Nonlinear Dyn (2009) 56: 357–367
DOI 10.1007/s11071-008-9402-y

ORIGINAL PAPER

Synchronization of four coupled van der Pol oscillators

Miguel A. Barrón · Mihir Sen

Received: 27 February 2008 / Accepted: 15 July 2008 / Published online: 8 August 2008

Abstract It is possible that self-excited vibrations in turbomachine blades synchronize due to elastic coupling through the shaft. The synchronization of four coupled van der Pol oscillators is presented here as a simplified model. For quasilinear oscillations, a stability condition is derived from an analysis based on linearizing the original equation around an unperturbed limit cycle and transforming it into Hill's equation. For the nonlinear case, numerical simulations show the existence of two well-defined regions of phase relationships in parameter space in which a multiplicity of periodic attractors is embedded. The size of these regions strongly depends on the values of the oscillator and coupling constants. For the coupling constant below a critical value, there exists a region in which a diversity of phase-shift attractors is present, whereas for values above the critical value an in-phase attractor is predominant. It is observed that the presence of an anti-phase attractor in the subcritical region is associated with sudden changes in the period of the coupled oscillators. The convergence of the coupled system to a particular periodic attractor is explored using several initial conditions. The study is extended to non-identical oscillators, and it is found that there is synchronization even over a wide range of difference among the oscillator constants.

M.A. Barrón (✉)
Departamento de Materiales, Universidad Autónoma Metropolitana-Azcapotzalco, Av. San Pablo 180, México, D.F. 02200, Mexico
e-mail: bmma@correo.azc.uam.mx

M. Sen
Department of Aerospace and Mechanical Engineering, University of Notre Dame, Notre Dame, IN 46556, USA

Keywords Coupling behavior · Periodic attractor · Phase-shift · Synchronization · Van der Pol oscillator

1 Introduction

Synchronization among self-excited oscillators has long been studied [1–3]. Common examples are biological [4], chemical [5], thermal [6], and electrical [7] oscillators. There are few mechanical systems, however, that have been analyzed. One prominent exception is that of pendulums swinging from a common structure that was studied by Huygens who, in 1665, noticed that they eventually entered into synchrony due to interaction through the structure [8, 9]. Since vibrations in mechanical systems are ubiquitous, it can be expected that this phenomenon is much more common than it appears from the literature. Though there are many other examples, rotating machinery is one place where one can expect to find self-excited oscillations which could be potentially problematic. It is known that unbalanced rotating shafts, such as in turbomachines, that are connected to a common structure may synchronize [10]. This system is, however, forced by the mass imbalance in each shaft rather than being individually self-excited [11]. It is also possible that the motion of turbomachine blades synchronizes in some way due to interaction between them. It has been found, for instance, that there is phase-locking of the turbines in a wind farm [12]. Since the rotation of each turbine is determined by the wind, there is no external frequency that is being imposed. We propose here that, even for turbomachines with a single rotor on a shaft, the vibrations of the blades can become synchronized among themselves by coupling through the fluid or the elasticity of the shaft. Observations of this have not been recorded in the literature, but the present is a theoretical argument for making these measurements.

A cantilever beam in the presence of a cross-flow undergoes self-excited oscillations due to the interaction between vortex shedding in the wake and the beam itself. The frequency of vortex shedding f behind a circular cylinder of diameter D in the presence of a uniform flow of velocity U is quantified by the Strouhal number $fD/U \approx 0.21$ over a wide range of Reynolds numbers. Computation of the vortex-induced vibration of a slender flexible structure can be carried out using direct numerical simulation. The problem of multiple cylinders attached to a single shaft has not been attempted so far. In the simplest model possible that captures the essentials, one can take each cylinder to be a system of a single degree of freedom, and couple them all by linear elastic interaction through the shaft. Modeling the fluid–mass interaction is extremely complicated and should, strictly speaking, involve the numerical solution of the Navier–Stokes equation on the fluid side [13]. At this stage, to demonstrate the possibility of synchronization, a simple reduced-order approach will be sought. There is some support for the use of a van der Pol term to represent the fluid–structure interaction, though this may be done in several different ways [14–18]. In the present work, the interactive force will be modeled as a van der Pol damping term.

The van der Pol equation has long been studied as a quintessential example of a self-excited oscillator [19] and has been used to represent oscillations in a wide variety of applications. The study of coupled oscillators provides information on emergent properties of the coupled system [20], such as synchronization, clustering, oscillatory modes and stability, all of which are of interest to the problem of blade oscillation. The most common coupling between van der Pol oscillators that has been examined is that between a pair of oscillators [21–23]. Studies of three coupled oscillators are not common [24], but the problem of four coupled van der Pol oscillators has been looked at by several authors. We are interested in four oscillators for two reasons: the presence of four blades on a rotating shaft is not uncommon, and four is the smallest number for which the governing equation for the motion of a single oscillator coupled with its immediate neighbors does not include the motion of *all* the others and can thus be expected to have some generality. The oscillatory modes of rings of three and four coupled quasilinear van der Pol oscillators with errors in their parameters have been analyzed through the averaging method [25, 26]. For the four-oscillator ring, a lack of synchronization between two single-oscillation modes is reported as a result of the asymmetry introduced by the errors. A weakly-coupled ring of identical oscillators has been experimentally studied, and some authors report the presence of oscillatory modes in which the phase cannot be precisely determined [27]. In their theoretical analysis they assume that the coupled and uncoupled amplitudes are the same, an assumption that is not correct as is shown in [28, 29] and corroborated in the present work. The synchronization of a ring of four identical quasilinear van der Pol oscillators has also been studied by linearization around the unperturbed limit cycle [7]. Three domains of stability were reported, and the numerical results were experimentally corroborated in a later work [30]. Unfortunately, the method is only valid for a small region of parameter space where the values of the oscillator constant are near zero. In a recent work [31], the presence of numerous phase-locked motions in an array of four coupled phase-only oscillators was reported. An important drawback of this work is in the difficulty of correlating phase-only oscillators with systems of practical interest such as turbomachine blades and thermal systems.

