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ORIGINAL PAPER

Robust adaptive nonlinear control for uncertain control-affine systems and its applications

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Abstract This paper addresses the robust tracking control problem for a class of uncertain nonlinear systems with time-varying parameters, perturbed by external disturbances. The unknown time-varying parameters and disturbances are neither required to be periodic nor to have known bounds. Depending on the characteristics of disturbance signals, two adaptive-based control algorithms are developed. First, an adaptive H_∞ control is designed that achieves: (i) an H_∞ tracking performance when the external disturbances are L_2 signals, and (ii) the convergence of tracking error to zero if the disturbances are bounded and L_2 signals. Then a novel adaptive control algorithm is proposed, only with the assumption of boundedness of disturbances, to drive the tracking error to zero. The designed tracking controllers are then used for controlling a cart-pendulum system, as an underactuated mechanical system, and chaos synchronization of uncertain Genesio–Tesi chaotic system. Numerical simulations are also given to demonstrate the effectiveness of the proposed control schemes.

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1 Introduction

Designing robust tracking control for uncertain nonlinear systems is considered as a challenging problem in the field of control. In many applications, a nominal model can be derived for the system, but model uncertainties and parameter variations cause the desired performance not be achieved. In the past years, considerable research efforts have been devoted to tackle this problem. Developing tracking controllers for servo systems [6, 28], magnetic levitation [29], some classes of chaotic systems [5, 15], cart pendulum system [17, 23], optical disk drives [14] and a class of underactuated mechanical systems [11] are samples of tremendous efforts devoted to practical control problems. The assumptions made on the system uncertainties motivate researchers to propose various tracking control methodologies. Among the reported methods, adaptive-based control techniques are powerful tools, especially when the variations of unknown parameters are slow enough [1, 12, 18]. In fact, conventional adaptive methods including adaptive control laws together with some parameter adjusting mechanisms may fail for the case of arbitrarily fast time-varying perturbations. Investigating into this field, several results have been reported when the time-varying parameters are periodic (see, e.g., [8, 25] and the references therein). Besides, a priori knowledge in most of the existing adaptive control designs is that the system uncertainties can be exactly linearly parameterized [4, 9, 16, 25]. A known upper bound may be also assumed for the norm of parameter vector either for constant or time-varying parameters [4, 9].

In practice, external disturbances may also affect the performance of dynamical systems. Two major techniques are investigated to deal with this problem: disturbance attenuation and disturbance rejection. The first policy is mostly considered if the bound of disturbance signal is unknown. The robust H_∞ control method has been widely used to attenuate the influence of disturbance in this case [2, 20, 22]. These conventional H_∞ control techniques, however, would meet some difficulties if the plant model have large uncertainties. Moreover, in these control schemes, the disturbance signal is implicitly assumed to belong to the L_2 space. Concerning with the rejection purpose, several adaptive approaches have been introduced for the case of sinusoidal or, more generally, periodic disturbances (see, e.g., [3, 8] and the references therein). As a result, to tackle nonlinear systems affected by parametric uncertainties and external disturbances, the combination of tools from both robust and adaptive approaches may yield better designs than those produced by either method alone.

However, most of the results reported for the above mentioned problem suffer from at least one of the following restrictions: (i) the bound of system uncertainties and time-varying perturbations are respectively specified by known functions and real constants, (ii) the external disturbances and time-varying parameters are assumed to be periodic, (iii) the uncertainties are exactly linearly parameterized, and (iv) a known constant is assumed as the lower bound of control coefficient. Nevertheless, the convergence of tracking error to zero, without any prior knowledge of the exact upper bound or the periodicity of perturbations, has remained as a challenging problem.

In this paper, an adaptive-based H_∞ control law is first developed for the uncertain systems perturbed by L_2 disturbances to achieve a prescribed H_∞ tracking performance. For the case of bounded external disturbances, this controller assures the convergence of tracking error to zero. Then removing the assumption that disturbance is an L_2 signal, a novel adaptive tracking controller is proposed. Besides possessing the simplicity and universality properties, this control scheme guarantees that all the closed-loop signals are bounded and tracking error is driven to zero despite the system uncertainties and bounded external disturbances. Some specific properties of the developed tracking controllers are: (i) the bound of system uncertainties is specified by an uncertain time-varying parametric function (not only a known real function), (ii) the unknown time-varying parameters and disturbances are neither required to be periodic nor to have known bounds, (iii) rather than a real constant, a state-dependent function is assumed as the lower bound of control coefficient.

From a practical point of view, a large class of nonlinear plants, such as cart-pendulum system, single link flexible joint manipulator, mass-spring damper system, magnetic levitation system and van der Pol's oscillator can be transformed to or originally described by the model considered here. Hence, the designed tracking controllers can be easily applied to these systems. Furthermore, the dynamic equations of some chaotic systems, e.g., Duffing–Holmes chaotic system [5], chaotic nonlinear gyros [27] and Genesio–Tesi system [10], can be also described by underlying model. In fact, chaos control and synchronization of chaotic systems can be viewed as another important application of the control methods developed by this paper.

