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Lyapunov type stability and Lyapunov exponent for exemplary multiplicative dynamical systems

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Abstract This paper presents analysis of Lyapunov type stability for multiplicative dynamical systems. It has been formally defined and numerical simulations were performed to explore nonlinear dynamics. Chaotic behavior manifested for exemplary multiplicative dynamical systems has been confirmed by calculated Lyapunov exponent values.

Keywords Multiplicative calculus · Lyapunov stability · Lyapunov exponent

1 Introduction

Chaotic behavior can be observed in systems behavior from all fields of science. Our interests concern chaos occurring in process of defects growth in materials. Models of fractal defects evolution presented in [1–3] link the energy uniformly distributed over fractal and its measure ν_D using essential material characteristics, which is energy density $a(D)$ depending on fractal dimension:

$$\mathcal{E} = a(D)\nu_D.$$

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In order to verify this theoretical model, stochastic numerical simulations of breaking fibers in composite were performed and described in [2]. The idea we would like to examine is if growing defects may behave in a chaotic way. Describing the evolution of defects treated as fractals implies usage of multiplicative derivatives, because ordinary additive derivative of function depending on fractal dimension or measure does not exist. Therefore, multiplicative calculus presented in [4] and restored in [1, 3] must be applied.

The goal of the paper is chaos examination in multiplicative dynamical systems described with multiplicative derivatives. Derived and tested methods will be employed to systems of fractal defects evolution. In this paper, calculations are performed for multiplicative counterparts of well-known dynamical systems: Lorenz system, which we will name multiplicative Lorenz system. It can be described with multiplicative derivatives, and also with additive derivatives, using the relation between additive and multiplicative derivatives presented in [1]. Analysis of nonautonomous multiplicative Lorenz system described with additive derivatives has been executed using the Lyapunov stability theory [5] and presented in [6].

This paper contains derivation of stability theory of the Lyapunov type for system of autonomous multiplicative differential equations. Obtained formula is tested for multiplicative Lorenz system described with multiplicative derivatives.

This paper also contains a proposed definition of a Lyapunov exponent for the multiplicative dynamical system. In order to verify it, numerical calculations were performed for one-dimensional multiplicative version of a logistic equation. For the multiplicative Lorenz system described with multiplicative derivatives, the largest Lyapunov exponent was obtained.

2 Stability of Lyapunov type

On the basis of the Lyapunov theorem about stability for ordinary differential equations [5], its counterpart for multiplicative dynamical system has been derived.

For the system of autonomous multiplicative differential equations:

$$\frac{\pi x_j}{\pi t} = f_j(x_1, \ldots, x_n), \quad j = 1, \ldots, n \tag{1}$$

equilibria are calculated from:

$$f_j(x_1, \ldots, x_n) = 1. \tag{2}$$

The solution close to fixed point $\boldsymbol{x}_0 = (x_1^{(0)}, x_2^{(0)}, \ldots, x_n^{(0)})$ is described by a small multiplicative shift $\epsilon_j(t)$:

$$x_j(t) = x_j^{(0)}\bigl(1 + \epsilon_j(t)\bigr). \tag{3}$$

Behavior of derivative $\frac{\pi \epsilon_j(t)}{\pi t}$ for long time determines if equilibrium $\boldsymbol{x}_0$ is stable. The value of this derivative can be obtained by comparison of derivatives of both sides of (3). The derivative of the right-hand side of (3) is equal:

$$\frac{\pi}{\pi t}\bigl(x_j^{(0)}\bigl(1 + \epsilon_j(t)\bigr)\bigr) = \left(\frac{\pi \epsilon_j(t)}{\pi t}\right)^{\frac{\epsilon_j(t)}{1+\epsilon_j(t)}}. \tag{4}$$