It is clear that many aspects remain to be clarified for four coupled oscillators. In this work the influence of system parameters on attractor multiplicity and synchronization behavior is addressed. The rest of the paper is structured in the following way. In the first part a mathematical model of coupled self-excited oscillators is presented and non-dimensionalized. In the second, a small value of the oscillator constant is assumed to enable linearization of the equations and the quasilinear behavior to be obtained. The third part deals with

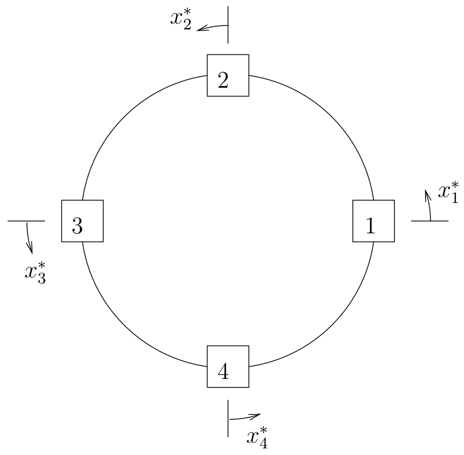

Fig. 1 Ring of four masses coupled by springs

larger values of the constant for which the problem is nonlinear and must be numerically explored. Finally, the issue of non-identical oscillators is addressed.

2 Mathematical model

Consider four masses in the form of a ring connected by springs as shown in Fig. 1. Each of the masses is a self-excited van der Pol oscillator. In this representation the coupling between adjacent masses is through a spring between them, so that the stretching of the spring provides the coupling force. We will assume the spring to be linear, so that

$$m\ddot{x}_i^* + c\left(x_i^{*2} - a^2\right)\dot{x}_i^* + kx_i^* = \sigma\left(x_{i-1}^* - 2x_i^* + x_{i+1}^*\right), \quad (1)$$

for $i = 1, \ldots, 4$, where x_i^* is the linear displacement of each oscillator, and the derivatives are with respect to time t^*. Since the four masses form a ring, the displacement $x_0^* = x_4^*$ and $x_5^* = x_1^*$. Similar models have been analyzed for electrical oscillators [7]. For the moment the four oscillators are all identical, with m being the mass and k the spring constant. The parameters c and a define the nonlinear van der Pol damping; damping is positive or negative depending on whether the oscillations are large or small, respectively, and thus regulates the amplitude of the limit cycle which is of $O(2a)$ [32]. σ is the strength of the coupling between the oscillators; the interaction is assumed to be linear, and since its sum is zero, there is no net external force on the system due to the interaction.

The non-dimensional time and displacement variables can be defined as $t = t^*(k/m)^{1/2}$ (based on the undamped natural period) and $x_i = x_i^*/a$ (based on the amplitude of the oscillations). Thus, we get

$$\ddot{x}_i + A\left({x_i}^2 - 1\right)\dot{x}_i + x_i = B(x_{i-1} - 2x_i + x_{i+1}), \quad (2)$$

where the two non-dimensional parameters are $A = ca^2/(mk)^{1/2}$ and $B = \sigma/k$. We will refer to A as the oscillator constant, and B the coupling constant. Initial conditions $x_i(0)$ and $\dot{x}_i(0)$ are also needed.

The four terms in (2), from the left to right, are inertial force $F_i^1 = \ddot{x}_i$, van der Pol damping force $F_i^2 = A({x_i}^2 - 1)\dot{x}_i$, elastic force $F_i^3 = x_i$, and coupling force $F_i^4 = B(x_{i-1} - 2x_i + x_{i+1})$, all per unit mass. The total energy of the system is $E = \sum_{i=1}^4(\dot{x}_i^2 + x_i^2)/2$, being the sum of the kinetic and potential energies, respectively. Multiplying (2) by $\dot{x}_i$, summing and then integrating over a period, we find that the terms corresponding to F_i^1, F_i^3 and F_i^4 vanish, indicating that over one period the energy coming in due to van der Pol damping is also equal to that going out.

The periods and phase shifts can be determined from the numerically-calculated long-time results. In this work the periodic motions obtained in this way are considered to be attractors [33]. T_0 is the uncoupled period of any of the oscillators for $B = 0$, and T_B is the coupled period for any other value of B. In all the examples treated here (except possibly when the oscillators are non-identical) it is found that, for the same parameter A, the coupled period is the same for all four oscillators and thus independent of i. In order to measure the extent of change in the oscillator period by the coupling phenomena, a relative period is introduced as $R = T_B/T_0$. The phase shift $\Delta\phi_{ij}$, being the phase shift in degrees between oscillators i and j, is also found. $\Delta\phi_{ij} > 0$ indicates that the peaks of oscillator i are reached *before* those of oscillator j. If two oscillators have the same phase, we will say that they belong to the same *cluster*. The term *synchronization* will be taken to mean that the oscillators have the same period, while *full synchronization* implies that they are also all in phase.