The organization of the paper is as follows. Section 2 specifies the underlying uncertain nonlinear systems and presents the problem formulation. In Sect. 3, the above adaptive-based robust controllers are derived. In Sect. 4, the proposed robust adaptive controllers are developed for two practical problems including: (i) tracking control of a cart-pendulum system as an underactuated mechanical system, and (ii) chaos synchronization for uncertain Genesio–Tesi chaotic system. Some numerical simulations are also presented in this section to illustrate the effectiveness of the methods. Finally, the concluding remarks are given in Sect. 5.

Throughout the paper, $|\cdot|$ denotes the absolute value and $\|\cdot\|$ indicates the Euclidean vector norm. For a $n \times 1$ vector V, $\|V\|_Q^2 := V^T Q V$ with the weighting matrix Q. Furthermore, $V \in L_2[0, T]$ if $\int_0^T \|V(t)\|^2 \, dt < \infty$, $T \in [0, \infty)$, and $V \in L_\infty[0, \infty)$ if $\|V(t)\| < \infty$ for all $t \in [0, \infty)$.

2 Problem formulation

In this section, the problem of tracking control in the presence of system uncertainties and external disturbances is formulated for a class of uncertain control-affine systems described by

$$\begin{aligned}
\dot{x}_1 &= x_2, \\
\dot{x}_2 &= x_3, \\
&\vdots \\
\dot{x}_n &= f(X,t) + g(X,t)u(t) + d(t), \\
y &= x_1
\end{aligned} \tag{1}$$

where f and g are unknown bounded nonlinear continuous functions, $X = [x_1, x_2, \dots, x_n]^T$ is the state vector of the system, u, y, and d denote, respectively, the input, the output, and the external disturbance.

Equivalently, the uncertain system (1) can be represented by an nth-order nonlinear perturbed system of the form

$$y^{(n)} = f(X,t) + g(X,t)u(t) + d(t). \tag{2}$$

The objective is to design a robust controller for uncertain nonlinear system (2) such that the output y tracks the desired reference trajectory y_r as close as possible despite the system uncertainties and external disturbances. For a given smooth reference trajectory y_r, the tracking error vector is defined as $E = [e, \dot{e}, \dots, e^{(n-1)}]^T$ with $e = y - y_r$. In order to obtain the tracking error dynamic, let

$$\Lambda = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}_{n\times n}, \qquad B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}_{n\times 1}.$$

Therefore,

$$\dot{E} = \Lambda E + B\big[f(X,t) + g(X,t)u(t) + d(t) - y_r^{(n)}\big]. \tag{3}$$

Choosing a vector as $K = [k_1, k_2, \dots, k_n]^T$ such that $A := \Lambda - BK^T$ is Hurwitz, the error dynamic (3) can be expressed as

$$\begin{aligned}\dot{E} = AE + B\big[&K^T E + f(X,t) + g(X,t)u(t) \\ &+ d(t) - y_r^{(n)}\big].\end{aligned} \tag{4}$$

The following assumptions are made regarding system (2).

Assumption 1 *There exists a positive function $b(X)$ such that $|g(X,t)| \geq b(X)$, $\forall t \geq 0$. In order to derive the control law, without loss of generality, we assume that $g(X,t) > 0$. However, the design procedure can be easily modified for the case $g(X,t) < 0$.*

Assumption 2 *The unknown function $f(X,t)$ is bounded by some continuous function $L(X)$ and a linearly parameterized time-varying uncertainty, i.e.,*

$$f(X,t) \leq L(X) + \phi^T(X)\theta(t) \tag{5}$$

where $\phi(X)$ is a well-known regressor vector with appropriate dimension and $\theta(t) \in \Re^p$ is an unknown time-varying parameter vector belonging to a compact set $\Omega = \{\theta(t) \mid \|\theta(t)\| \leq \alpha\}$ in which $\alpha > 0$ is an unknown constant parameter.

Remark 1 Assumption 1 assures that system (1) is controllable [24]. Compared with some previous investigations which assume a known lower constant bound $b_{\min} > 0$ for $g(X,t)$ [7, 13, 25], Assumption 1 is less conservative.

Remark 2 The inequality (5) implies that the system uncertainty is not required to be exactly equal to a linearly parameterized term. In fact, the equality version of (5), adopted in some previous works [4, 9, 16], is relaxed here to an inequality which can be easier satisfied.

3 Design of adaptive-based robust controllers

This section is devoted to design two adaptive control schemes, dealing with the robust tracking problem for the uncertain system (2). The effectiveness of each method depends on the characteristics of external disturbance $d(t)$. More precisely, the assumptions of $d \in L_2[0,\infty)$ and/or $d \in L_\infty[0,\infty)$ can result in different consequents.

