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On energy transfer between vibrating systems under linear and nonlinear interactions

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Abstract The present study deals with energy transfer in a dissipative mechanical system. Numerical results are given by considering two different potentials and periodical excitation. Specifically, we show energy transfer from linear oscillator to another one, depending on initial conditions. Also, energy transfer from linear to nonlinear (energy pumping), as well as from nonlinear to linear, oscillator is analyzed, under linear and nonlinear interactions.

Keywords Cubic and linear interaction · Energy transfer · Energy pumping · Essential stiffness nonlinearity

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1 Introduction

In last years, various theoretical, numerical, and experimental results of energy transfer, particularly nonlinear energy pumping, which consists in passive irreversible transfer of energy from a linear structure to a nonlinear one, have been discussed. Previous works in this area concerned mainly energy transfer with linear absorber with important mass, or from linear to nonlinear oscillator [1, 2]. As discussed in [3, 4], this energy transfer is closely related to nonlinear normal modes of undamped/unforced system and nonlinear resonance mechanism. In a recent work [5], the phenomenon was studied for different excitations: transient and periodical. Besides, in that work, advantages of such a system were carried out in particular efficiency of the phenomenon.

The dynamics of passive energy transfer from a damped linear oscillator to an essentially nonlinear end attachment was studied numerically and analytically in [6]. It was showed that the transfer of energy is caused by either fundamental or subharmonic resonance capture [7], and in some cases is initiated by nonlinear beat phenomena. The authors showed that, due to the essential nonlinearity, the end attachment is capable of passively absorbing broadband energy at both high and low frequencies, acting, in essence, as a passive broadband boundary controller.

Energy pumping, synchronization and beat phenomenon in transients and steady-state vibrations (in a non-ideal structure) having an attachment, coupled to an oscillator containing essential nonlinearities, were studied in [8]. It investigated the effectiveness of the nonlinear energy sink in vibration attenuation to a non-ideal structure by using numerical simulations.

A multifrequency energy transfer from a two-mode, initially excited linear system to a multidegree-of-freedom essentially nonlinear attachment was analyzed in [9]. The study was performed utilizing numerical wavelet transforms and the authors showed that the transfer of energy occurs through simultaneous resonance interactions of both linear modes with a set of nonlinear normal modes (NNMs) of the attachment.

In works like [4, 10, 11] it was shown that a weakly coupled, essentially nonlinear attachment, if properly designed, may passively absorb energy from the linear nonconservative structure, acting as nonlinear energy sink. Then, energy pumping from the linear structure to the nonlinear attachment takes place. In [12], a design procedure for enhancing nonlinear energy pumping from a mode of a linear damped substructure to a weakly coupled, essentially nonlinear oscillator, was presented.

Additional study of dynamics and performance of essentially nonlinear attachments coupled to the linear system substructure was performed in [13]. In [14], it was shown theoretically and experimentally that an essentially nonlinear attachment of a single-degree-of-freedom linear system subjected to external force is capable of absorbing steady-state vibration energy from the linear oscillator localizing the energy far from the forced subsystem.

The energy pumping phenomena in two- and three-degrees-of-freedom systems of coupled linear and nonlinear oscillators was discussed in [15]. Different designs of systems allowing energy pumping are examined both analytically and numerically with the aim of designing the mechanical systems providing most effective and fast energy pumping.

All of these works have analyzed the energy transfer from the linear to nonlinear oscillator. Moreover, some works in this area try to solve the problem of minimal energy level necessary to reach the stable orbit responsible for energy pumping considering homogeneous initial conditions for the displacements for oscillators and for the initial velocity of the nonlinear oscillator. An impulse is applied as initial velocity (the energy of the system at $t = 0^+$) for the linear oscillator. So, depending on the level of energy applied, energy pumping occurs (see [1, 3, 4]).

The main distinction of the current work from the results presented in the mentioned works is that here we take into account *a more general dynamical phenomenon than energy pumping*. In this paper we extend previous results obtained in [16–18] about energy transfer between oscillators. Specifically, in these papers the authors provide analytical conditions for occurrence of energy transfer between oscillators considering a damped system. The interest is in the oscillations of the energy of each oscillator calculated on the orbits of the perturbed system. For oscillators coupled by springs, expansions of the energies, in a small parameter ε, were obtained.

Here we show numerically that depending on the initial conditions (not necessarily the case null-impulsive initial conditions), there is an energy transfer from linear to nonlinear oscillator with essential stiffness nonlinearity, or from nonlinear to linear oscillator, according to the theory developed in [16, 17]. In our numerical investigations we consider only the resonant case, since if the non-resonant condition holds, then there is no energy transfer [17].

The paper is organized as follows. In Sect. 2 we present a brief definition of energy transfer, which will be used in the remainder of the work. In Sect. 3 we perform a numerical study based on the results first developed by Dantas and Balthazar [17], in order to show the effectiveness of their theory for the case of nonlinear interaction between the oscillators. In Sect. 4 we show numerically the occurrence of energy transfer for oscillators under linear and cubic interactions, proving the analytical results obtained in [16]. Finally, in Sect. 5 we present the concluding remarks.

2 An overview of energy transfer

All results presented in this paper are based on the following definition of energy transfer [16]:

Let H_i, $i = 1, 2$, and R_j, $j = 1, \ldots, 4$, be functions adequately smoothly defined on open sets of $\mathbb{R}^2$ and $\mathbb{R}^6$, respectively. It is assumed that each open set contains the origin and each R_j is T-periodic in the variable t.

