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3-scroll and 4-scroll chaotic attractors generated from a new 3-D quadratic autonomous system

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Abstract This article introduces a new chaotic system of 3-D quadratic autonomous ordinary differential equations, which can display 2-scroll chaotic attractors. Some basic dynamical behaviors of the new 3-D system are investigated. Of particular interest is that the chaotic system can generate complex 3-scroll and 4-scroll chaotic attractors. Finally, bifurcation analysis shows that the system can display extremely rich dynamics. The obtained results clearly show that this is a new chaotic system which deserves further detailed investigation.

Keywords Chaotic attractor · 3-D quadratic autonomous system · 3-scroll · 4-scroll · Bifurcation

1 Introduction

Chaos as a very interesting complex nonlinear phenomenon has been intensively studied in the last four decades within the science, mathematics and engineering communities (see, for example, Refs. [1–5] and many references cited therein). Recently, chaos has been found to be very useful and has great potential in many technological disciplines, such as information and computer sciences, power systems protection, biomedical systems analysis, flow dynamics and liquid mixing, encryption and communications, and so on. It is not surprising, therefore, that academic research on chaotic dynamics has evolved from the traditional trend of analyzing and understanding chaos to the new direction of controlling and utilizing it.

In particular, generating complex multi-scroll or multi-wring chaotic attractors from 3-D autonomous systems has seen rapid development. In this endeavor, there are two major efforts: generalizing Chua's circuit with multi-scroll attractors and generalizing the Lorenz system with multi-wing attractors. Chua's circuit [6–8], as a paradigm of chaos and a bridge between electronic circuits and the chaos theory, has been widely studied and used as a platform for engineering applications. The Lorenz system and Lorenz-like system were also intensively studied in the last decades (see, for example, Refs. [3–5, 9–15] and many references cited therein).

It is noted that there exist a number of results about how to generate multi-scroll chaotic attractors, and hence it is no longer a very difficult task. However, how to generate multi-scroll chaotic attractors form a 3-D smooth system remains a technical challenge. In such an attempt, a 3-D autonomous quadratic system with five equilibria was proposed by Liu [16]. Two other autonomous quadratic chaotic systems with five equilibria were designed by Lü [4] and

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Qi [10], showing that generating 4-scroll chaotic attractors from 3-D smooth systems is possible. Very recently, Li [17] proposed a 3-D smooth system with three equilibria. It was found that a 3-scroll attractor can be generated. Moreover, it was shown that Li's chaotic attractor has a more complex topological structure than the classical 2-scroll Lorenz attractor.

This paper introduces a new 3-D smooth autonomous system with five equilibria, in which each equation contains a single quadratic term. Again, the system can generate 2-scroll chaotic attractor. However, a new 3-scroll chaotic attractor is detected in the new 3-D smooth system proposed in the current work. This 3-scroll chaotic attractor finally evolves into a 4-scroll chaotic attractor in some way.

2 The new 3-D chaotic system

A new 3-D chaotic system is proposed in this paper. The autonomous differential equations that describe the system are

$$\begin{aligned}\dot{x} &= a(x-y) - yz,\\ \dot{y} &= -by + xz,\\ \dot{z} &= -cz + dx + xy,\end{aligned} \tag{1}$$

where a, b, c and $d \in R^+$ are constant parameters of the system.

The system is found to be periodic in a wide parameter range and has many interesting complex dynamical behaviors. For example, as shown in Fig. 1, it is periodic with the parameters $a = 3.1, b = 9, c = 5$ and $d = 0.06$.