Theorem 1 *For the uncertain nonlinear system (1) or equivalently (2), suppose that Assumptions 1 and 2 hold and there exists a positive-definite symmetric matrix P satisfying the Riccati-like inequality*

$$A^T P + PA + Q + PB\left(\frac{1}{\rho^2}I - \frac{1}{r}I\right)B^T P \leq 0 \tag{6}$$

where $\rho > 0$ is a prescribed attenuation level, $Q = Q^T > 0$ is a prescribed weighting matrix and $r > 0$ is the H_∞ controller gain. Defining $M(X) = L(X) - y_r^{(n)} + K^T E$, the robust adaptive control law

$$u = -\frac{1}{b(X)}\left[\frac{M^2(X)B^T PE}{|M(X)B^T PE| + \delta_1 e^{-\sigma_1 t}} + u_a + u_r\right] \tag{7}$$

with

$$u_r = \frac{1}{2r} B^T PE, \tag{8}$$

$$u_a = \hat{\alpha}^2 \frac{\phi^T(X)\phi(X)B^T PE}{\|\phi(X)B^T PE\|\hat{\alpha} + \delta_2 e^{-\sigma_2 t}}, \tag{9}$$

$$\dot{\hat{\alpha}} = \gamma \|\phi(X)B^T PE\| \tag{10}$$

where δ_i and σ_i, $i = 1, 2$, are (small) positive constants specified by the designer, $\hat{\alpha}$ is the estimate of α and $\gamma > 0$ is the adaptation gain, guarantees that

(i) *The following H_∞ tracking performance is achieved if $d \in L_2[0, \infty)$,*

$$\int_0^T \|E(t)\|_Q^2 \, dt \le I_0 + \rho^2 \int_0^T \|d(t)\|^2 \, dt, \quad \forall 0 \le T < \infty \tag{11}$$

where I_0 is a positive constant that depends on $\delta_i, \sigma_i, i = 1, 2$, and the initial conditions.

(ii) *The tracking error is uniformly ultimately bounded (UUB) if $d \in L_\infty[0, \infty)$.*

(iii) *The tracking error converges to zero if $d \in L_2[0, \infty) \cap L_\infty[0, \infty)$.*

Proof Choose a Lyapunov function as

$$V(E, \tilde{\alpha}) = \frac{1}{2} E^T PE + \frac{1}{2\gamma}\tilde{\alpha}^2 \tag{12}$$

where $\tilde{\alpha} = \alpha - \hat{\alpha}$ denotes the estimation error. The time derivative of $V(E, \tilde{\alpha})$ along (4) is

$$\begin{aligned}\dot{V}(E, \tilde{\alpha}) = {} & \frac{1}{2} E^T \left(A^T P + PA\right)E \\ & + \left(f - y_r^{(n)} + K^T E\right)B^T PE + d^T B^T PE \\ & + u^T g B^T PE + \frac{1}{\gamma}\tilde{\alpha}\dot{\tilde{\alpha}}.\end{aligned} \tag{13}$$

Taking into account (5) and substituting control law (7) into (13) yields

$$\begin{aligned}\dot{V}(E, \tilde{\alpha}) \le {} & \frac{1}{2} E^T \left(A^T P + PA\right)E + M(X)B^T PE \\ & + \theta^T(t)\phi(X)B^T PE + E^T PB[d + u_r] \\ & - \frac{E^T PBM^2(X)B^T PE}{|M(X)B^T PE| + \delta_1 e^{-\sigma_1 t}} \\ & - \hat{\alpha}^2 \frac{E^T PB\phi^T(X)\phi(X)B^T PE}{\|\phi(X)B^T PE\|\hat{\alpha} + \delta_2 e^{-\sigma_2 t}} \\ & - \frac{1}{\gamma}\tilde{\alpha}\dot{\hat{\alpha}}.\end{aligned} \tag{14}$$

Using the inequality (6) and substituting u_r from (8), imply that

$$\begin{aligned}\dot{V}(E, \tilde{\alpha}) \le {} & -\frac{1}{2} E^T QE + \left|M(X)B^T PE\right| \\ & + \alpha \left\|\phi(X)B^T PE\right\| - \left|M(X)B^T PE\right| \\ & + \delta_1 e^{-\sigma_1 t} - \hat{\alpha}\left\|\phi(X)B^T PE\right\| + \delta_2 e^{-\sigma_2 t} \\ & - \frac{1}{2}\left(\frac{1}{\rho}B^T PE - \rho d\right)^T \\ & \times \left(\frac{1}{\rho}B^T PE - \rho d\right) \\ & + \frac{1}{2}\rho^2\|d\|^2 - \frac{1}{\gamma}\tilde{\alpha}\dot{\hat{\alpha}}.\end{aligned} \tag{15}$$

By the adaptation law (10) and some manipulations, one can obtain

$$\begin{aligned}\dot{V}(E, \tilde{\alpha}) \le {} & -\frac{1}{2} E^T QE + \frac{1}{2}\rho^2\|d\|^2 + \delta_1 e^{-\sigma_1 t} \\ & + \delta_2 e^{-\sigma_2 t}.\end{aligned} \tag{16}$$

The following results are then concluded.

(i) If $d \in L_2[0, T]$, integrating the inequality (16) from $t = 0$ to $t = T$ yields

$$\begin{aligned}& \frac{1}{2}\int_0^T \|E(t)\|_Q^2 \, dt + V\left(E(T), \tilde{\alpha}(T)\right) \\ & \quad \le V\left(E(0), \tilde{\alpha}(0)\right) + \frac{\delta_1}{\sigma_1}\left(1 - \delta_1 e^{-\sigma_1 T}\right) \\ & \qquad + \frac{\delta_2}{\sigma_2}\left(1 - \delta_2 e^{-\sigma_2 T}\right) + \frac{1}{2}\rho^2 \int_0^T \|d(t)\|^2 \, dt, \\ & \forall 0 \le T < \infty.\end{aligned} \tag{17}$$

Defining $I_0 = 2(V(E(0), \tilde{\alpha}(0)) + \frac{\delta_1}{\sigma_1} + \frac{\delta_2}{\sigma_2})$ shows that the H_∞ tracking performance (11) is achieved.