3 Quasilinear analysis for small A

For small values of the oscillator constant, a quasilinear behavior can be assumed and the nonlinear equations can be linearized around an unperturbed sinusoidal limit cycle; powerful analytical tools are available for this. For example, through a variable transformation, the linearized equations can be changed into the canonical form of a Hill's equation, $\ddot{x} + p(t)x = 0$, where $p(t)$ is a periodic function. From this equation, stability and synchronization conditions can be derived.

In order to analyze the stability and synchronization of the four coupled van der Pol oscillators for small values of A, the steps described in [7] for a ring of quasilinear van der Pol oscillators are employed. Equation (2) is linearized around a sinusoidal limit cycle, and after diagonal variables are introduced, a homogeneous second-order ordinary differential equation

$$\frac{d^2 z_N}{dt^2} + a_{1N}(t)\frac{dz_N}{dt} + a_{0N}(t) z_N = 0 \tag{3}$$

arises, where the main variable and the coefficients are defined as

$$z_N = \sum_{i=1}^{N} \delta_i \zeta_i, \tag{4}$$

$$a_{1N}(t) = A\left(A_m^2 \cos^2 \omega t - 1\right), \tag{5}$$

$$a_{0N}(t) = 1 + 4B - A\left(A_m^2 \omega \sin 2\omega t\right). \tag{6}$$

In (3) and (4), z_N is Nth diagonal variable (for the present case $N = 4$), δ_i is a parameter equal to 1 if i is even and -1 if i is odd, ζ_i is the ith linearized variable. In addition, in (5) and (6), A_m and ω are the amplitude and frequency of the unperturbed sinusoidal limit cycle, respectively. Using the transformation [34]

$$z_N = \rho_N \exp\left(-\frac{1}{2}\int_0^t a_{1N}(s)\,ds\right), \tag{7}$$

the canonical Hill's equation

$$\frac{d^2 \rho_N}{dt^2} + p(t)\rho_N = 0 \tag{8}$$

is obtained, where the periodic function $p(t)$ is defined as

$$p(t) = a_{0N}(t) - \frac{1}{4}a_{1N}^2(t) - \frac{1}{2}\frac{da_{1N}}{dt}. \tag{9}$$

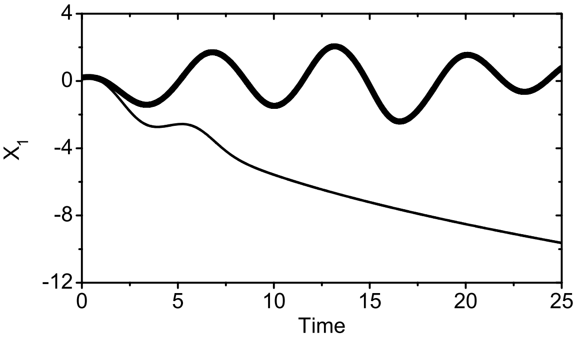

Fig. 2 Dynamics of the quasilinear oscillators with $A = 0.1$. $B = -0.2$ (stable, *thick line*) and $B = -0.3$ (unstable, *thin line*); similar behavior is exhibited by x_2, x_3, and x_4

Stability analysis can be carried out by means of periodic and non-periodic terms. Equation (8) can be written as [35]

$$\frac{d^2 \rho_N}{dt^2} + \left(\lambda + Q(t)\right)\rho_N = 0, \tag{10}$$

where λ is the non-periodic term given by

$$\lambda = 1 + 4B, \tag{11}$$

and where $Q(t)$ is the periodic term expressed as

$$Q(t) = -\frac{1}{4}A^2\left(A_m^2 \cos^2(\omega t) - 1\right)^2 - \frac{1}{2}AA_m^2 \omega \sin(2\omega t). \tag{12}$$

Under the assumption that the periodic term $Q(t)$ vanishes due to the small value of A in (12), the stability of the coupled quasilinear oscillators depends on the value of the non-periodic term λ. If $\lambda > 0$ the coupled system is oscillatory and becomes stable. If $\lambda = 0$ the coupled system is not oscillatory, and if $\lambda < 0$ the coupled system is unstable. Then, in accordance with (11), in order for oscillations of the coupled oscillators to be stable, the following condition must be satisfied: $B \in S$, where $S = (-0.25, \infty)$. For example, for $A = 0.1$ the stable and unstable behaviors of the coupled oscillators are shown in Fig. 2 when $B = -0.2$ and $B = -0.3$, respectively. In the first case $B \in S$, whereas in the second $B \notin S$. It is important to remark that fulfillment of this condition does not imply that full synchronization is guaranteed.