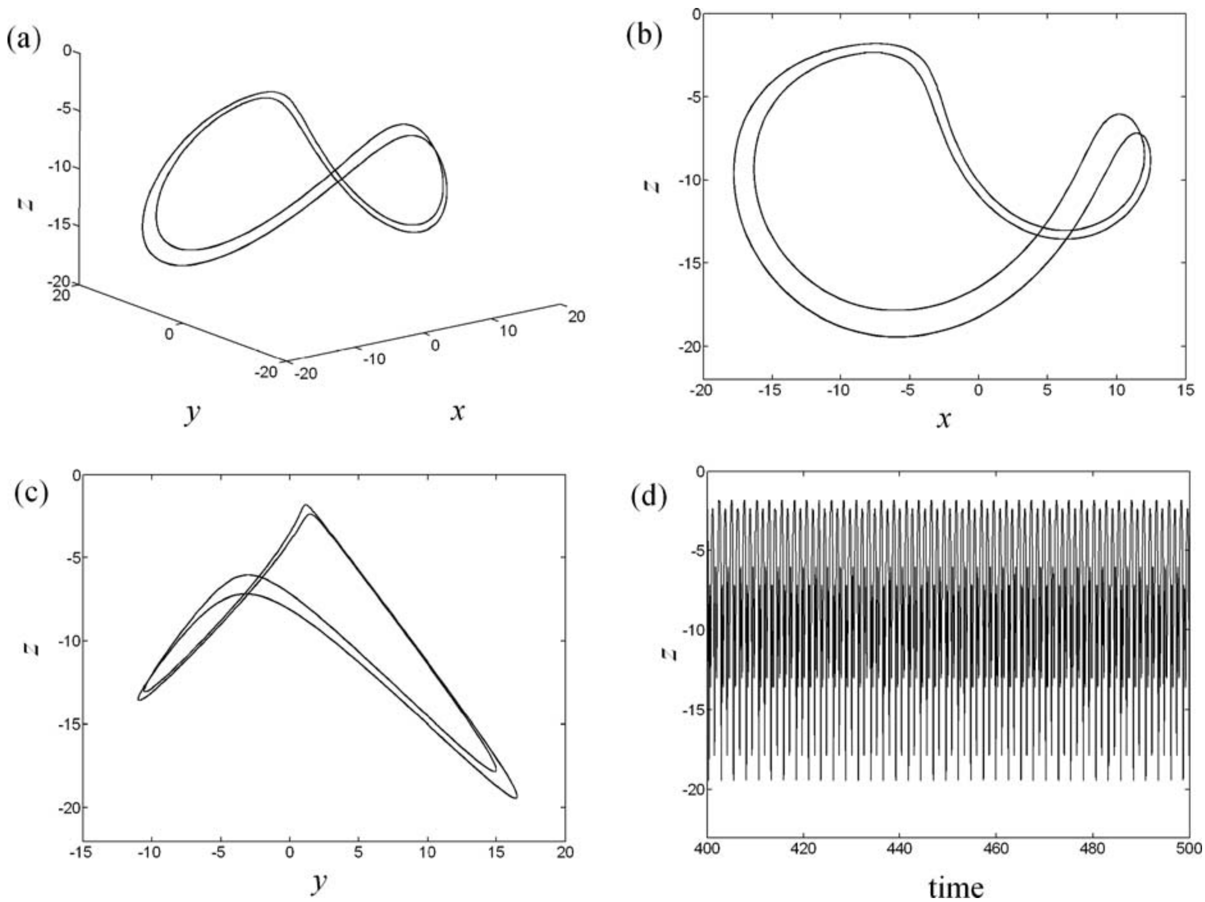


Fig. 1 Periodic orbits generated form system (1): $a = 3.1, b = 9, c = 5$ and $d = 0.06$

3 Some basic properties of the new 3-D system

Now, some basic properties of the system (1) are analyzed.

(1) The fixed points of system (1) can be easily found by solving the following system of equations:

$$\begin{aligned} &a(x-y)-yz=0, \qquad -by+xz=0,\\ &-cz+dx+xy=0. \end{aligned} \tag{2}$$

The system has five fixed points, which are respectively described as follows:

$$O(0,0,0), \qquad S_1(x_1,y_1,z_1), \qquad S_2(x_2,y_2,z_2),$$
$$S_3(x_3,y_3,z_3), \qquad S_4(x_4,y_4,z_4).$$

After operating the above three nonlinear algebraic equations, one obtains

$$x_1=\frac{-bd+\Gamma}{-a+\Delta}, \qquad x_2=\frac{-bd-\Gamma}{-a+\Delta},$$
$$x_3=\frac{-bd+\Lambda}{-a-\Delta}, \qquad x_4=\frac{-bd-\Lambda}{-a-\Delta},$$
$$z_1=z_2=-\frac{a}{2}+\frac{\Delta}{2}, \qquad z_3=z_4=-\frac{a}{2}-\frac{\Delta}{2},$$
$$y_1=\frac{x_1z_1}{b}, \qquad y_2=\frac{x_2z_2}{b},$$
$$y_3=\frac{x_3z_3}{b}, \qquad y_4=\frac{x_4z_4}{b},$$

where

$$\Delta=\sqrt{a^2+4ab}, \qquad \Gamma=\sqrt{(bd)^2+bc(-a+\Delta)^2},$$
$$\Lambda=\sqrt{(bd)^2+bc(-a-\Delta)^2}.$$