(ii) If $d \in L_\infty[0, \infty)$, there exists a $D > 0$ such that $\|d\| \leq D$. By inequality (16), $\dot{V}$ can be bounded as $\dot{V} \leq -\lambda_Q \|E\|^2 + \rho^2 D^2 + \delta_1 + \delta_2$ where λ_Q is the minimum eigenvalue of Q. Choosing $\lambda_Q > \frac{\rho^2 D^2 + \delta_1 + \delta_2}{\zeta^2}$ for any small $\zeta > 0$, there exists a $\kappa > 0$ such that $\dot{V} \leq -\kappa \|E\|^2 < 0$ for all $\|E\| > \zeta$. Thus, there is a $T > 0$ such that $\|E\| \leq \zeta$ for all $t \geq T$. This implies that the tracking error is UUB [12] and all the closed-loop signals are bounded.

(iii) Suppose that $d \in L_2[0, \infty) \cap L_\infty[0, \infty)$. Since all the closed-loop signals are bounded, error dynamic (3) along with control law (7) implies that $\dot{E}$ is also bounded. Using the proofs given for (i) and (ii) show that $\|E\|$ is bounded and square-integrable. Hence, Barbalat's lemma [12] implies that the tracking error converges to zero, despite the system uncertainties and external disturbances. □

As it was pointed out in Theorem 1, the proposed controller for the case of $d \in L_\infty[0, \infty)$ only guarantees that the tracking error is UUB. In the following, an adaptive control algorithm is developed for this class of disturbances that ensures the tracking error is driven to zero. It is assumed that $d(t)$ is bounded by an unknown constant parameter.

As a preliminary step to design the controller, define the augmented regressor vector $\phi_a(X)$ and the augmented time-varying vector $\theta_a(t)$ as

$$\begin{aligned} \phi_a(X) &:= \left[\phi^T(X), 1\right]^T, \\ \theta_a(t) &:= \left[\theta^T(t), d(t)\right]^T \end{aligned} \tag{18}$$

where $\theta_a(t) \in \Re^{p+1}$ belongs to a compact set $\Omega_a = \{\theta_a(t) \mid \|\theta_a(t)\| \leq \alpha_a\}$ and α_a is an unknown positive constant.

Theorem 2 *Consider the uncertain nonlinear system (1), perturbed by some bounded disturbance $d(t)$. Suppose that Assumptions 1 and 2 hold. The adaptive control law u, formed by (7) with $u_r = 0$ and*

$$u_a = \hat{\alpha}_a^2 \frac{\phi_a^T(X)\phi_a(X) B^T P E}{\|\phi_a(X) B^T P E\| \hat{\alpha}_a + \delta_2 e^{-\sigma_2 t}}, \tag{19}$$

$$\dot{\hat{\alpha}}_a = \gamma_a \left\|\phi_a(X) B^T P E\right\| \tag{20}$$

where P is a positive-definite matrix, $\hat{\alpha}_a$ is the estimated value of α_a and $\gamma_a > 0$ is the adaptation gain, ensures that the tracking error is converged to zero.

Proof Take a Lyapunov function candidate as

$$V(E, \tilde{\alpha}_a) = E^T P E + \frac{1}{2\gamma_a} \tilde{\alpha}_a^2 \tag{21}$$

where $\tilde{\alpha}_a = \alpha_a - \hat{\alpha}_a$ denotes the estimation error and $P = P^T > 0$ is the solution of the Lyapunov equation $A^T P + P A = -W$ for a given positive definite symmetric matrix W. Differentiating $V(E, \tilde{\alpha}_a)$ along the error trajectory (4) and using (5) yield

$$\begin{aligned} \dot{V}(E, \tilde{\alpha}_a) \leq{}& E^T \left(A^T P + P A\right) E + M(X) B^T P E \\ &+ \theta^T(t) \phi(X) B^T P E + d B^T P E \\ &+ u^T g B^T P E + \frac{1}{\gamma} \tilde{\alpha}_a \dot{\hat{\alpha}}_a. \end{aligned} \tag{22}$$

By definition (18) and substituting the proposed adaptive control law u into (22), one can obtain

$$\begin{aligned} \dot{V}(E, \tilde{\alpha}_a) \leq{}& -E^T W E + M(X) B^T P E \\ &+ \theta_a^T(t) \phi_a(X) B^T P E \\ &- \frac{E^T P B M^2(X) B^T P E}{|M(X) B^T P E| + \delta_1 e^{-\sigma_1 t}} \\ &- \hat{\alpha}_a^2 \frac{\phi_a^T(X) \phi_a(X) B^T P E}{\|\phi_a(X) B^T P E\| \hat{\alpha}_a + \delta_2 e^{-\sigma_2 t}} \\ &- \frac{1}{\gamma} \tilde{\alpha}_a \dot{\hat{\alpha}}_a. \end{aligned} \tag{23}$$

Using the adaptation law (20) and following the proof of Theorem 1 implies that

$$\dot{V}(E, \tilde{\alpha}_a) \leq -E^T W E + \delta_1 e^{-\sigma_1 t} + \delta_2 e^{-\sigma_2 t}. \tag{24}$$