Consider the following system

$$\begin{cases} \dot{q}_1 = \frac{\partial H_1}{\partial p_1}(q_1, p_1) + \varepsilon R_1(q_1, p_1, q_2, p_2, t, \varepsilon), \\ \dot{p}_1 = -\frac{\partial H_1}{\partial q_1}(q_1, p_1) + \varepsilon R_2(q_1, p_1, q_2, p_2, t, \varepsilon), \\ \dot{q}_2 = \frac{\partial H_2}{\partial p_2}(q_2, p_2) + \varepsilon R_3(q_1, p_1, q_2, p_2, t, \varepsilon), \\ \dot{p}_2 = -\frac{\partial H_2}{\partial q_2}(q_2, p_2) + \varepsilon R_4(q_1, p_1, q_2, p_2, t, \varepsilon), \end{cases} \quad (1)$$

and define

$$\begin{cases} E_1(t, \varepsilon, a, b, c, d) = H_1\big(q_1(t, \varepsilon), p_1(t, \varepsilon)\big), \\ E_2(t, \varepsilon, a, b, c, d) = H_2\big(q_2(t, \varepsilon), p_2(t, \varepsilon)\big), \end{cases} \quad (2)$$

where $(q_1(t,\varepsilon),\ p_1(t,\varepsilon),\ q_2(t,\varepsilon),\ p_2(t,\varepsilon))$ is the solution of (1) such that $(q_1(0,\varepsilon),\ p_1(0,\varepsilon),\ q_2(0,\varepsilon),\ p_2(0,\varepsilon)) = (a, b, c, d)$.

It is said that there is a transfer of energy, from oscillator 1 to oscillator 2, in the point (a, b, c, d) of the phase space of the system given by (1), if the following condition is satisfied:

There is $T_0 = T_0(a, b, c, d) \geq 0$ such that for all finite time interval $[T_1, T]$, $T_1 \geq T_0$, there exists $\varepsilon_0 = \varepsilon_0(a, b, c, d, T) > 0$ such that

$$\begin{aligned} &E_1(t, \varepsilon, a, b, c, d) < E_1(0, \varepsilon, a, b, c, d), \\ &E_2(t, \varepsilon, a, b, c, d) > E_2(0, \varepsilon, a, b, c, d), \end{aligned} \quad (3)$$

for all $t \in [T_1, T]$ and $\varepsilon \in (0, \varepsilon_0)$.

Thus it follows from (2) that

$$\begin{aligned} &E_1(0, \varepsilon, a, b, c, d) = H_1(a, b), \\ &E_2(0, \varepsilon, a, b, c, d) = H_2(c, d). \end{aligned} \quad (4)$$

Note that $E_1(t, \varepsilon, a, b, c, d)$ and $E_2(t, \varepsilon, a, b, c, d)$ are the energies of the unperturbed oscillators, obtained from (1) taking $\varepsilon = 0$, computed on the perturbed trajectories. Moreover, in order to use correctly the Regular Perturbation Theory, it is necessary to work in a finite time interval. So, this restriction was included in the above definition.

3 Investigations on energy transfer of coupled linear oscillators with cubic interaction

First we consider the 2-DOF system composed of two coupled and damped oscillators [16, 18]:

$$\begin{cases} \ddot{x} + f(x) + \varepsilon\big(c_0\dot{x} + \frac{\partial V}{\partial x}(x, y) = 0\big), \\ \ddot{y} + g(y) + \varepsilon\big(c_0\dot{y} + \frac{\partial V}{\partial y}(x, y)\big) = 0, \end{cases} \quad (5)$$

where c_0 is the coefficient of the viscous damping. It is assumed that the bodies 1 and 2 have masses equal to 1. Here x is the displacement of the body 1 from its equilibrium position and y is the displacement of the body 2. Moreover, V is the potential energy associated to the coupling spring. It is assumed that $V(0, 0) = 0$.

Assuming that $f(x) = \omega_1^2 x$ and $g(y) = \omega_2^2 y$ in (5), the unperturbed energies are given by

$$\begin{aligned} H_1(q_1, p_1) &= \frac{p_1^2 + \omega_1^2 q_1^2}{2}, \\ H_2(q_2, p_2) &= \frac{p_2^2 + \omega_2^2 q_2^2}{2}. \end{aligned} \quad (6)$$

Now, taking the following change of variables

$$q_1 = x, \qquad p_1 = \dot{x}, \qquad q_2 = y, \qquad p_2 = \dot{y}, \quad (7)$$

and assuming that

$$V(q_1, q_2) = \frac{1}{4}\big(q_1^4 - 4q_1^3 q_2 + 6q_1^2 q_2^2 - 4q_1 q_2^3\big), \quad (8)$$

equation (5) can be written as a following set of four first-order equations:

$$\begin{cases} \dot{q}_1 = p_1, \\ \dot{p}_1 = -\omega_1^2 q_1 - \varepsilon\big(c_0 p_1 + \frac{\partial V}{\partial q_1}\big), \\ \dot{q}_2 = p_2, \\ \dot{p}_2 = -\omega_2^2 q_2 - \varepsilon\big(c_0 p_2 + \frac{\partial V}{\partial q_2}\big). \end{cases} \quad (9)$$

Now, we transform the equations of motion utilizing the following change of variables (action-angle variables):

$$\begin{aligned} q_1 &= \sqrt{\tfrac{2I}{\omega_1}}\sin\theta, & p_1 &= \sqrt{2\omega_1 I}\cos\theta, \\ q_2 &= \sqrt{\tfrac{2J}{\omega_2}}\sin\varphi, & p_2 &= \sqrt{2\omega_2 J}\cos\varphi. \end{aligned} \quad (10)$$

It therefore follows, from (2), (6) and (10), that

$$\begin{aligned} E_1 &= \omega_1 I, \\ E_2 &= \omega_2 J. \end{aligned} \quad (11)$$

Using (10) in (9), we get

$$\begin{cases} \dot{I} = \varepsilon\Big(-2c_0 I\cos^2\theta - \sqrt{\frac{2I}{\omega_1}}\cos\theta\frac{\partial V}{\partial q_1}\Big), \\ \dot{\theta} = \omega_1 + \varepsilon\Big(c_0\sin\theta\cos\theta + \frac{\sin\theta}{\sqrt{2\omega_1 I}}\frac{\partial V}{\partial q_1}\Big), \\ \dot{J} = \varepsilon\Big(-2c_0 J\cos^2\varphi - \sqrt{\frac{2J}{\omega_2}}\cos\varphi\frac{\partial V}{\partial q_2}\Big), \\ \dot{\varphi} = \omega_2 + \varepsilon\Big(c_0\sin\varphi\cos\varphi + \frac{\sin\varphi}{\sqrt{2\omega_2 J}}\frac{\partial V}{\partial q_2}\Big), \end{cases} \quad (12)$$

where $V = V(\sqrt{\frac{2I}{\omega_1}}\sin\theta, \sqrt{\frac{2J}{\omega_2}}\sin\varphi)$. From the basic theorem on the Differentiability of the Flow of Autonomous Equations, given $T > 0$, there is $\varepsilon_0 > 0$ such that

$$\begin{aligned} I &= I_0 + \varepsilon I_1 + O(\varepsilon^2), \\ \theta &= \theta_0 + \varepsilon\theta_1 + O(\varepsilon^2), \\ J &= J_0 + \varepsilon J_1 + O(\varepsilon^2), \\ \varphi &= \varphi_0 + \varepsilon\varphi_1 + O(\varepsilon^2) \end{aligned} \tag{13}$$

hold for all $t \in [0, T]$ and $0 < \varepsilon < \varepsilon_0$.

