qquad - \tilde{\gamma} x_{3\tau} + k_3 e_3. \end{cases} \tag{16}$$

The parameter $\varepsilon_i = 1$, and the feedback gain $K$ is

$$\begin{cases} \dot{k}_1 = -e_1^2, \\ \dot{k}_2 = -e_2^2, \\ \dot{k}_3 = -e_3^2. \end{cases} \tag{17}$$

The estimate of parameters $\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}, \tilde{a}, \tilde{b}, \tilde{c}$ obey the update laws as follows:

$$\begin{cases} \dot{\tilde{\alpha}} = x_{1\tau} e_3, \\ \dot{\tilde{\beta}} = x_{2\tau} e_3, \\ \dot{\tilde{\gamma}} = x_{3\tau} e_3, \end{cases} \tag{18}$$

and

$$\begin{cases} \dot{\tilde{a}} = (-y_1+y_2) e_1, \\ \dot{\tilde{b}} = -y_3 e_3, \\ \dot{\tilde{c}} = y_1 e_2. \end{cases} \tag{19}$$

The chaotic systems initial values are:

$$x_1(0)=3, \qquad x_2(0)=-4, \qquad x_3(0)=2,$$
$$y_1(0)=-4, \qquad y_2(0)=5, \qquad y_3(0)=3,$$

hence the initial errors are: $-7$, 9, 1. The systems initial parameters are:

$$\tilde{\alpha}(0)=1, \qquad \tilde{\beta}(0)=1, \qquad \tilde{\gamma}(0)=1,$$
$$\tilde{a}(0)=1, \qquad \tilde{b}(0)=1, \qquad \tilde{c}(0)=1,$$
$$k_1(0)=1, \qquad k_2(0)=1, \qquad k_3(0)=1, \qquad \tau=2.$$

In Fig. 1, the evolution of the synchronization errors between systems (12) and (13) can be seen. In Fig. 2, we can see that the drive system and the response system are lag-synchronized. The response system $y_i(t)$ lags 2 seconds behind the drive

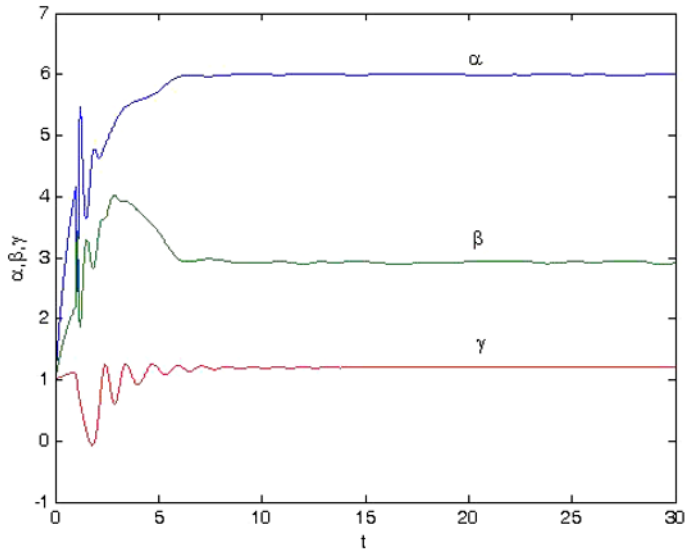


**Fig. 3** The estimate of unknown parameters $\hat{\alpha}, \hat{\beta}, \hat{\gamma}$

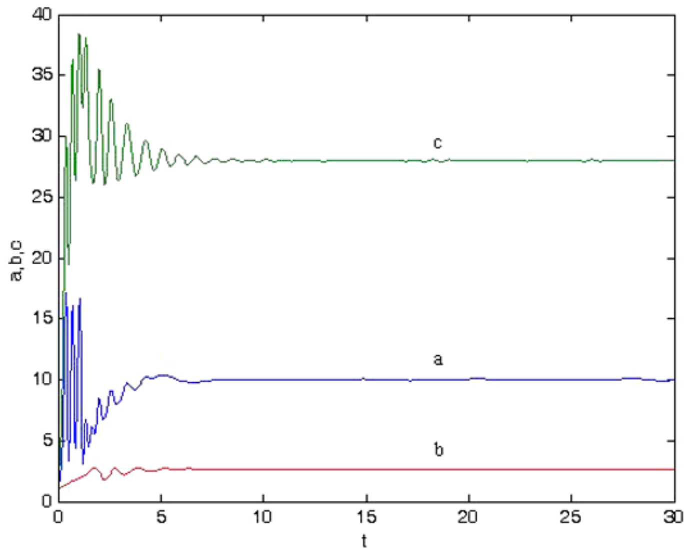


**Fig. 4** The estimate of unknown parameters $\hat{a}, \hat{b}, \hat{c}$

system $x_i(t)$. Moreover, Figs. 3 and 4 display the unknown parameters $\widehat{\alpha} = 6$, $\widehat{\beta} = 2.92$, $\widehat{\gamma} = 1.2$, $\widehat{a} = 10$, $\widehat{b} = 8/3$, $\widehat{c} = 28$ when $t \to \infty$.

## 4 Conclusion

In this paper, a global synchronization method of two strictly different uncertain chaotic systems with time-delay is discussed. Comparing with other researches, we consider the systems with unknown parameters and time-delay. Furthermore, we propose an adaptive control scheme, containing the updated laws for the estimate of the unknown parameters and the feedback gain. The result is rigorously approved by Lyapunov stability theorem. Moreover, numerical simulation is presented to demonstrate the effectiveness and feasibility of the proposed scheme.

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