Now, integrating the inequality (24) from $t = 0$ to $t = T$ yields

$$\begin{aligned} &\int_0^T \|E(t)\|_W^2 \, dt + V\left(E(T), \tilde{\alpha}_a(T)\right) \\ &\quad \leq V\left(E(0), \tilde{\alpha}_a(0)\right) + \frac{\delta_1}{\sigma_1}\left(1 - \delta_1 e^{-\sigma_1 T}\right) \\ &\qquad + \frac{\delta_2}{\sigma_2}\left(1 - \delta_2 e^{-\sigma_2 T}\right) \end{aligned} \tag{25}$$

for all $0 \leq T < \infty$ that shows $E \in L_2[0, \infty)$. On the other hand, by inequality (24), it can be concluded that

$\dot{V} \leq -\lambda_W \|E\|^2 + \delta_1 + \delta_2$, where λ_W denotes the minimum eigenvalue of W. Following the procedure given for the proof of Theorem 1, Barbalat's lemma ensures that the tracking error converges to zero despite the perturbations. □

4 Application examples

This section presents some practical aspects of the proposed control schemes for uncertain control-affine systems. Magnetic levitation system, cart pendulum system, mass-spring-damper, some oscillators, and a large class of chaotic systems are some examples that can be described by the model considered in this paper. As a result, the proposed control algorithms can be easily developed for such systems.

In order to verify and demonstrate the effectiveness of the methods, tracking control is first applied to a cart-pendulum system as an underactuated mechanical system. This system is frequently used to validate the efficiency of nonlinear control techniques. Another interesting application of the proposed control algorithms is chaos control and synchronization. Chaotic systems are nonlinear dynamical systems with some specific characteristics, e.g., irregular identities of the motion in phase-plane and excessive sensitivity to initial conditions. Since chaotic behavior can be observed in many real-world plants, chaos control and synchronization have been extensively studied in such applications as information processing, secure communication, power electronic circuits, power system collapse prevention, chemical reactions, laser systems, etc. Motivated by the aforementioned points, Genesio–Tesi system, originally conceived by Genesio and Tesi [10], is considered here since it captures many features of chaotic systems [19].

4.1 Application to a cart-pendulum system

Consider an inverted pendulum on a cart. The objective is to design a controller such that the pole's angular position tracks the desired trajectory. The dynamic equations of the system are stated as [21]

$$\dot{x}_1 = x_2,$$
$$\dot{x}_2 = \frac{g\sin x_1 - \cos x_1 (\frac{m_p}{m_c+m_p} l x_2^2 \sin x_1 - \frac{1}{m_c+m_p} u)}{l(\frac{4}{3} - \frac{m_p}{m_c+m_p}\cos^2 x_1)} + d(t) \tag{26}$$

where $x_1 = \theta$, $|\theta| < \frac{\pi}{2}$, denotes the angular position and $x_2 = \dot{\theta}$ is the angular velocity of the pole. Moreover, m_c and m_p represent the mass of the cart and the mass of the pole, respectively, l is half of the pole's length and g is the gravitational acceleration. From a practical point of view, the values of the masses may be time-varying. For instance, the manipulator is carrying a metal load and moving in a magnetic field, or is carrying a tank and watering. So, consider the masses as unknown time-varying ones, i.e., $m_c(t)$ and $m_p(t)$, and define the time-varying parameter $\eta(t) = \frac{m_p(t)}{m_p(t)+m_c(t)}$. Writing the dynamical equations (26) in the form of (1) implies that

$$f(X,t) = \frac{g\sin x_1}{l(\frac{4}{3} - \eta(t)\cos^2 x_1)} - \frac{\eta(t) x_2^2 \sin x_1 \cos x_1}{(\frac{4}{3} - \eta(t)\cos^2 x_1)},$$
$$g(X,t) = \frac{\frac{\cos x_1}{m_c+m_p}}{l(\frac{4}{3} - \eta(t)\cos^2 x_1)}.$$

It can be shown that for a real constant $a \geq 1$ and a positive variable $\xi < 1$ the inequality $\frac{a}{a-\xi} \leq \frac{1}{a-1}\xi + 1$ holds. Using the fact $\eta(t)\cos^2 x_1 < 1$, one obtains an upper bound for $f(X,t)$ as

$$\begin{aligned} f(X,t) &\leq \frac{g}{l}|\sin x_1|\left(\frac{3}{4} + \left(\frac{9}{4}\cos^2 x_1\right)\eta(t)\right) \\ &\quad + x_2^2 |\sin x_1 \cos x_1| \\ &\quad \times \left(\frac{3}{4} + \left(\frac{9}{4}\cos^2 x_1\right)\eta(t)\right) \\ &= \underbrace{\frac{3g}{4l}|\sin x_1| + \frac{3}{4}x_2^2|\sin x_1\cos x_1|}_{L(X)} \\ &\quad + \underbrace{\left(\frac{9}{4}|\sin x_1|\cos^2 x_1\right)\left(\frac{g}{l} + x_2^2|\cos x_1|\right)}_{\phi(X)} \\ &\quad \times \eta(t). \end{aligned} \tag{27}$$

In order to obtain a state-dependent lower bound for $g(X,t)$, suppose that $m_p \leq m_c \leq m_{\max}$. It implies

$$g(X,t) \geq \frac{\cos x_1}{m_c + m_p} \cdot \frac{3}{4l} \geq \frac{3\cos x_1}{8 l m_{\max}} = b(X).$$

Hence, the controllers proposed in this paper, can be easily developed for the uncertain inverted pendulum system perturbed by external disturbances. In the following, some simulation results are presented, using the controller developed in Theorem 1.