Table 1 Classification of periodic attractors for small A

Case	Attractor type	Clusters	Phase shifts
1	In-phase	1	$\Delta\phi_{ij} = 0\ \forall\ i, j$
2	Anti-phase	2	$\Delta\phi_{31} = \Delta\phi_{42} = 0$, $\Delta\phi_{21} = 180°$
3	Anti-phase	0	$\Delta\phi_{31} = 180°$, $\Delta\phi_{42} = 180°$

Table 2 Examples of initial conditions for different periodic attractors for small A; $A = 0.1$, $B = 0.1$

Case	$x_1(0)$	$\dot{x}_1(0)$	$x_2(0)$	$\dot{x}_2(0)$	$x_3(0)$	$\dot{x}_3(0)$	$x_4(0)$	$\dot{x}_4(0)$
1	5	0	5	0	5	0	5	0
2	10	−10	−10	10	10	−10	−10	10
3	5	0	8	0	−5	0	−8	0
3	9	8	7	6	5	4	3	2
1	0.2	0.2	1	1	1	1	1	1
2	2	−5	3	5	−8	1	9	−3
1	2	3	4	5	6	7	8	9

By applying Floquet theory and determining its corresponding Fourier modes, the following synchronization domains are reported [7, 36]:

$$D_1 = (-0.25, -0.0011] \cup [0.004, \infty), \quad (13)$$

$$D_2 = (-0.0011, -0.0006] \cup [0.002, 0.004), \quad (14)$$

$$D_3 = (-0.0006, 0) \cup (0, 0.002). \quad (15)$$

In accordance with [7], for $B \in D_1$ the four coupled quasilinear van der Pol oscillators exhibit full synchronization; for $B \in D_2$ the oscillators are in a correlated state, i.e. they present a tight—both in strength and direction—linear relationship. Besides, for $B \in D_3$ the desynchronization is complete but stable.

Table 1 shows the type of attractors found. Three types of periodic attractors are detected for $A = 0.1$ and $B = 0.1$, as is shown in Fig. 3: a full synchronization and two anti-phase attractors; no other phase-shifted attractors are detected. The type 1 attractor illustrates full synchronization between all 4 oscillators. The type 2 anti-phase attractor exhibits two clusters, namely $\Delta\phi_{13} = \Delta\phi_{24} = 0$, respectively, with one of the clusters in anti-phase with the other one, i.e. $\Delta\phi_{12} = \Delta\phi_{23} = 180°$. Thus oscillators x_1 and x_3 exhibit full synchronization with each other and an out-of-phase state with oscillators x_2 and x_4. Similar full-synchronization behavior is exhibited by x_2 and x_4 and anti-phase behavior with respect to x_1 and x_3. On the other hand, the type 3 anti-phase attractor exhibits no clusters, i.e. $\Delta\phi_{13} = \Delta\phi_{24} = 180°$. Table 2 shows some initial conditions from diverse regions of the state space and the corresponding type of attractor.

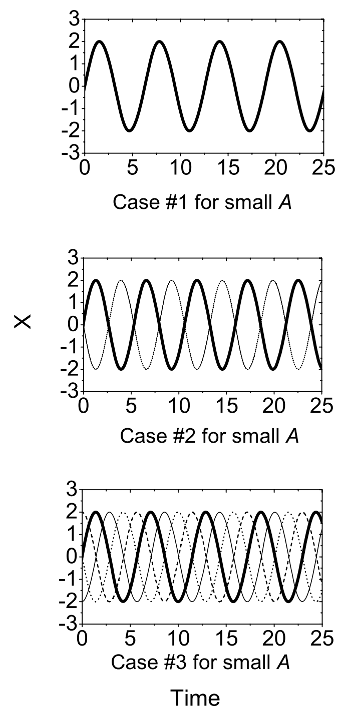

Fig. 3 Periodic attractors of the quasilinear case for $A = 0.1$ and $B = 0.1$. There are three types of stable behavior: full synchronization (*top graph*), 2 pairs of oscillators in full synchronization with $\Delta\phi_{13} = \Delta\phi_{24} = 0$ (*middle graph*), and synchronization only and out-of-phase for pairs of oscillators (*bottom graph*)

4 Numerical analysis for A not small

The value of A in an actual application depends on parameters of the system, and could vary over a wide range. It is important to know, for design purposes, what the possible dynamic behaviors are so as to avoid undesirable operation. For this reason, it is important to understand the response of the system for different values of A. A strong limitation of the analysis in the previous section is that the value of the oscillator constant must be small. For other values of A, the coupled system is strongly nonlinear and the analysis is invalid. Therefore, this analysis has to be done using numerical simulations. Integration is carried out using the fourth-order Runge–Kutta method. It is found that the short- and long-term dynamical behaviors of the coupled oscillators are very sensitive to the time step-size and the final integration time. Sometimes, quasiperiodic motions and unexpected attractors are obtained, particularly for time steps $\leq 10^{-2}$ and for integration times less than 25 cycles of the coupled system. Due to this, double precision calculations, time steps of 10^{-4} time units, and integration times above 10^3 cycles are employed in the computer simulations in order to prevent undesirable parasitic dynamics and attractors induced by the numerical procedure.

4.1 Parametric study

We will look at the range $A \in [1, 10]$ corresponding to strongly nonlinear behavior. In the numerical simulations the following procedure is employed: initially, the four oscillators are maintained at a condition in which $x_i(0) = \dot{x}_i(0) = 1$; then, the first oscillator is perturbed to $x_1(0) = \dot{x}_1(0) = 0.2$ and the equations are integrated until steady oscillations are achieved. The coupling constant, B, is gradually increased from 0 up to a critical value B_c, which depends on the oscillator constant, above which all the oscillators are fully in-phase.