Substituting the initial conditions

$$\begin{aligned} I(0) &= e_1, \qquad \theta(0) = \alpha, \\ J(0) &= e_2, \qquad \varphi(0) = \beta, \end{aligned} \tag{14}$$

and (13) in (12), we obtain that

$$\begin{aligned} &I(t, e_1, \alpha, e_2, \beta, \varepsilon) \\ &\quad = e_1 + \varepsilon I_1(t, e_1, \alpha, e_2, \beta) + O(\varepsilon^2), \\ &J(t, e_1, \alpha, e_2, \beta, \varepsilon) \\ &\quad = e_2 + \varepsilon J_1(t, e_1, \alpha, e_2, \beta) + O(\varepsilon^2), \end{aligned} \tag{15}$$

where

$$\begin{aligned} I_1 &= -\int_0^t \biggl(2c_0 e_1 \cos^2(\omega_1 s + \alpha) \\ &\quad + \sqrt{\frac{2e_1}{\omega_1}} \cos(\omega_1 s + \alpha) \frac{\partial V}{\partial q_1}(\mathcal{A}, \mathcal{B})\biggr) ds, \\ J_1 &= -\int_0^t \biggl(2c_0 e_2 \cos^2(\omega_2 s + \beta) \\ &\quad + \sqrt{\frac{2e_2}{\omega_2}} \cos(\omega_2 s + \beta) \frac{\partial V}{\partial q_2}(\mathcal{A}, \mathcal{B})\biggr) ds, \end{aligned} \tag{16}$$

and

$$\begin{aligned} \mathcal{A} &= \mathcal{A}(s) = \sqrt{\frac{2e_1}{\omega_1}} \sin(\omega_1 s + \alpha), \\ \mathcal{B} &= \mathcal{B}(s) = \sqrt{\frac{2e_2}{\omega_2}} \sin(\omega_2 s + \beta). \end{aligned} \tag{17}$$

Now, we assume the cubic potential given by (8), and the resonance condition given by

$$\omega_1 = \omega_2. \tag{18}$$

From (10), (11), (14), (15) and (18) it is deduced that

$$\begin{aligned} E_1 &= e_1 + \varepsilon\bigl(-c_0 e_1 t + K_{11}(t, e_1, \alpha, e_2, \beta)\bigr) + O(\varepsilon^2), \\ E_2 &= e_2 + \varepsilon\bigl(-c_0 e_2 t + L_{11}(t, e_1, \alpha, e_2, \beta)\bigr) + O(\varepsilon^2), \end{aligned} \tag{19}$$

where E_1 and E_2 depend on $(t, \varepsilon, \sqrt{\frac{2e_1}{\omega_1}}\sin\alpha, \sqrt{2\omega_1 e_1}\cos\alpha, \sqrt{\frac{2e_2}{\omega_2}}\sin\beta, \sqrt{2\omega_2 e_2}\cos\beta)$, and

$$\begin{aligned} K_{11} &= -3\left(\sqrt{\frac{2e_1}{\omega_1}}\right)^3 \sqrt{\frac{2e_2}{\omega_2}} \\ &\quad \times \int_0^t \left(\frac{1}{8}\sin(\omega_1 - \omega_2)s + \alpha - \beta\right) ds \\ &\quad + 3\left(\sqrt{\frac{2e_1}{\omega_1}}\right)^3 \sqrt{\frac{2e_2}{\omega_2}} \\ &\quad \times \int_0^t \left(\frac{1}{8}\sin(3\omega_1 - \omega_2)s + 3\alpha - \beta\right) ds \\ &\quad + 3\left(\sqrt{\frac{2e_1}{\omega_1}}\sqrt{\frac{2e_2}{\omega_2}}\right)^2 \\ &\quad \times \int_0^t \left(\frac{1}{8}\sin(2\omega_1 - 2\omega_2)s + 2\alpha - 2\beta\right) ds \\ &\quad + \sqrt{\frac{2e_1}{\omega_1}}\left(\sqrt{\frac{2e_2}{\omega_2}}\right)^3 \\ &\quad \times \int_0^t \left(\frac{1}{8}\sin(\omega_1 - 3\omega_2)s + \alpha - 3\beta\right) ds \\ &\quad - \sqrt{\frac{2e_1}{\omega_1}}\left(\sqrt{\frac{2e_2}{\omega_2}}\right)^3 \\ &\quad \times \int_0^t \left(\frac{3}{8}\sin(\omega_1 - \omega_2)s + \alpha - \beta\right) ds \\ &\quad + I_{11}(t, e_1, \alpha, e_2, \beta), \end{aligned} \tag{20}$$

$$\begin{aligned} L_{11} &= \left(\sqrt{\frac{2e_1}{\omega_1}}\right)^3 \sqrt{\frac{2e_2}{\omega_2}} \\ &\quad \times \int_0^t \left(\frac{1}{8}\sin(\omega_2 - 3\omega_1)s - 3\alpha + \beta\right) ds \end{aligned}$$