Defining $y(t) = x_1(t)$, the desired angle trajectory is assumed to be $y_r(t) = 0.12\sin(\frac{\pi}{3}t)$. The system parameters are taken as $l = 0.3$ (m), $g = 9.8$ (m/s^2) and initial states as $x_1(0) = 0.05$ and $x_2(0) = 0$. Motivated by the structure proposed in [26] for time-varying masses of manipulators, we take $m_c(t) = 1 + 0.1\sin(2t)$ and $m_p(t) = 0.15 + (t^2 + 2t + 0.1)/(t^3 + 1.5t^2 + 3t + 1)$. The gain vector and the weighting matrix are, respectively, taken as $K = [2\,2]^T$ and $Q = 2I_{2\times 2}$. For attenuation levels $\rho = 1$ and $\rho = 0.3$, the H_∞ controller gains are computed as $r = 1$ and $r = 0.09$, respectively. The Riccati-like inequality (6) gives

$$P = \begin{bmatrix} 20 & 2 \\ 2 & 6 \end{bmatrix}$$

By selecting the adaptation gain $\gamma = 0.5$ and the constants $\delta_i = 0.5$, and $\sigma_i = 0.1$, $i = 1, 2$, the robust tracking controller developed in Theorem 1 can be easily constructed. Two cases are considered here.

Case 1: Bounded disturbance $d(t) = 2\sin 5t$. The simulation results are illustrated in Fig. 1. The angular position $x_1(t)$ tracks the desired trajectory for both attenuation levels. Figures 1(a) and 1(c) show that choosing a lower level of attenuation can more effectively override the effect of external disturbances.

Case 2: Bounded and square-integrable disturbance $d(t) = 2e^{-0.5t}\sin 5t$. Figure 2 demonstrates the simulation results in the presence of system uncertainties and the external disturbance $d(t)$. As expected, the effects of perturbations are diminished and tracking error converges to zero. Besides, choosing a proper attenuation level ρ gives a better tracking performance.

4.2 Application to Genesio–Tesi chaotic system

The dynamic equations of unforced Genesio–Tesi chaotic system are given by [19]

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= x_3, \\ \dot{x}_3 &= -ax_1 - bx_2 - cx_3 + x_1^2 \end{aligned} \tag{28}$$

where a, b, and c are three positive parameters. It is well known that this system exhibits chaotic behavior for $a = 6$, $b = 2.92$ and $c = 1.2$. For the unforced system (28), known as the drive system, the response system is defined as

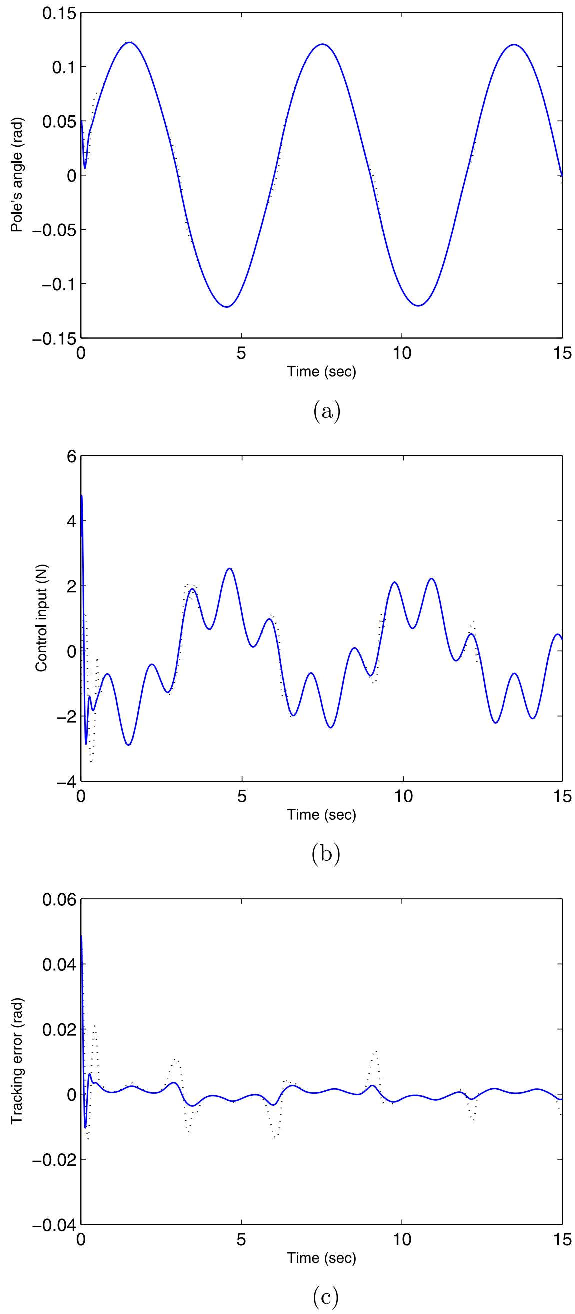

Fig. 1 Tracking control of cart-pendulum system—Case 1, $\rho = 1$ (··) and $\rho = 0.3$ (–), (**a**) Output response; (**b**) Control input; and (**c**) Tracking error