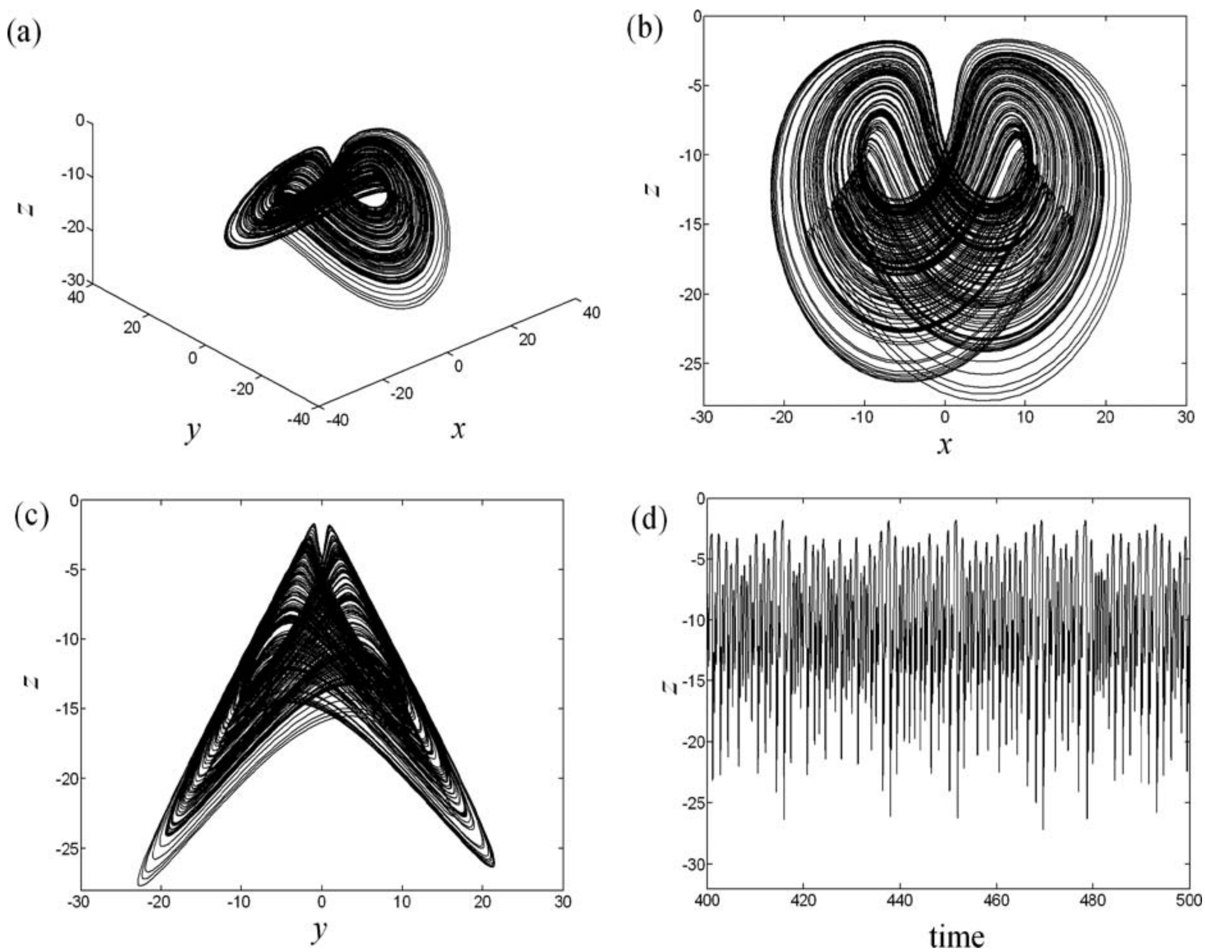


Fig. 2 The lower two-scroll attractor; $a=4.15$, $b=10$, $c=4$ and $d=0.1$

For the equilibrium $O(0,0,0)$, system (1) is linearized and the Jacobian matrix is defined as

$$J_0 = \begin{bmatrix} a & -a & -y \\ z & -b & x \\ d+y & x & -c \end{bmatrix} = \begin{bmatrix} a & -a & 0 \\ 0 & -b & 0 \\ d & 0 & -c \end{bmatrix}.$$

In order to gain its eigenvalues, we let

$$|\lambda I - J_0| = 0.$$

The eigenvalues corresponding to equilibrium $O(0, 0, 0)$ are obtained as follows:

$$\lambda_1 = a, \qquad \lambda_2 = -b, \qquad \lambda_3 = -c.$$

As $a, b, c \in R^+$, λ_1 is a positive real number, λ_2 and λ_3 are two negative real numbers. This means the equilibrium $O(0,0,0)$ is a saddle point. Hence, the equilibrium $O(0,0,0)$ is unstable and yields a possibility for chaos.

(2) In order to ensure for system (1) to be chaotic, just like the typical 3-D autonomous chaotic systems such as the Lorenz system, it is required thats

- system (1) be dissipative, that is

$$\Delta V = \frac{\partial \dot{x}}{\partial x} + \frac{\partial \dot{y}}{\partial y} + \frac{\partial \dot{z}}{\partial z} = a - b - c < 0;$$

- all the five equilibria of system (1) be unstable.

It is noticed that a 3-D conservative quadratic autonomous system can also display chaos. Indeed, Sprott [11] found an example of a 3-D conservative autonomous chaotic system. Here, only the dissipative system case is discussed.