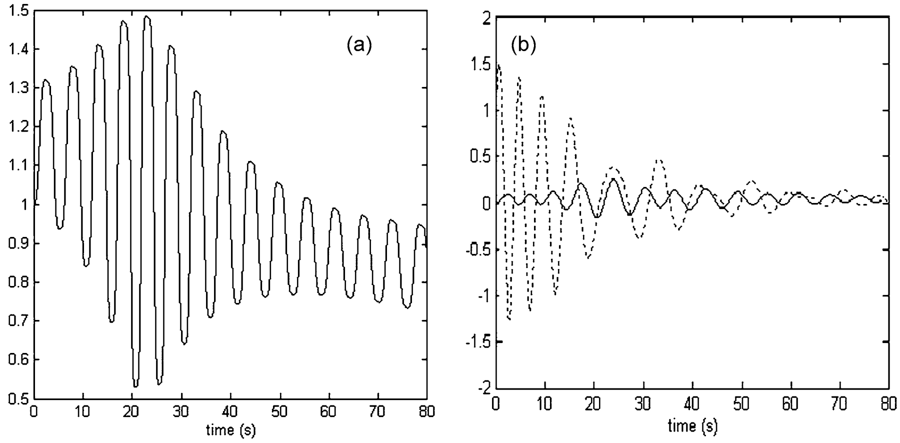


Fig. 1 (**a**) Energy of the oscillator 2. (**b**) Time history for oscillators satisfying the condition (**b**) in this subsection (the *dotted line* denotes the displacement of the oscillator 1 and the *solid line* of the oscillator 2)

$$
\begin{aligned}
&+\left(\sqrt{\frac{2e_1}{\omega_1}}\right)^3\sqrt{\frac{2e_2}{\omega_2}} \\
&\times\int_0^t\left(-\frac{3}{8}\sin(\omega_2-\omega_1)s-\alpha+\beta\right)ds \\
&-3\left(\sqrt{\frac{2e_1}{\omega_1}}\sqrt{\frac{2e_2}{\omega_2}}\right)^2 \\
&\times\int_0^t\left(-\frac{1}{8}\sin(2\omega_2-2\omega_1)s-2\alpha+2\beta\right)ds \\
&+3\sqrt{\frac{2e_1}{\omega_1}}\left(\sqrt{\frac{2e_2}{\omega_2}}\right)^3 \\
&\times\int_0^t\left(-\frac{1}{8}\sin(\omega_2-\omega_1)s-\alpha+\beta\right)ds \\
&+3\sqrt{\frac{2e_1}{\omega_1}}\left(\sqrt{\frac{2e_2}{\omega_2}}\right)^3 \\
&\times\int_0^t\left(\frac{1}{8}\sin(3\omega_2-\omega_1)s-\alpha+3\beta\right)ds \\
&-c_0e_2t+J_{11}(t,e_1,\alpha,e_2,\beta),
\end{aligned}
\tag{21}
$$

and I_{11}, J_{11} are bounded functions.

There are three possible resonances, 1:1, 1:3 and 3:1. Here we only consider the case of 1:1 resonance, $\omega_1=\omega_2=\omega$. Then, from (19), (20) and (21), it follows that

$$
\begin{aligned}
E_1&=e_1+\varepsilon\left[(-c_0e_1+\Psi)t+\bar{I}_{11}\right]+O\left(\varepsilon^2\right),\\
E_2&=e_2+\varepsilon\left[(-c_0e_2-\Psi)t+\bar{J}_{11}\right]+O\left(\varepsilon^2\right),
\end{aligned}
\tag{22}
$$

where

$$
\begin{aligned}
\Psi=&-\frac{3}{2\omega^2}\big[e_1e_2\sin(-2\beta+2\alpha)\\
&-\sqrt{e_1e_2}(e_1+e_2)\sin(-\beta+\alpha)\big],
\end{aligned}
\tag{23}
$$

and $\bar{I}_{11}$, $\bar{J}_{11}$ are bounded functions.

The problem of energy transfer reduces itself to the signal analysis of the coefficients of t in (22). So, three cases are possible:

(a) $\Psi<c_0e_1$ and $\Psi>-c_0e_2$: Both oscillators lose energy.
(b) $\Psi>c_0e_1$ and $\Psi>-c_0e_2$: The oscillator 1 loses energy and the oscillator 2 has an increase of energy.
(c) $\Psi<c_0e_1$ and $\Psi<-c_0e_2$: The oscillator 2 loses energy and the oscillator 1 has an increase of energy.

In the sequel, we present numerical simulations that confirm the above theoretical results. It is the first time in the literature that numerical results are presented as regards this theory.

Figure 1 shows the energy of the oscillator 2. In Fig. 1(b) the oscillator 1 loses energy and the oscillator 2 has an increase of energy. The simulations were performed with the parameters $\omega=1.0$, $c_o=1.0$, $\varepsilon=0.025$, and initial conditions $e_1=0.001$, $e_2=1.0$, $\alpha=\pi$ and $\beta=1.0$.

Considering initial conditions $e_1=0.001$, $e_2=0.001$, $\alpha=3.5$ and $\beta=0.5$, both oscillators perform damped free oscillations and no energy transfer occurs. This result is plotted in Fig. 2 for $\omega=1.01$, $c_o=1.0$ and $\varepsilon=0.005$.

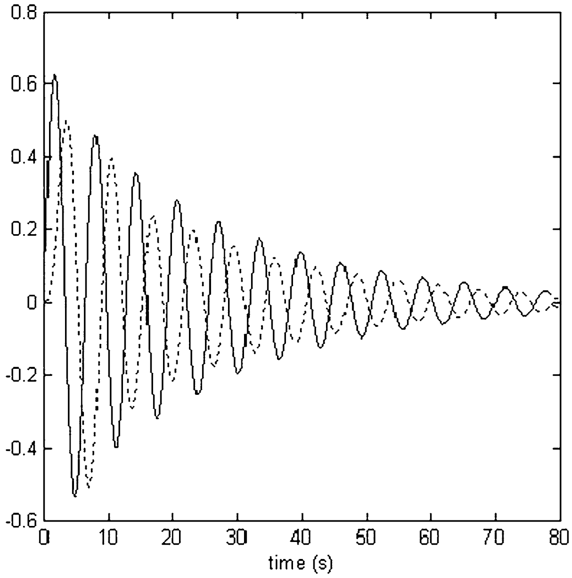


Fig. 2 Time history for oscillators satisfying the condition (a) in this subsection (the *dotted line* for oscillator 1 and the *solid line* for oscillator 2)

Note that we are interested in energy transfer according to the definition given by (3).