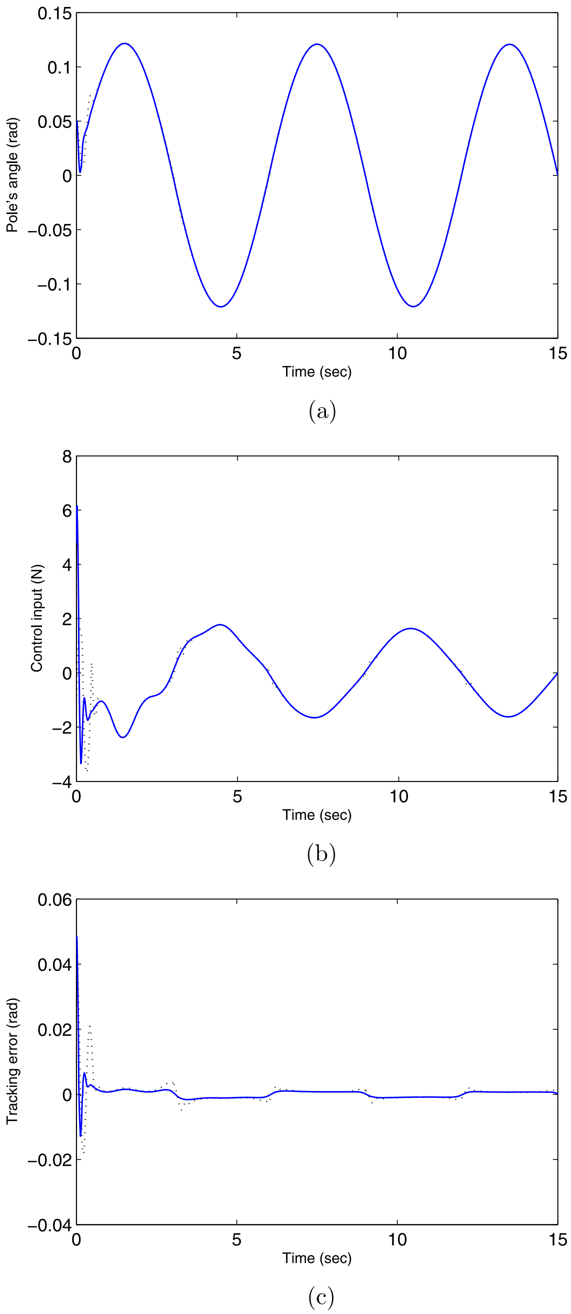

Fig. 2 Tracking control of cart-pendulum system—Case 2, $\rho = 1$ (··) and $\rho = 0.3$ (–), (**a**) Output response; (**b**) Control input; and (**c**) Tracking error

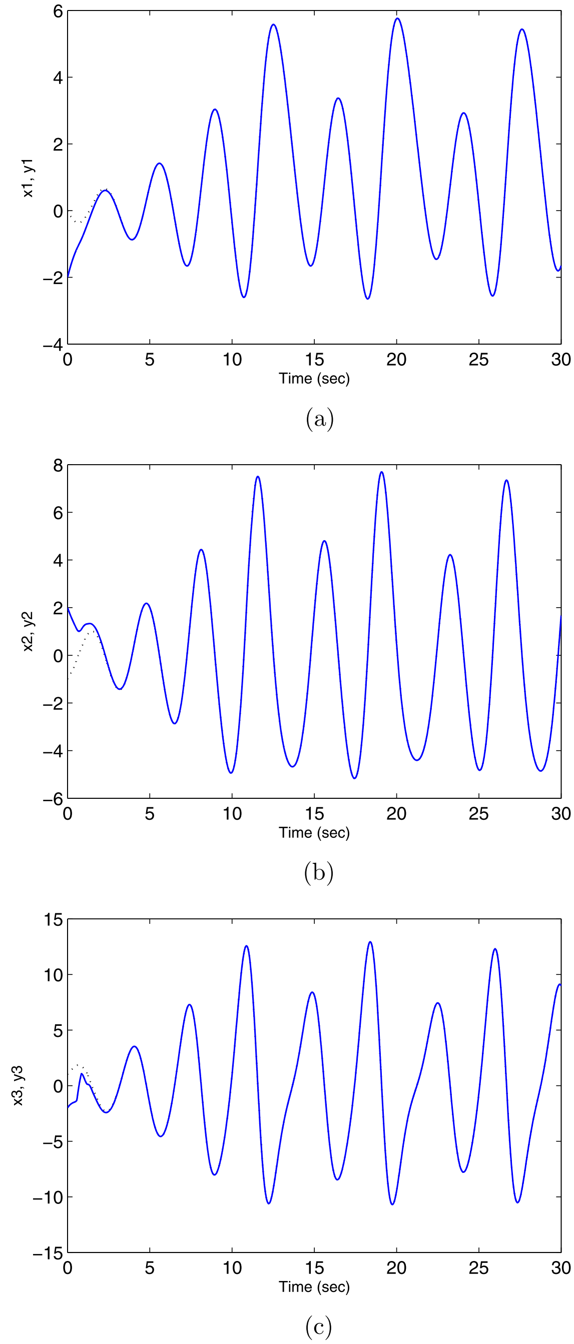

Fig. 3 The states of Genesio–Tesi chaotic system; drive system: *dotted line*, response system: *solid line* (**a**) x_1 and y_1, (**b**) x_2 and y_2, and (**c**) x_3 and y_3