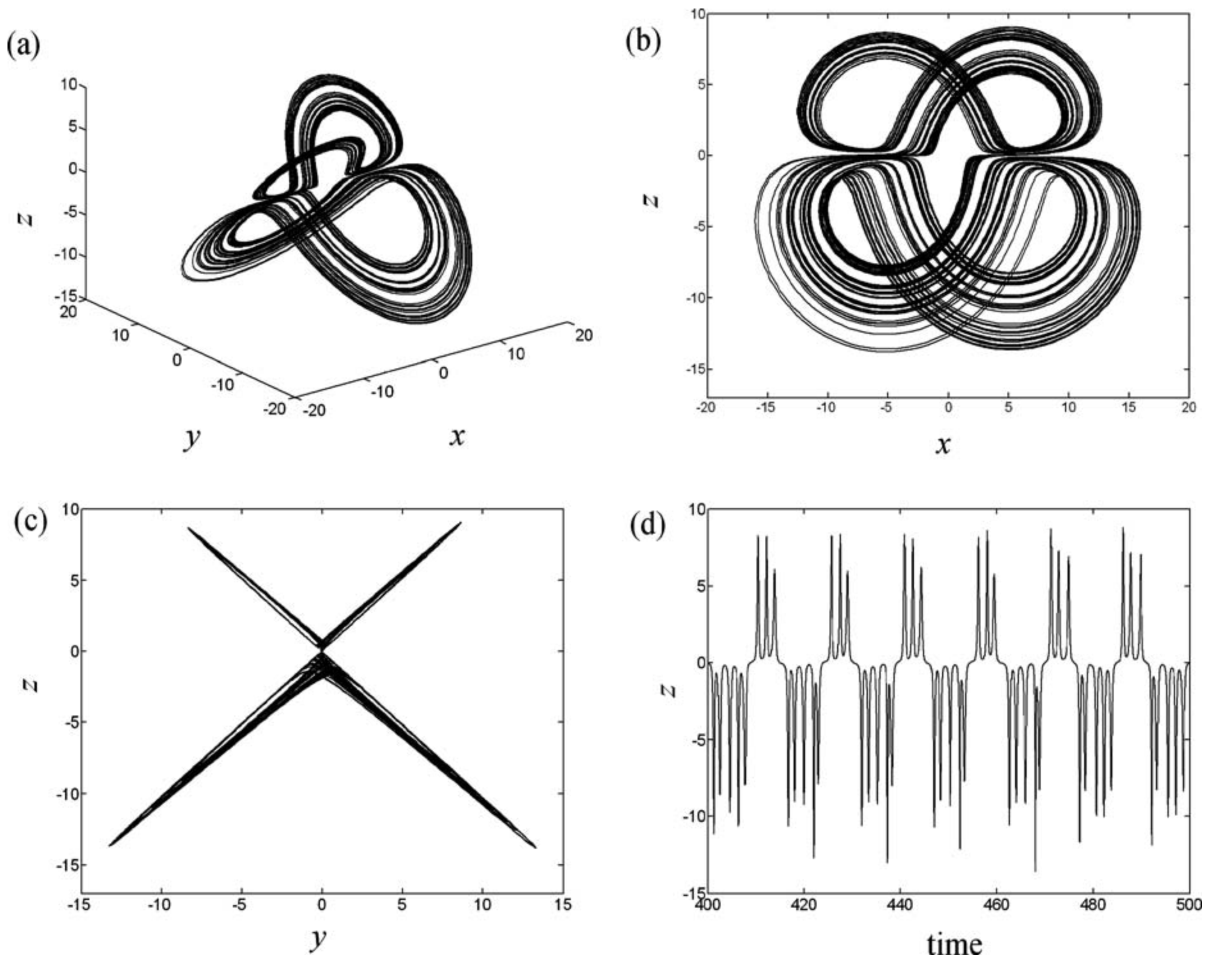


Fig. 3 The four-scroll attractor; $a = 1, b = 5.7, c = 5$ and $d = 0.06$

When $a, b, c, e \in R^+$, system (1) has five equilibria. To have chaotic behavior, these equilibria cannot be stable, that is, the Jacobian should have at least one unstable eigenvalue when it is evaluated at each of these equilibria.

4 Multi-scroll attractors generated from system (1)

4.1 The 2-scroll chaotic attractor

This system has been found to be chaotic over a wide range of parameters and has many interesting complex dynamical behaviors. For example, when $a = 4.15, b = 10, c = 4$ and $d = 0.1$, it is chaotic. Figure 2 shows the observed dynamics. The 2-scroll chaotic attractor shown in Fig. 2 is called the lower-left attractor or the lower-right attractor, according to its geometric location.

Since $a = 5, b = 10, c = 4$ and $d = 0.1$ are all positive real numbers in system (1), it is found that $x_1, x_2, x_3, x_4, y_1, y_2, y_3$ and y_4 are real numbers, that is, the system (1) has five real equilibria.

4.2 The 4-scroll chaotic attractor

With parameters $a = 1, b = 5.7, c = 5$ and $d = 0.06$, Fig. 3 shows the 4-scroll chaotic attractor. It can be seen that there exist many orbits freely running not only around $S_{1,2}$, but also around $S_{3,4}$. In Fig. 4, a similar 4-scroll chaotic attractor is given with parameters $a = 1.46, b = 9, c = 5$ and $d = 0.06$. From the 3-D view (Fig. 4a), it is clearly observed that the four sub-attractors look like the inner corns of four connected eddies. The orbits between any two equilibria