4 Coupled linear and nonlinear oscillators with linear interaction

Now assume that $f(x)=\omega_1^2 x,\ g(y)=y^3$ and that the oscillator 2 has mass m_1 in (5) [16]. Consider the system in the state variables (q_1, p_1, q_2, p_2),

$$\begin{cases} \dot q_1 = p_1, \\ \dot p_1 = -\omega_1^2 q_1 - \varepsilon\big(c_0 p_1 + \frac{\partial V}{\partial q_1} + A\ \sin(\omega t)\big), \\ \dot q_2 = p_2, \\ \dot p_2 = -\frac{1}{m_1} q_2^3 - \varepsilon \frac{1}{m_1}\big(c_0 p_2 + \frac{\partial V}{\partial q_2}\big), \end{cases} \tag{24}$$

and assume also that

$$V(q_1, q_2) = \frac{(q_1-q_2)^2}{2}. \tag{25}$$

Now we use the following change of variables: $q_1 = \sqrt{\frac{2I}{\omega_1}}\sin\theta,\ p_1=\sqrt{2\omega_1 I}\cos\theta,\ q_2=\sqrt{m_1}J cn\varphi$ and $p_2 = -\sqrt{m_1}J^2 sn\varphi\, dn\varphi = \sqrt{m_1}J^2 cn'\varphi$, where $cn[t,k]$, $sn[t,k]$ and $dn[t,k]$ are the elliptic Jacobian functions with argument $k=\frac{1}{\sqrt{2}}$.

It therefore follows from the new change of variables and (2) that

$$E_1 = I, \qquad E_2 = m_1 \frac{J^4}{4}. \tag{26}$$

Substituting the new variables in (24), one obtains

$$\begin{cases} \dot I = \varepsilon\big(-2c_0 I\cos^2\theta - \sqrt{\frac{2I}{\omega_1}}\cos\theta\frac{\partial V}{\partial q_1} \\ \qquad - A\sqrt{\frac{2I}{\omega_1}}\cos\theta\sin(\omega t)\big), \\ \dot\theta = \omega_1 + \varepsilon\big(c_0\sin\theta\cos\theta + \frac{\sin\theta}{\sqrt{2\omega_1 I}}\frac{\partial V}{\partial q_1} \\ \qquad + A\frac{1}{\sqrt{2\omega_1 I}}\sin\theta\sin(\omega t)\big), \\ \dot J = \varepsilon\big(-\frac{c_0 J}{m_1}cn'\varphi^2 - \frac{1}{J\sqrt{m_1^3}}cn'\varphi\frac{\partial V}{\partial q_2}\big), \\ \dot\varphi = J + \varepsilon\big(\frac{c_0}{m_1}cn\varphi\, cn'\varphi + \frac{1}{J^2\sqrt{m_1^3}}cn\varphi\frac{\partial V}{\partial q_2}\big). \end{cases} \tag{27}$$

Taking $I(0)=e_1,\ \theta(0)=\alpha,\ J(0)=e_2$ and $\varphi(0)=\beta$ as initial conditions, we get similar relations as (13) and (15), where in (15) the terms I_1 and J_1, in this case, are given by

$$\begin{aligned} &I_1(t, e_1, \alpha, e_2, \beta) \\ &= -\frac{c_0 e_1}{2}t + \frac{2K}{\pi}\sqrt{\frac{2m_1 e_1}{\omega_1}}\sum_{m=0}^{\infty} a_m \cos \\ &\quad\times\left(\alpha - \frac{\omega_1\beta}{e_2}\right)\Lambda_1 \\ &\quad - \frac{2AK}{\pi e_2}\sqrt{\frac{2e_1}{\omega_1}}\Lambda_2 + b_1(t,\alpha,\beta) \end{aligned} \tag{28}$$

and

$$\begin{aligned} &J_1(t, e_1, \alpha, e_2, \beta) \\ &= -\frac{c_0 e_2 k}{2m_1}t - \frac{1}{e_2^2}\frac{1}{\sqrt{m_1^3}}\sqrt{\frac{2e_1}{\omega_1}}\cos\left(\alpha - \frac{\omega_1\beta}{e_2}\right) \\ &\quad\times\sum_{m=0}^{\infty}(2m+1)a_m\Lambda_3 + b_2(t,\alpha,\beta), \end{aligned} \tag{29}$$

where

$$\Lambda_1 = \int_{\frac{\pi\beta}{2K}}^{\frac{\pi e_2}{2K}(t+\frac{\beta}{e_2})} \frac{1}{2}\cos\left(\left(-2\frac{K\omega_1}{\pi e_2} + 2m+1\right)u\right)du,$$

$$\Lambda_2 = \int_{\frac{\pi\beta}{2K}}^{\frac{\pi}{2K}(e_2 t+\beta)} \cos\left(\frac{2K\omega_1}{\pi e_2}u - \frac{\omega_1\beta}{e_2} + \alpha\right)$$

$$\times\left(\sin\left(\frac{2K\omega}{\pi e_2}u-\frac{\beta\omega}{e_2}\right)\right)du, \tag{30}$$

$$\Lambda_3=\int_{\frac{\pi\beta}{2K}}^{\frac{\pi}{2K}(e_2t+\beta)}\left(\frac{1}{2}\cos\left(\left(2m+1-\frac{2K\omega_1}{\pi e_2}\right)u\right)\right)du.$$

Here b_1 and b_2 are limited functions.

In (28) and (29) if the term

$$-2\frac{K\omega_1}{\pi e_2}+2m+1 \tag{31}$$

is zero for some m, an unbounded term is obtained after integration, and it has effect on the energy transfer. This case is called *internal resonance*. If $\omega_1=\omega$ in (28), there is another unbounded term, and we have the *external resonance*. Next, the external resonant case will be considered.