$$\begin{aligned} \dot{y}_1 &= y_2, \\ \dot{y}_2 &= y_3, \\ \dot{y}_3 &= -a(t)y_1 - b(t)y_2 - c(t)y_3 + y_1^2 + u(t) + d(t) \end{aligned} \tag{29}$$

where $d(t)$ denotes the external disturbance and $a(t)$, $b(t)$, and $c(t)$ are some unknown parameters. The unknown time-varying parameter vector of the system is formed as $\theta(t) = [a(t)\ b(t)\ c(t)]^T$. The ob-

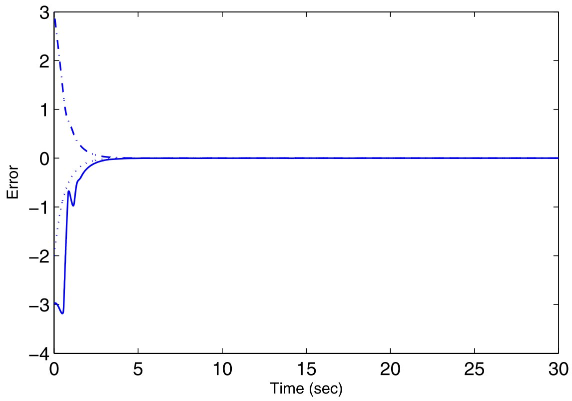

Fig. 4 Tracking errors, y_1–x_1: *dotted line*, y_2–x_2: *dash dotted line*, $y_3 - x_3$: *solid line*

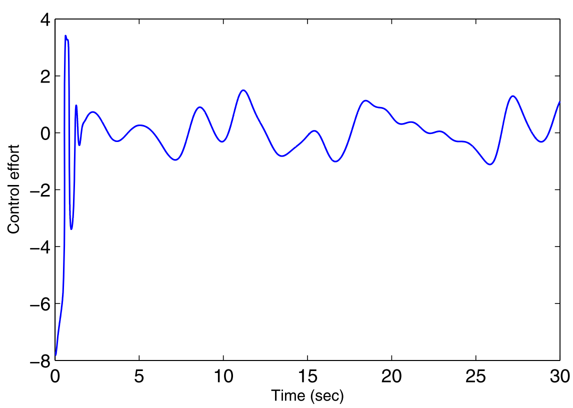

Fig. 5 The control input applied to Genesio–Tesi chaotic system

jective is to design an appropriate control signal $u(t)$ such that the states of the response system (29) track the trajectories produced by those of the drive system (28), i.e., $\lim_{t\to\infty} \|E(t)\| = 0$, where $E(t) = [y_1(t) - x_1(t)\ y_2(t) - x_2(t)\ y_3(t) - x_3(t)]^T$. In the following, the robust adaptive control scheme developed in Theorem 2 is used to solve the underlying synchronization problem. Comparing with inequality (5), one can conclude $\phi(X) = [-x_1\ -x_2\ -x_3]^T$ and $L(X) = x_1^2$. To present some simulation results, the time-varying parameter vector and external disturbance signal are selected as $\theta(t) = [6\sin t\ 2(1+\cos t)\ 1.2 - \sin t]^T$ and $d(t) = 0.5\sin 2t$. The initial states of the drive system and the response system are taken as $x_1(0) = 0$, $x_2(0) = -1$, $x_3(0) = 1$, $y_1(0) = -2$, $y_2(0) = 2$, and $y_3(0) = -2$. Choosing the gain vector $K = [2\ 2\ 2]^T$ and determining the positive definite matrix

$$P = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 10 & 0 & 5 \end{bmatrix}$$

the proposed adaptive controller of Theorem 2 can be developed. The simulation results for $\delta_i = 2$, and $\sigma_i = 0.1$, $i = 1, 2$, and $\gamma_a = 0.2$ are given in Figs. 3, 4 and 5. The results demonstrate the feasibility and effectiveness of the proposed adaptive controller in chaos synchronization.

5 Conclusions

The so-called control-affine systems, perturbed by time-varying parameters and external disturbances are considered. An adaptive-based H_∞ control scheme is first developed for the uncertain systems perturbed by L_2 disturbances to achieve a prescribed H_∞ tracking performance. For the case of bounded external disturbances, this controller assures the convergence of tracking error to zero. Then removing the assumption that disturbance is an L_2 signal, a novel adaptive tracking controller is proposed to converge the tracking error to zero in the presence of bounded time-varying parameters and external disturbances. An important contribution of the methods presented in this paper is that the unknown time-varying parameters and disturbances are neither required to be periodic nor to have known bounds. The proposed control schemes are employed for controlling a cart-pendulum system and chaos synchronization of uncertain Genesio–Tesi chaotic system. Numerical simulations illustrate the effectiveness of the proposed control algorithms.

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