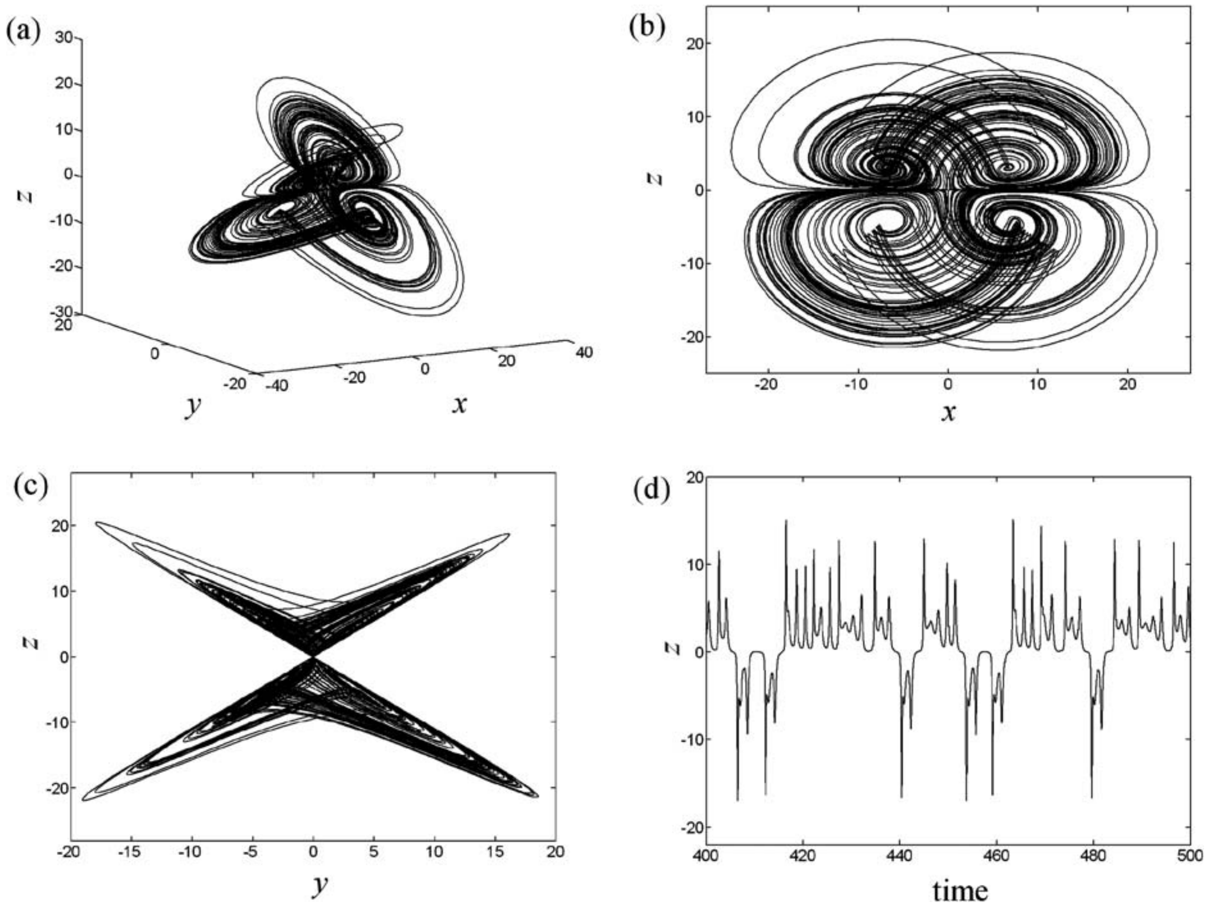


Fig. 4 The four-scroll attractor; $a = 1.46, b = 9, c = 5$ and $d = 0.06$

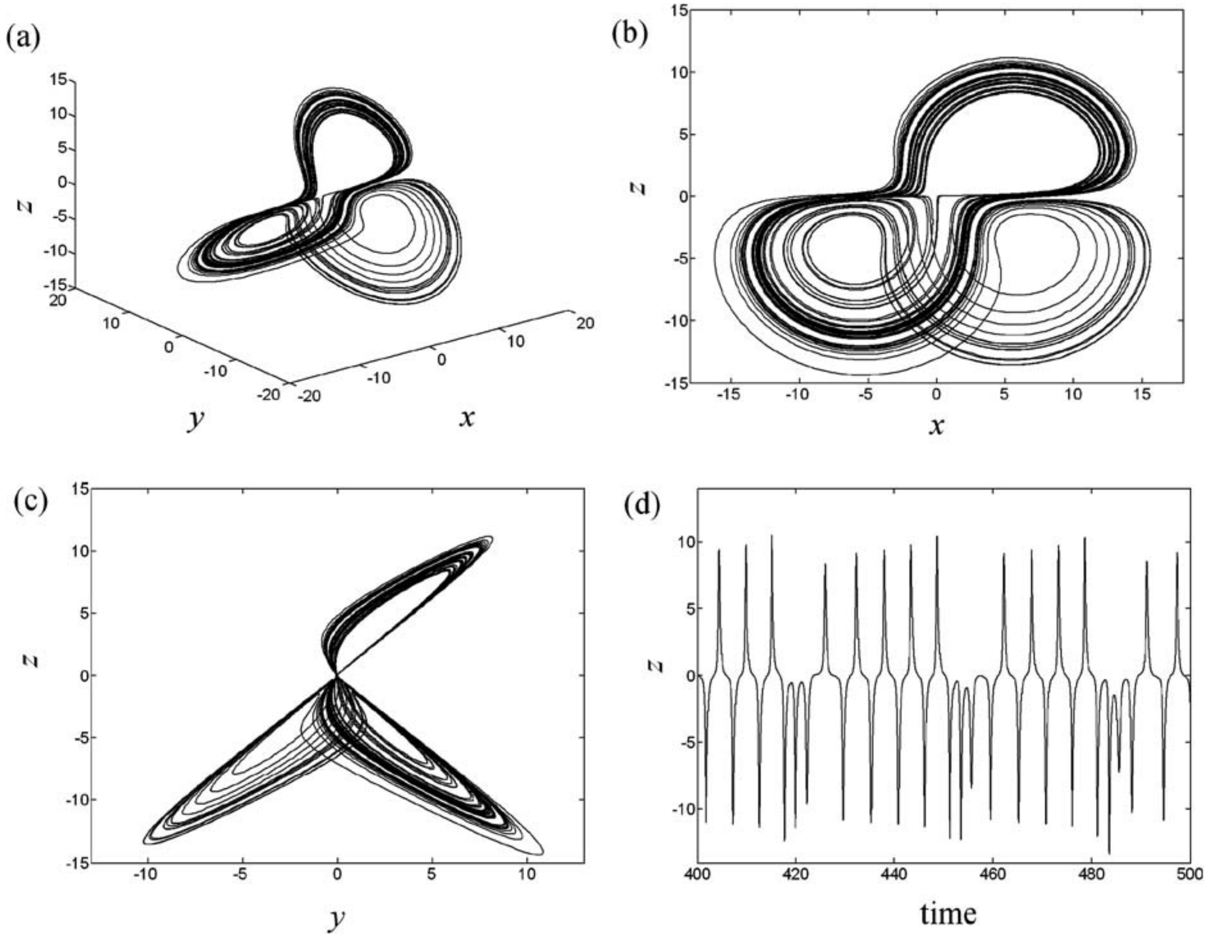
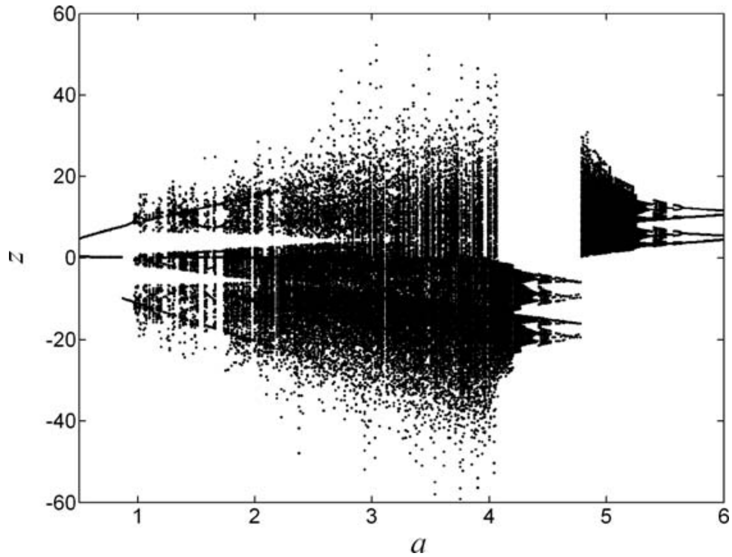


Fig. 5 The three-scroll attractor; $a = 0.977$, $b = 10$, $c = 4$ and $d = 0.1$

Fig. 6 Bifurcation diagram for the range $0.5 < a < 6$; $b = 10$, $c = 4$ and $d = 0.1$



resemble the butterfly shape of the Lorenz chaotic attractor, which as a whole forms a singular tornado-like shape with four inner holes.