It is assumed that for some $m_0\in\mathbb{N}$, the following conditions are satisfied:

$$e_2=\frac{2K\omega_1}{(2m_0+1)\pi},\qquad \omega_1=\omega, \tag{32}$$

where K is given by

$$K=\int_0^1\left(1-t^2\right)^{-\frac{1}{2}}\left(1-\frac{1}{2}t^2\right)^{-\frac{1}{2}}dt.$$

Note that if there is no external excitation, the first condition in (32) means that the unperturbed system, obtained taking $\varepsilon=0$ in (24), has a periodic solution. Indeed, the above condition can be interpreted as a $(2m_0+1):1$ internal resonance of the unperturbed system in order to get a periodic motion. Note that in this system there are two oscillators, a linear one and a nonlinear one. For each oscillator, periodic solutions are given by $x=A\sin t$, $y=Bcn(Bt)$. In order for $t\to(x(t),y(t))$ to be periodic, the condition given in (32) is sufficient. If $m_0=0$ and $\alpha=\pi/2$, $\beta=0$ in (14) then that periodic solution is a nonlinear normal mode in the sense of Rosenberg [19].

From (28), (29) and (32), one obtains that

$$E_1=e_1+\varepsilon\left(-\frac{c_0e_1}{2}+a_{m_0}e_2\sqrt{\frac{m_1e_1}{2\omega}}\cos\left(\alpha-\frac{\omega\beta}{e_2}\right)+A\sqrt{\frac{e_1}{2\omega}}\sin\alpha+O(1)\right)t+O\left(\varepsilon^2\right), \tag{33}$$

$$E_2=m_1\frac{e_2^4}{4}+m_1e_2^3\varepsilon\left(-\frac{c_0e_2k}{2m_1}-\cos\left(\alpha-\frac{\omega\beta}{e_2}\right)\times\frac{(2m_0+1)\pi a_{m_0}}{4Ke_2}\sqrt{\frac{2e_1}{\omega m_1^3}}+O(1)\right)t+O\left(\varepsilon^2\right),$$

where k and a_{m_0} depend on the Fourier expansion of the $cn(t)$.

From now on, in this subsection, it is assumed that $A=0$. So, in (33), ω is replaced by ω_1.

Using the same steps as were used in the arguments of the previous subsection, the problem of energy transfer reduces itself to the signal analysis of the coefficients of t in (33). Define

$$\begin{aligned}\Psi_1&=-\frac{c_0e_1}{2}+a_{m_0}e_2\sqrt{\frac{m_1e_1}{2\omega_1}}\cos\left(\alpha-\frac{\omega_1\beta}{e_2}\right),\\ \Psi_2&=-\frac{c_0e_2k}{2m_1}-\cos\left(\alpha-\frac{\omega_1\beta}{e_2}\right)\times\frac{(2m_0+1)\pi a_{m_0}}{4Ke_2}\sqrt{\frac{2e_1}{\omega_1m_1^3}}.\end{aligned} \tag{34}$$

So, there are three cases to be considered:

(a) If $\Psi_1<0$ and $\Psi_2>0$ then energy pumping occurs.
(b) If $\Psi_1>0$ and $\Psi_2<0$ then the linear oscillator suffers an increase of energy and the nonlinear oscillator loses energy.
(c) If $\Psi_1<0$ and $\Psi_2<0$ then both oscillators lose energy.

Observe that from (32), the conditions (a), (b) and (c) are equivalent to, respectively,

$$\cos\left(\alpha-\frac{\omega_1\beta}{e_2}\right)<-\frac{4c_0kK^3\sqrt{2\omega_1m_1}}{\pi^3(2m_0+1)^3a_{m_0}\sqrt{e_1}}, \tag{35}$$

$$\cos\left(\alpha-\frac{\omega_1\beta}{e_2}\right)>\frac{\pi c_0(2m_0+1)}{4a_{m_0}K}\sqrt{\frac{2e_1\omega_1}{m_1}}, \tag{36}$$

$$-\frac{4c_0kK^3\sqrt{2\omega_1m_1}}{\pi^3(2m_0+1)^3a_{m_0}\sqrt{e_1}}<\cos\left(\alpha-\frac{\omega_1\beta}{e_2}\right)<\frac{\pi c_0(2m_0+1)}{4a_{m_0}K}\sqrt{\frac{2e_1\omega_1}{m_1}}. \tag{37}$$

Now, numerical results, presented for the first time in this text, are performed to confirm some of the above analytical results.

For all numerical simulations in this section we have used the following parameters: $\omega=1.01$,

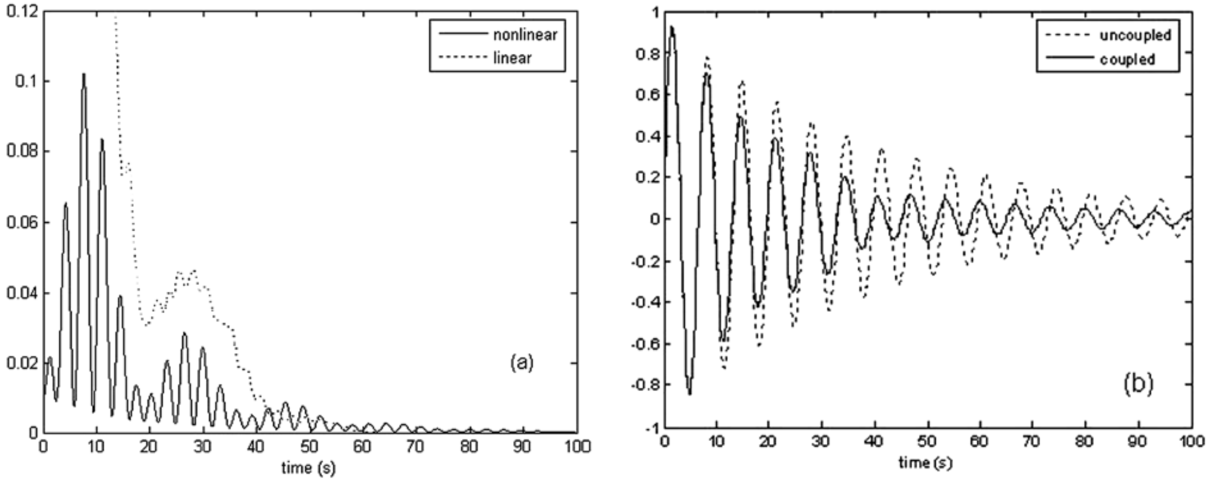


Fig. 3 (**a**) Time history for energies I and J. (**b**) Time history for displacements (24)