Figure 4 shows the relative period, R, as a function of B for $A = 1$. As B increases in the interval $B \in [0, 0.12]$, it can be observed that R decreases. The corresponding phase-shift is shown in Fig. 5. This region corresponds to a phase-shift attractor with phase-locking between oscillators 1 and 3, and between 2 and 4, the two sets of oscillators being in anti-phase, i.e. $\Delta\phi_{13} = \Delta\phi_{24} = 0$ and $\Delta\phi_{12} = \Delta\phi_{23} = \Delta\phi_{34} = \Delta\phi_{41} = 180°$. However, there is a sudden change in

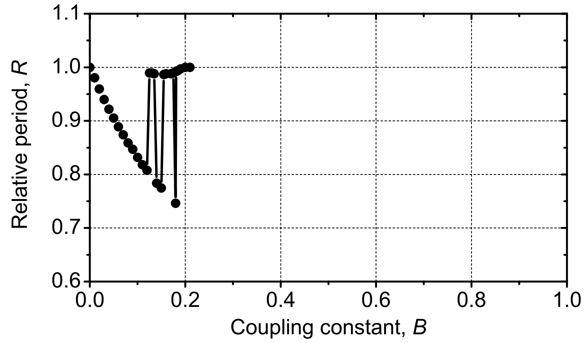

Fig. 4 Relative period, R, as a function of the coupling constant, B, for $A = 1.0$; $B \geq 0.2$, $R = 1$

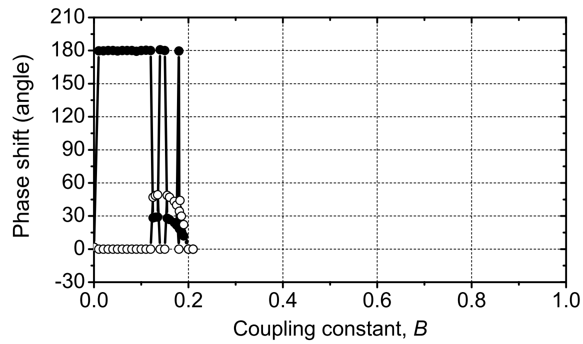

Fig. 5 Phase shifts for $A = 1.0$; *empty circles* $\Delta\phi_{31}$, *black circles* $\Delta\phi_{21} = \Delta\phi_{41}$; $B \geq 0.2$, $\Delta\phi_{31} = \Delta\phi_{21} = \Delta\phi_{41} = 0$

R from 0.81 to 0.99 at $B \approx 0.12$. This is associated with a change in the qualitative nature of the motion: from an anti-phase attractor the coupled system goes to a phase-shifted attractor. There are three sudden changes of this kind until B reaches a value of $B_c = 0.20$. Above this, there is phase-locking of all oscillators so that $R = 1$.

Figures 6 and 7 show R and the phase shifts, respectively, for $A = 5.0$. Two small intervals are detected, $B \in [0.14, 0.15]$ and $B \in [0.18, 0.19]$, in which an anti-phase attractor with two clusters emerges. It is found that $B_c = 0.75$ for $A = 5.0$.

The relative periods and phase shifts for $A = 10.0$ are shown in Figs. 8 and 9. It is found that $B_c = 0.85$. When $B \in [0.85, \infty)$, the in-phase attractor is present and the coupled system exhibits full synchronization regardless of the value of A. It is observed that as the coupled system becomes strongly nonlinear, the sudden changes in the period and phase shifts tend to disappear. Given that sudden changes in the period are seemingly associated with the emergence of anti-phase attractors, it is expected that for strong nonlinearity of the coupled system the anti-phase attractors are unlikely to rise.

A parametric diagram is constructed using the above information, and is shown in Fig. 10. The thick black line in this figure is $B_c(A)$, above which the coupled oscillators are in-phase for the initial conditions described before. In this diagram, the region below B_c is a subcritical region, whereas the region above B_c is supercritical. In the subcritical region, one can find a mix of phase-shift attractors, anti-phase attractors

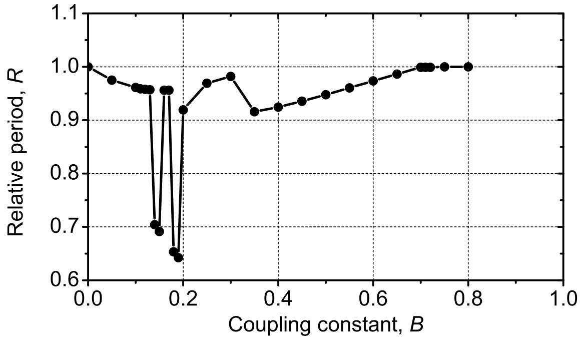

Fig. 6 Relative period of coupled oscillators for $A = 5.0$; $B \geq 0.8$, $R = 1$