4.3 The 3-scroll chaotic attractor

With parameters $a = 0.977, b = 10, c = 4$ and $d = 0.1$, Fig. 5 shows that the 3-D quadratic autonomous system (1) can exhibit an attractor with three scrolls. Hence, it has a more complex topological structure than the classical 2-scroll Lorenz attractor. Notably, the third scroll, the so-called upper-right attractor, connects the other two scrolls (i.e. the lower two scrolls).

5 Bifurcation analysis

In the following, the new 3-D system is investigated by means of bifurcation diagram, phase plots and power spectrum diagrams.

Figure 6 shows the bifurcation diagram of the state variable z. In the bifurcation diagram of Fig. 6, the variable parameter is a; the other parameters are chosen to be: $b = 10, c = 4$ and $d = 0.1$. Macroscopically, the dynamical behaviors of system (1) are extremely rich. Over a wide range of parameter a, chaos can be found. As seen in Fig. 6, there are relatively large regions of periodic motions embedded within the chaotic region; e.g., for $1.52 < a < 1.57$ and $1.67 < a < 1.72$, periodic solutions are noted. It is of interest

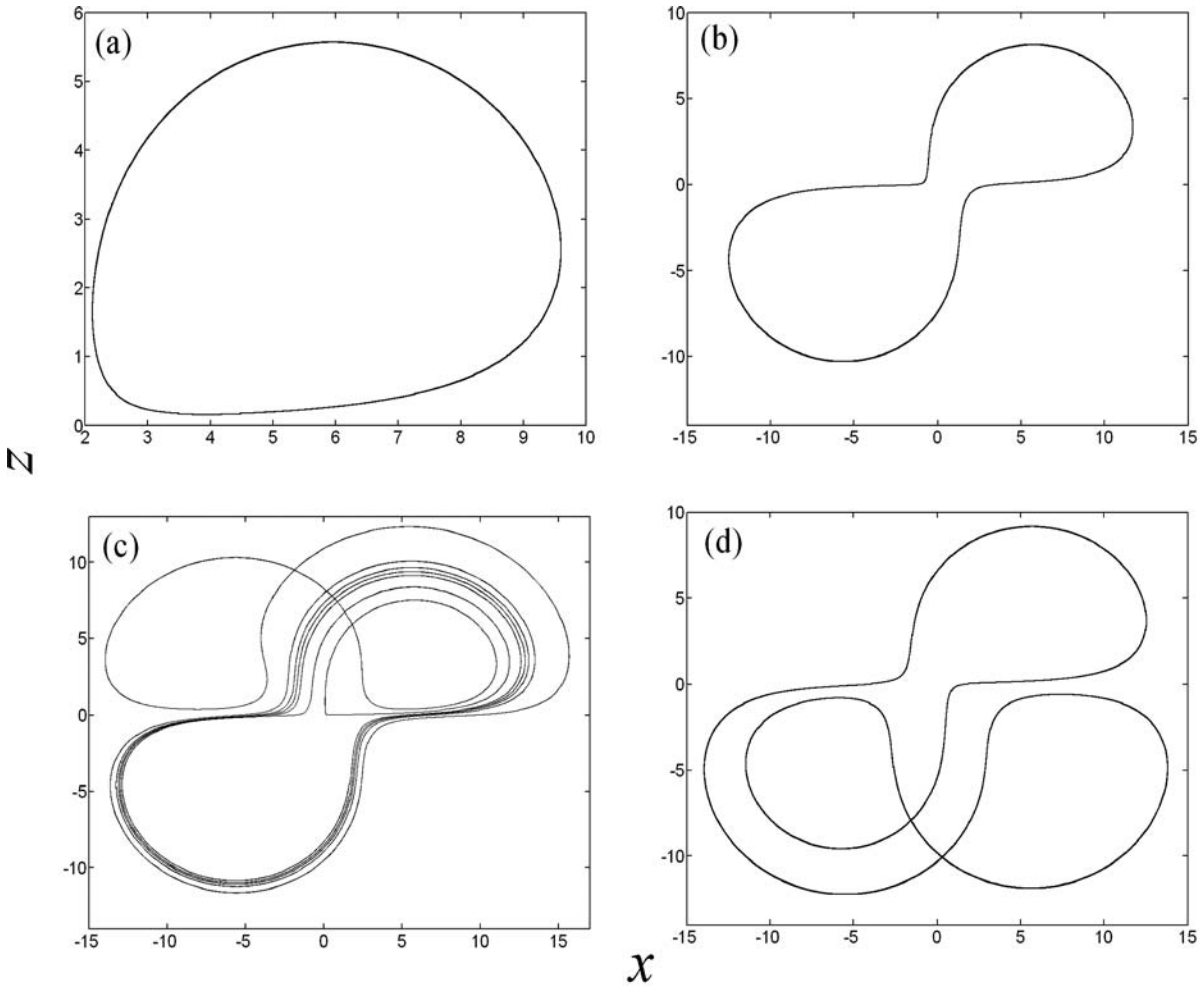


Fig. 7 Phase portraits on x–z plane for various a. (**a**) $a = 0.6$; (**b**) $a = 0.92$; (**c**) $a = 0.976$; (**d**) $a = 1.09$; (**e**) $a = 3.5$; (**f**) $a = 5.42$; (**g**) $a = 5.439$; (**h**) $a = 5.442$

that at several values of a (e.g. $a \approx 4.785$), however, discontinuities (jumps) arise, the nature of which is not understood.