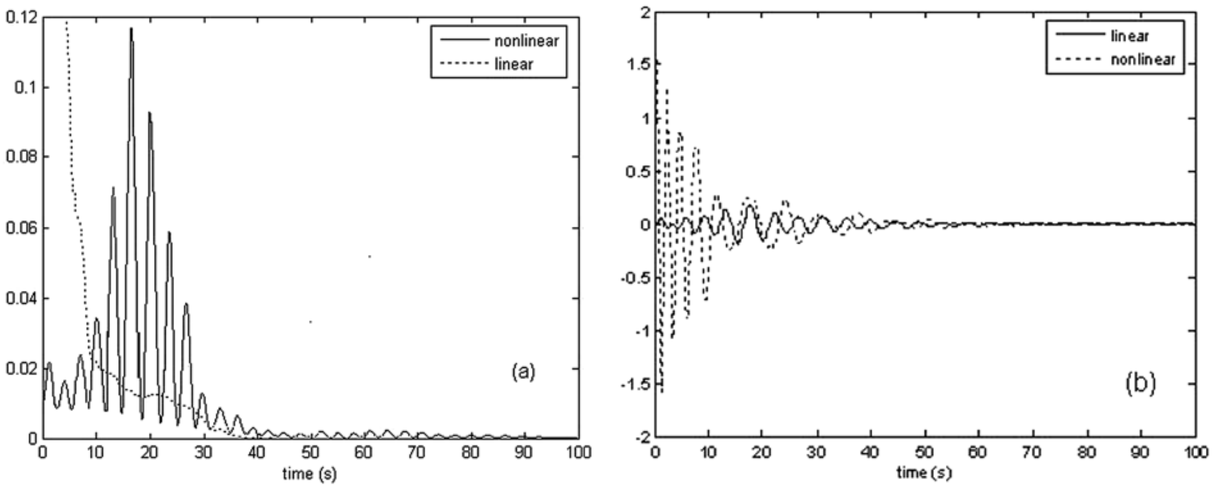


Fig. 4 Energy transfer from nonlinear to linear oscillator. (**a**) Energies of the oscillators. (**b**) Transient response of (24)

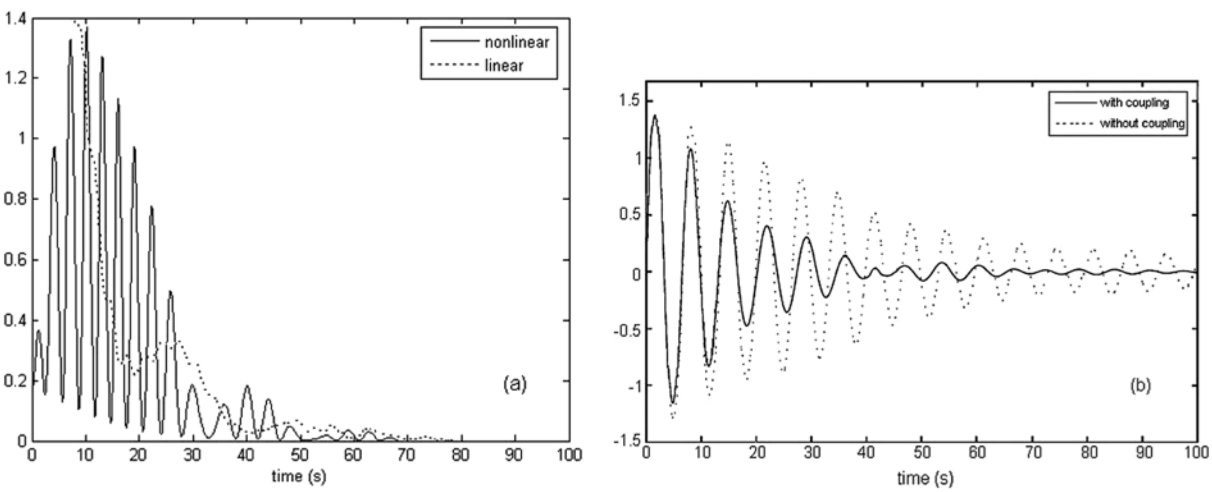
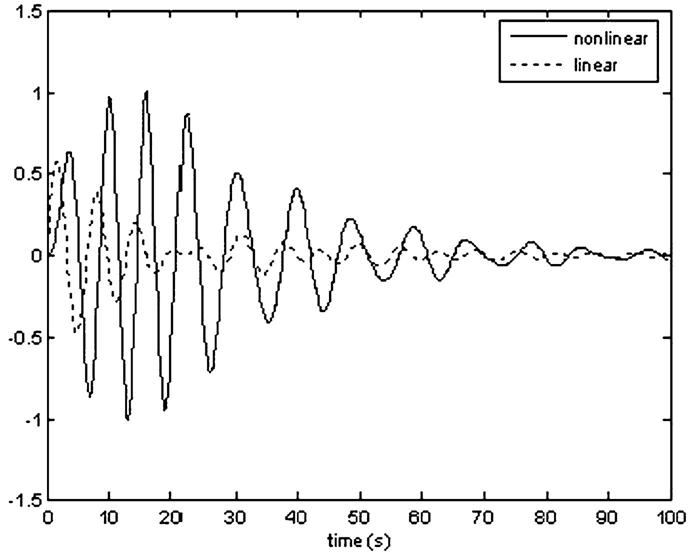


Fig. 5 Energy transfer from linear to nonlinear oscillator. (**a**) Energies. (**b**) Transient response of (24)

Fig. 6 Transient response of each oscillator when energy pumping occurs



$c_0 = 1.0$, $k^2 = 0.5$, $K = 1.854075$, $m_0 = 3.0$, $a_{m_0} = 0.95$ and $\varepsilon = 0.025$. Particular cases satisfying the above conditions (a), (b) and (c) were obtained for the system subjected to different initial conditions. For the plots in Fig. 3 we choose the initial conditions $e_1 = 1.0$, $e_2 = 0.01$ and $\alpha = \pi$, $\beta = \frac{\pi}{4}$. This situation corresponds to $\Psi_1 < 0$ and $\Psi_2 < 0$, that is, both oscillators perform damped free oscillations and no energy transfer occurs. In Fig. 3, we observe that the energies I and J (see (26) and (27)) decay nearly exponentially to zero indicating the absence of resonance capture.