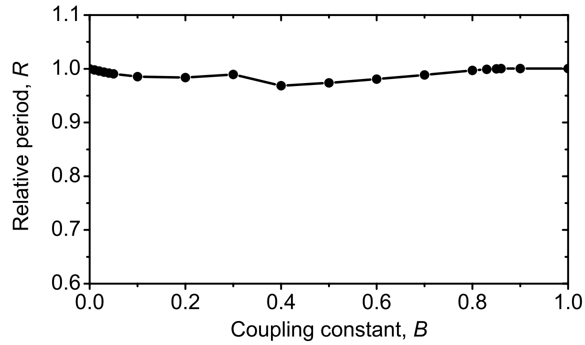

Fig. 8 Relative period of coupled oscillators for $A = 10.0$

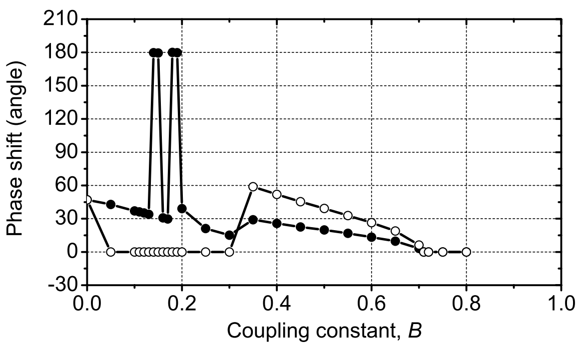

Fig. 7 Phase shifts for $A = 5.0$; *empty circles* $\Delta\phi_{31}$, *black circles* $\Delta\phi_{21} = \Delta\phi_{41}$; $B \geq 0.8$, $\Delta\phi_{31} = \Delta\phi_{21} = \Delta\phi_{41} = 0$

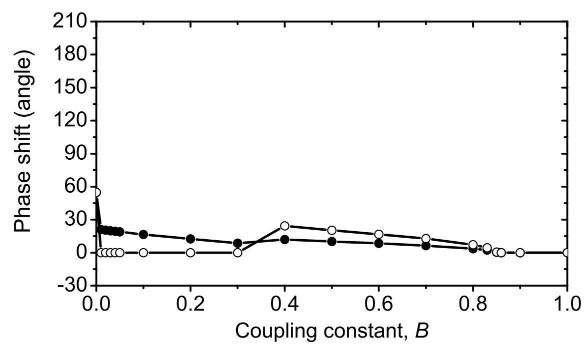

Fig. 9 Phase shifts for $A = 10.0$; *empty circles* $\Delta\phi_{31}$, *black circles* $\Delta\phi_{21} = \Delta\phi_{41}$

Fig. 10 Regions of periodic attractors for coupled identical oscillators

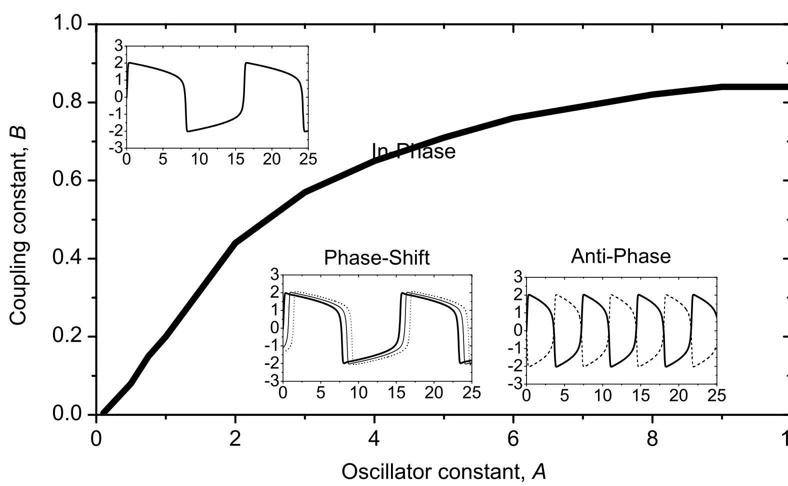

Table 3 Classification of periodic attractors for A not small

Case	Attractor type	Clusters	Phase shifts
1	In-phase	1	$\Delta\phi_{ij} = 0\ \forall i, j$
2	Anti-phase	2	$\Delta\phi_{31} = \Delta\phi_{42} = 0$, $\Delta\phi_{21} = 180°$
3	Anti-phase	0	$\Delta\phi_{31} = 180°$, $\Delta\phi_{42} = 180°$
4	Phase-shifted	1	$\Delta\phi_{42} = 0$, $\Delta\phi_{21} > 0$, $\Delta\phi_{31} > 0$
5	Phase-shifted	1	$\Delta\phi_{42} = 0$, $\Delta\phi_{21} < 0$, $\Delta\phi_{31} < 0$
6	Phase-shifted	1	$\Delta\phi_{31} = 0$, $\Delta\phi_{21} < 0$, $\Delta\phi_{41} > 0$
7	Phase-shifted	1	$\Delta\phi_{31} = 0$, $\Delta\phi_{21} > 0$, $\Delta\phi_{41} < 0$

Table 4 Examples of initial conditions for different periodic attractors for A not small; $A = 8.0$, $B = 0.4$

Case	$x_1(0)$	$\dot{x}_1(0)$	$x_2(0)$	$\dot{x}_2(0)$	$x_3(0)$	$\dot{x}_3(0)$	$x_4(0)$	$\dot{x}_4(0)$
1	5	0	5	0	5	0	5	0
2	10	−10	−10	10	10	−10	−10	10
3	5	0	8	0	−5	0	−8	0
4	9	8	7	6	5	4	3	2
5	0.2	0.2	1	1	1	1	1	1
6	2	−5	3	5	−8	1	9	−3
7	2	3	4	5	6	7	8	9

or in-phase attractors, depending on the values of the coupling and oscillator constants. In the supercritical region, the coupled oscillators are fully synchronized. As will be explained later, for certain initial conditions in small regions of the state space, some anti-phase attractors may appear in the supercritical region.