It is instructive to look at the phase plots associated with various values of a, corresponding to different dynamical behaviors, as is discussed in the foregoing. The results are shown in Fig. 7. In Fig. 7a, which is for $a < 0.87$ ($a = 0.6$), it is seen that the trajectory is toward the stable limit cycle, which is asymmetric; moreover, the signals of z are exactly positive. In Fig. 7b, which is for $0.88 < a < 0.97$, the trajectory is still toward the stable limit cycle; in this case, however, the signals of z may be either positive or negative. It is of interest that, at $a = 0.976$ and $a = 1.09$, periodic motions occur. In Fig. 7c, the periodic orbit is running around the upper-left, the upper-right and the lower-left equilibria. However, in Fig. 7d, the periodic orbit is running around the upper-left, the lower-left and the lower-right equilibria. The remaining parts of Fig. 7, from which the initial transients have been omitted for clarity, show the following: (e) a 4-scroll chaotic attractor; (f) an upper chaotic attractor; (g) and (h) periodic motions; the transition from (f) to (h) is through a inverse period-doubling sequence. The power spectra of signal z for $a = 0.977$ and $a = 3.5$ are given in Fig. 8a and b, for the 3-scroll and 4-scroll chaotic attractors, respectively.

Of course, similar bifurcation diagrams and phase plots can be constructed if b, c or d were chosen as the variable parameters, but without giving any new

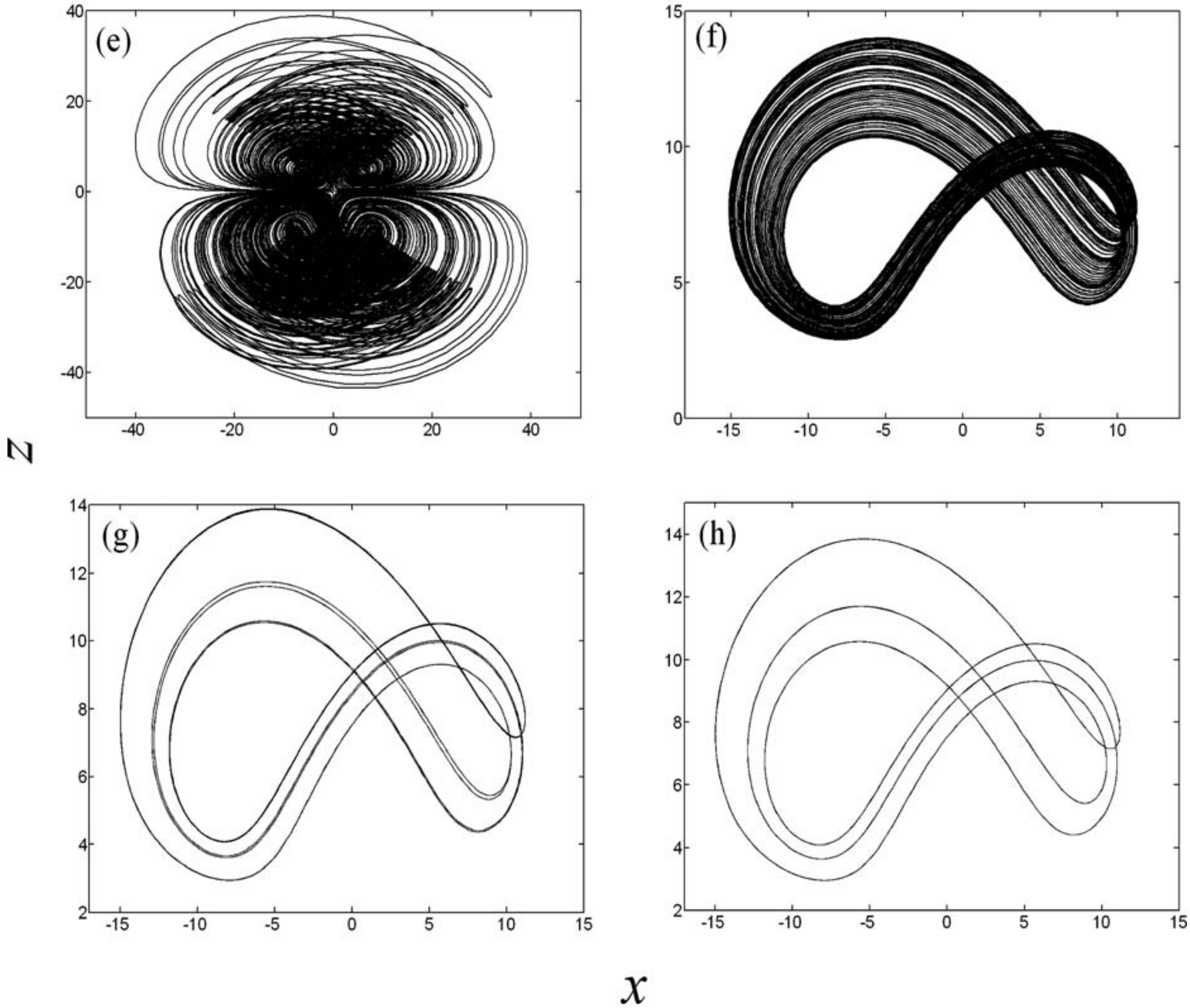


Fig. 7 (*Continued*)