In Fig. 4 we present numerical results for the case $\Psi_1 > 0$ and $\Psi_2 < 0$. Here we have used the initial conditions $e_1 = e_2 = 1.0$, $\alpha = \pi$ and $\beta = \frac{\pi}{4}$, and condition (b) in this subsection is satisfied. In Fig. 4(a) we can observe that energy transfer from the nonlinear to the linear oscillator occurs. This is indicated by the fact that as time progresses the energy I surpasses energy J (see (26) and (27)). Finally, in Fig. 5 the irreversible energy transfer (pumping) from the linear to the nonlinear oscillator takes place. The numerical time decays of energies are depicted in Fig. 5(a), for initial conditions $e_1 = 1.0$, $e_2 = 0.002$ and $\alpha = \pi$ and $\beta = \frac{\pi}{4}$. Figure 5(b) depicts the transient response of the linear oscillator considering coupled and uncoupled system (24), and Fig. 6 depicts the motion of the linear oscillator, together with the nonlinear attachment.

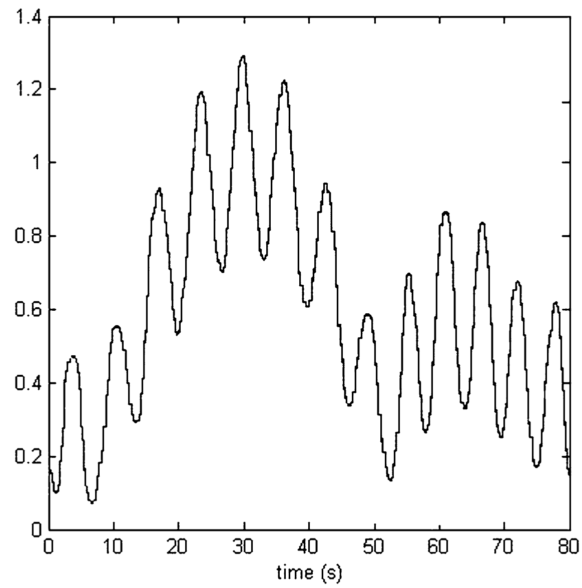


Fig. 7 Energy of the nonlinear attachment

4.1 Cubic interaction

If the potential V is given by (8), similar results can be obtained by using exactly the earlier approach. The resonance condition is given by

$$\begin{aligned}&\text{(a) } e_2 = \frac{6K\omega_1}{\pi(2m_0+1)} \text{ for some } m_0 \in \mathbb{N},\\&\text{(b) } \omega_1 = \omega.\end{aligned} \tag{38}$$

There are several possibilities for resonances: we have to consider the cases $\gcd(2m_0 + 1, 3) = 1$ or 3, where

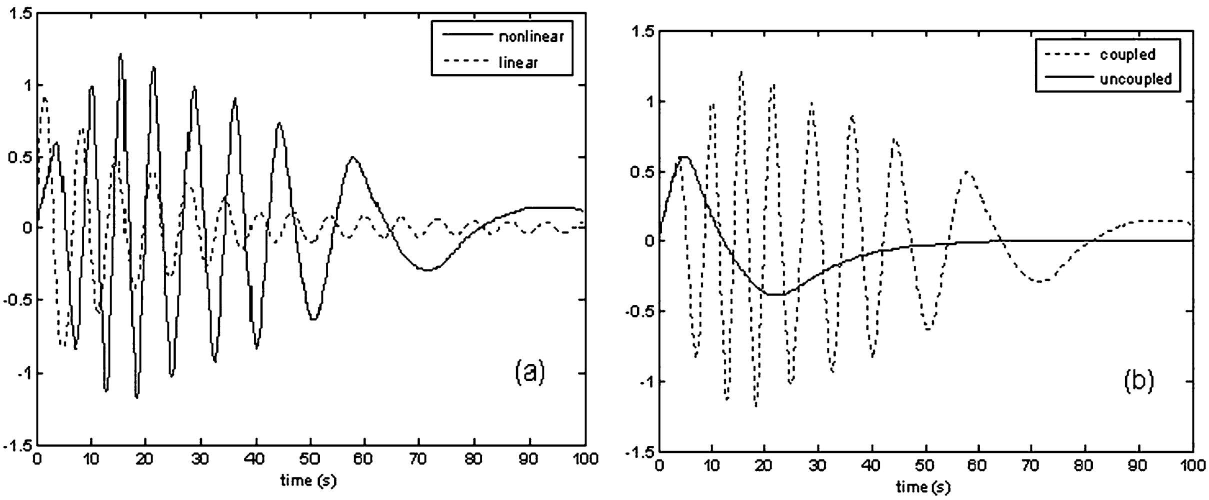


Fig. 8 (**a**) Resonance of the nonlinear oscillator and energy pumping occurrence. (**b**) Time history for nonlinear oscillator

gcd (*z, w*) refers to the *greatest common divisor for z and w*. Anyway, although the computations to obtain this and other estimations are very extensive, the results are similar to those in the previous section.

Results considering (8) and condition (38) are showed in Figs. 7 and 8 for the case of energy pumping occurrence.

Figures 7 and 8 show the responses of the system when subjected to nonlinear interaction. Choosing the parameter value $m_1 = 1$ and initial conditions $e_1 = 0.35$, $e_2 = 0.1$, $\alpha = 0.18$ and $\beta = 0.9$, energy pumping occurs as shown in Fig. 8.

5 Conclusions

In this paper we present results of the first of a series of works that are being developed in this area by the authors. The novel analytical approach for prediction of energy transfer in a dissipative mechanical system, based on the assumption of 1:1 internal resonance, obtained from references [16, 17], were being demonstrated numerically. Two types of interaction between oscillators were considered: the linear and the cubic one. We showed that, by choosing numerical parameters that satisfy certain initial conditions, there is energy transfer from linear to nonlinear oscillator, from nonlinear to linear oscillator, and between linear oscillators. In this work we have analyzed the energy transfer, particularly, the energy pumping phenomenon, for a more general case than the impulsively excited, damped system where the initial conditions are given all zero values except for the initial velocity of the linear oscillator. For all cases considered we have showed that the simulations are in complete agreement with the theoretical results.

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