4.2 Attractors in the subcritical region

Up to seven different periodic attractors are found in the nonlinear subcritical region, and Table 3 shows the types detected. Each attractor has its own basin of attraction, so that the appearance of a particular attractor will depend on the initial condition chosen. For these numerical simulations, initial conditions are randomly selected from every corner of phase space. As an example, Table 4 shows the initial conditions to get the attractors shown in Table 3 with parameter values $A = 8.0$ and $B = 0.4$. The corresponding time series are shown in Fig. 11, in which for cases 4–5 x_2 is in phase with x_4, and for cases 6–7 x_1 is in phase with x_3. Type 1 is an in-phase attractor in which all the four coupled oscillators are in full synchronization. The period of the coupled system, as mentioned before, depends on the particular values of the oscillator and coupling constants, A and B. The type 2 attractor exhibits two clusters: one of them between the first and the third oscillators, and the other between the second and the fourth oscillators. The two clusters stay in an anti-phase state. More complex is the behavior of the type 3 attractor, in which the first oscillator is in anti-phase with the third one, and the second one is in anti-phase with the fourth one. On the other hand, the remaining four cases in Fig. 11 are of the phase-shifted type. This means that some oscillators have a phase difference of neither zero nor 180°. Besides, every attractor presents some kind of clustering. In type 4 and 5 attractors, the second and fourth oscillators are locked in phase. However, in the type 4 attractor the second and the third oscillators are delayed from the first one, whereas in the type 5 attractor the opposite holds. As a matter of fact, the type 5 attractor is a mirror-image of the type 4 attractor. For type 6 and 7 attractors, the first and the third oscillators are in phase. In the type 6 attractor the second and the fourth oscillators are ahead or behind the first oscillator, respectively. The opposite occurs in the type 7 attractor where the second oscillator is delayed and the fourth is ahead of the first. It is observed that phase-shift attractors, in addition to changes in the phase, present small but perceptible differences in the amplitude, shown in Fig. 12, as has been previously reported [28].

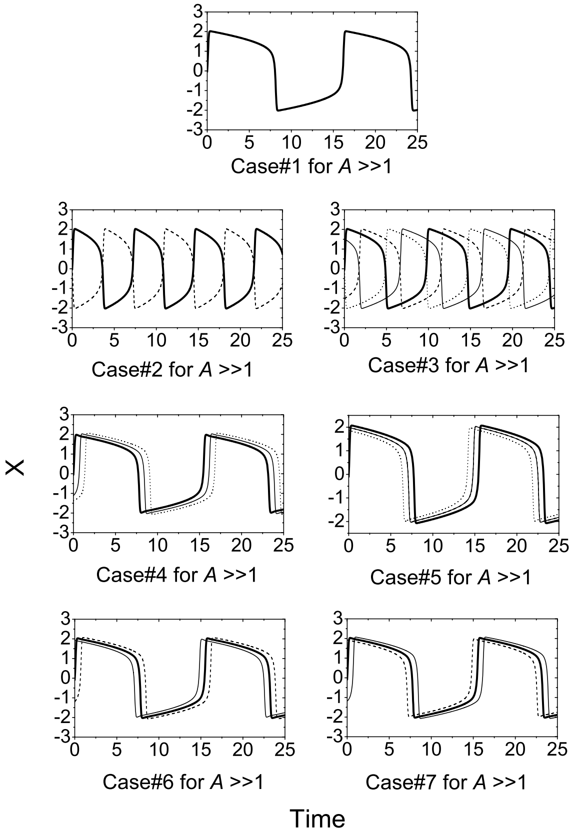

Fig. 11 Periodic attractors for $A = 8.0$ and $B = 0.4$ with initial conditions indicated in Table 4; in cases 4–5 x_2 (*thin line*) is in phase with x_4; in cases 6–7 x_1 (*thick line*) is in phase with x_3

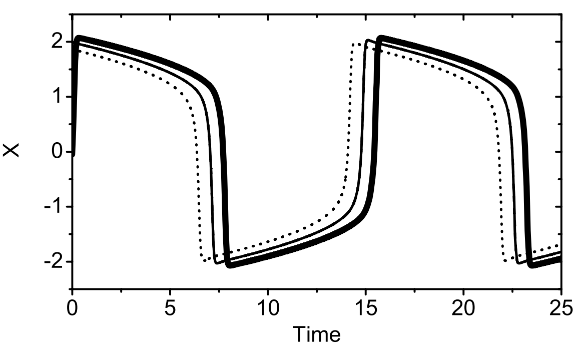

Fig. 12 Difference in amplitudes in a phase-shift type 5 attractor for $A = 8.0$ and $B = 0.4$; the *thick*, *thin* and *dotted lines* correspond to the first, second and third oscillators, respectively

4.3 Attractors in the supercritical region

In the supercritical region of the phase diagram in Fig. 10, numerical simulations showed the presence of a predominant in-phase, type 1, periodic attractor. Two