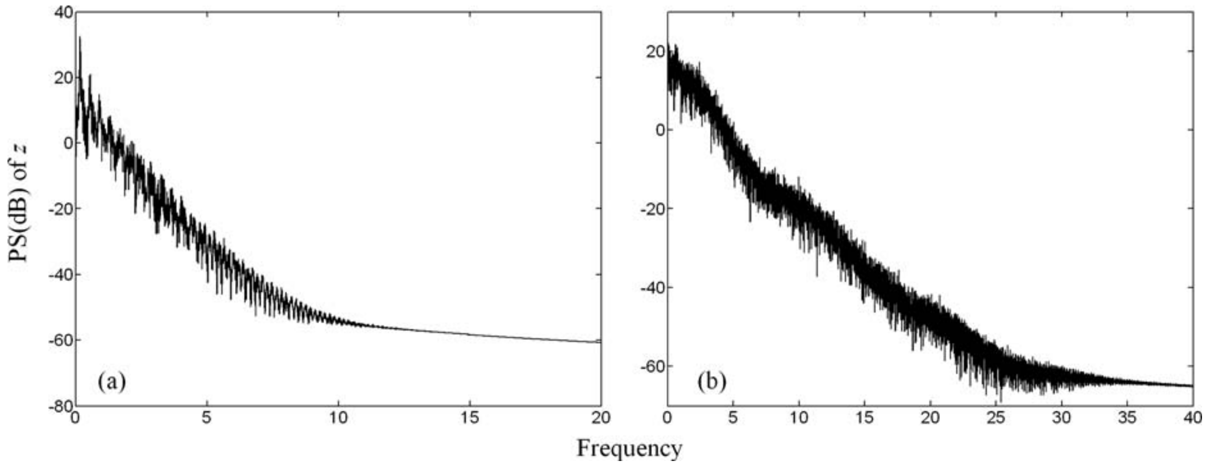


Fig. 8 Power spectrum of signal z. (**a**) $a = 0.977$; (**b**) $a = 3.5$

insight to the problem. Therefore, no additional bifurcation diagrams will be given.

6 Conclusions

This paper has proposed a new 3-D autonomous quadratic system, which can generate 3-scroll and 4-scroll chaotic attractors with complicated topological structures, and many other complex dynamics, over a wide range of parameters. This chaotic system has been analyzed in terms of bifurcation diagram, phase plots and power spectrum diagrams.

However, there remain two unanswered questions. First, the route to chaos for the present system is still unclear, although the inverse period-doubling bifurcations can be observed. As discussed in the foregoing, the 2-scroll, 3-scroll and 4-scroll chaotic attractors have been observed; however, the mechanism of switching in multi-scroll state (2-scroll to 3-scroll and back, 2-scroll to 4-scroll and back, 3-scroll to 4-scroll and back) is still unclear. The second question is that the discontinuities (jumps) observed in the bifurcation diagram of Fig. 6 cannot be explained theoretically. Thus, the finding of this particular example of dissipative chaotic system deserves further detailed investigation on its topological structure in the near future.

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References

1. Chen, G., Dong, X. (eds.): From Chaos to Order: Methodologies, Perspectives and Applications. World Scientific, Singapore (1998)
2. Chen, G., Yu, X. (eds.): Chaos Control: Theory and Applications. Springer, Berlin (2003)
3. Lü, J., Lu, J., Chen, S. (eds.): Chaotic Time Series Analysis and Its Applications. Wuhan University Press, Wuhan (2002) (in Chinese)
4. Lü, J.H., Chen, G.R., Cheng, D.Z.: A new chaotic system and beyond: the generalized Lorenz-like system. Int. J. Bifurc. Chaos **14**(5), 1507–1537 (2004). doi:10.1142/S021812740401014X
5. Lü, J.H., Chen, G.R.: Generating multiscroll chaotic attractors: theories methods and applications. Int. J. Bifurc. Chaos **16**(4), 775–858 (2006). doi:10.1142/S0218127406015179
6. Chua, L.O. (ed.): CNN: A Paradigm for Complexity. World Scientific, Singapore (1998)
7. Chua, L.O., Komuro, M., Matsumoto, T.: The double scroll family. IEEE Trans. Circuits Syst. **33**, 1072–1118 (1986). doi:10.1109/TCS.1986.1085869
8. Kennedy, M.P.: Three steps to chaos, part II: a Chua's circuit primer. IEEE Trans. Circuits Syst.-I **40**, 657–674 (1993)
9. Qin, Q., Lin, W., Qiao, N.: Dynamical behaviours of Liu system with time delayed feedbacks. Chin. Phys. B **17**(2), 569–572 (2008). doi:10.1088/1674-1056/17/2/035
10. Qi, G.Y., Chen, G.R, van Wyk, M.A., van Wyk, B.J., Zhang, Y.H.: A four-wing chaotic attractor generated from a new 3-D quadratic autonomous system. Chaos Solitons Fractals **38**(3), 705–721 (2008). doi:10.1016/j.chaos.2007.01.029
11. Sprott, J.C.: Simplest dissipative chaotic flow. Phys. Lett. A **228**(4–5), 271–274 (1997). doi:10.1016/S0375-9601(97)00088-1
12. Lü, J.H., Chen, G.R.: A new chaotic attractor coined. Int. J. Bifurc. Chaos **12**(3), 659–661 (2002)

13. Li, T.C., Chen, G.R., Tang, Y., Yang, L.J.: Hopf bifurcation of the generalized Lorenz canonical form. Nonlinear Dyn. **47**(4), 367–375 (2007). doi:10.1007/s11071-006-9036-x
14. Zhou, T.S., Chen, G.R., Celikovský, S.: Si'lnikov chaos in the generalized Lorenz canonical form of dynamical systems. Nonlinear Dyn. **39**(4), 319–334 (2005). doi:10.1007/s11071-005-4195-8
15. Wang, L., Ni, Q., Liu, P., Huang, Y.Y.: Chaos and its forming mechanism of a new Lorenz-like system. J. Dyn. Control **3**, 1–6 (2005) (in Chinese)
16. Liu, W.B., Chen, G.R.: Can a three-dimensional smooth autonomous quadratic chaotic system generate a single four-scroll attractor? Int. J. Bifurc. Chaos **14**(4), 1395–1403 (2004). doi:10.1142/S0218127404009880
17. Li, D.Q.: A three-scroll chaotic attractor. Phys. Lett. A **372**(4), 387–393 (2008)