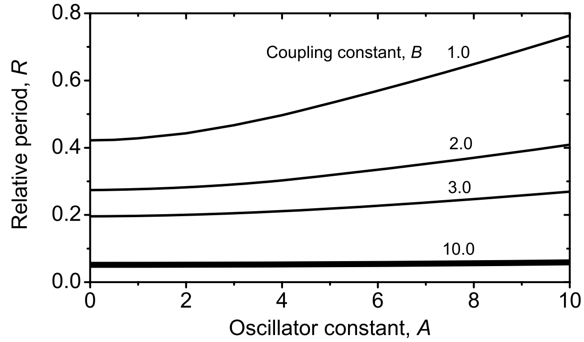

Fig. 13 Relative period of periodic anti-phase attractor in the supercritical region; initial conditions corresponding to case 2 in Table 4 were employed

anti-phase, type 2 and 3, periodic attractors, not shown in Fig. 10, with small basins of attraction are also detected. The in-phase attractor is undoubtedly that with the largest basin of attraction, as was corroborated using the same initial conditions for the subcritical attractors shown in Table 2. From the seven initial conditions considered in Table 4, those corresponding to type 1, 4, 5, 6 and 7 attractors drove the coupled system with $A = 8.0$ and $B = 2.0$ to the in-phase attractor, rather than to the phase-shifted attractors. Besides, initial conditions that carried the coupled attractors to the type 2 and 3 anti-phase attractors are also shown in Table 4. The type 2 anti-phase attractor in the supercritical region is characterized by a small period compared with the period of an uncoupled oscillator, as can be observed in Fig. 13. The period of the attractor depends both on the values of the oscillator and coupling constants. However, it is the coupling constant that affects mostly the period of the system.

5 Non-identical oscillators

Synchronization would not be of practical interest if it occurred only when all the oscillators were exactly the same. In order to analyze the behavior of non-identical oscillators in the strongly nonlinear region, the constant of *one* of the oscillators is perturbed as

$$A_p = A(1 + h), \quad (16)$$

where A_p is the perturbed value of the oscillator constant and h is the detuning parameter. Although in practice more than one oscillator could be different, we will, for simplicity, restrict ourselves to just

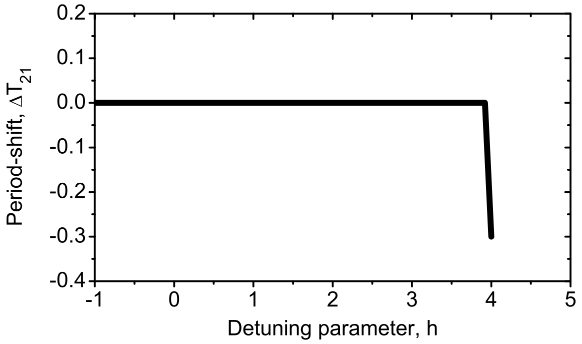

Fig. 14 Period-shift of oscillators 1 and 2 as a function of h, for $A = B = 1.0$; $h > 4$, $\Delta T_{21} \to -\infty$

one. For identical oscillators with the initial conditions $x_i = 5$ and $\dot{x}_i = 0$ for $i = 1, \ldots, 4$, it is verified that $x_1 = x_2 = x_3 = x_4$ for $h = 0$ and $B = 1$, i.e. the coupled system exhibits full synchronization. The oscillator $i = 1$ is arbitrarily chosen to be perturbed, and the values $A = B = 1$ are assumed. Similarly to the phase-shift, a period-shift ΔT_{ij} is defined as the difference in period between oscillators i and j. No multiple periods of individual oscillators are detected. Figure 14 depicts the behavior of ΔT_{21} as a function of h, where it can be observed that the synchronization of the attractor is conserved for $h \in [-1, 3.92]$. This kind of behavior has been observed before in the synchronization of other oscillators [2, 37]. Figure 14 shows that for $h = 4$, $\Delta T_{21} = -0.3$, but for $h > 4$, ΔT_{21} becomes very large. It is important to remark that for non-identical oscillators, $\Delta T_{ij} = 0$ does not imply that $\Delta\phi_{ij} = 0$ even for $B > B_c$.

6 Conclusions

A study of the coupled behavior of four van der Pol oscillators has been carried out, starting with identical oscillators. Using a small A approximation, a stability condition was analytically derived and no phase-shifted attractors were detected. For larger A, computer simulations in parameter space show the presence of a multiplicity of periodic attractors divided into two well-defined regions. A critical value of the coupling constant, B_c, as a function of the oscillator constant A was observed above which the coupled system is in full synchronization. Even though the four oscillators are identical, computer simulations show a complex behavior due to coupling.

For non-identical oscillators, it was shown that synchronization in frequency occurred even when the oscillator constant of one of the oscillators was almost four times that of the others. This raises the possibility that blades which are not absolutely identical could become vibrationally frequency-locked. The stresses on the shaft would then depend on how much each blade was out of phase with respect to its neighbors, with the consequent dangers of long-term fatigue effects.

Future work will include the continuum nature of turbomachine blades, the infinite degrees of freedom that each blade has, and the synchronization between them. It is also recommended, on the basis of this study, that experiments be conducted to establish this phenomenon.

Acknowledgements This work was done while Miguel A. Barrón was on sabbatical from the *Universidad Autónoma Metropolitana-Azcapotzalco* (*UAM-A*) and visiting the *University of Notre Dame*. He gratefully acknowledges financial support from the *Programa de Apoyo a Estancias Sabáticas de Investigación* of the *UAM-A*.

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