First order expansion of multiplicative derivative of the left-hand side of (3) is equal:

$$\begin{aligned}\frac{\pi x_j(t)}{\pi t} &= f_j(\boldsymbol{x}_0) \prod_{k=1}^{n} \left(\frac{\pi f_j(\boldsymbol{x}_0)}{\pi x_k}\right)^{\epsilon_k(t)} \\ &= \prod_{k=1}^{n} \left(\frac{\pi f_j(\boldsymbol{x}_0)}{\pi x_k}\right)^{\epsilon_k(t)}.\end{aligned} \tag{5}$$

Comparison of (4) and (5) for $\epsilon_j(t) \to 0$ gives:

$$\left(\frac{\pi \epsilon_j(t)}{\pi t}\right)^{\epsilon_j(t)} = \prod_{k=1}^{n} \left(\frac{\pi f_j(\boldsymbol{x}_0)}{\pi x_k}\right)^{\epsilon_k(t)}, \tag{6}$$

and logarithm of both sides of (6) equals:

$$\epsilon_j(t) \ln \frac{\pi \epsilon_j(t)}{\pi t} = \sum_{k=1}^{n} \epsilon_k(t) \ln \frac{\pi f_j(\boldsymbol{x}_0)}{\pi x_k}. \tag{7}$$

For every basis in (4), we calculate derivatives along the direction of coordinates. Therefore, the left-hand side of (7) is always diagonal. Choosing the particular basis in which the matrix $[\frac{\pi f_j(\boldsymbol{x}_0)}{\pi x_k}]$ is also diagonal, we obtain:

$$\ln \frac{\pi \epsilon_j(t)}{\pi t} = \lambda_j \;\rightarrow\; \frac{\pi \epsilon_j(t)}{\pi t} = e^{\lambda_j}, \tag{8}$$

where λ_j denotes the corresponding jth eigenvalue. Application of the relationship between additive and multiplicative derivative [3]:

$$\frac{\pi f(x)}{\pi x} = \exp\left\{x \frac{f'(x)}{f(x)}\right\} \tag{9}$$

to (8) gives us an additive ordinary differential equation:

$$\frac{\pi \epsilon_j(t)}{\pi t} = \exp\left\{t \frac{\dot{\epsilon}_j(t)}{\epsilon_j(t)}\right\} = e^{\lambda_j} \implies \dot{\epsilon}_j(t) = \frac{1}{t}\lambda_j \epsilon_j(t), \tag{10}$$

where $\dot{\epsilon}_j(t)$ is an ordinary additive derivative. Its solution equals:

$$\epsilon_j(t) = \epsilon_j^{(0)} t^{\lambda_j}, \tag{11}$$

where $\epsilon_j^{(0)}$ is determined by initial conditions. According to result (11), stability of fixed point $\boldsymbol{x}_0$ depends on eigenvalues of matrix $[\ln \frac{\pi f_j(\boldsymbol{x}_0)}{\pi x_k}]$, which is counterpart of stability Jacobian matrix obligatory for multiplicative dynamical system. For complex eigenvalues $\Re(\lambda_j) + i\Im(\lambda_j)$, their changes depend on absolute shift $\epsilon_j(t)$:

$$\begin{aligned}\bigl|\epsilon_j(t)\bigr| &= \bigl|\epsilon_j^{(0)}\bigr|\bigl|t^{\Re(\lambda_j)+i\Im(\lambda_j)}\bigr| \\ &= \bigl|\epsilon_j^{(0)}\bigr|\bigl|t^{\Re(\lambda_j)} e^{i\Im(\lambda_j)\ln t}\bigr| = \bigl|\epsilon_j^{(0)}\bigr| t^{\Re(\lambda_j)}.\end{aligned} \tag{12}$$

Similarly, as in the case of additive systems, if the real part of eigenvalue $\Re(\lambda_j)$ is negative, for long time $\epsilon_j(t) \to 0$ and $\boldsymbol{x}_0$ is stable. If $\Re(\lambda_j)$ is positive, value of $\epsilon_j(t)$ increases according to power function of time and $\boldsymbol{x}_0$ is unstable.

3 Lyapunov type stability for the multiplicative Lorenz system

Lyapunov type stability for multiplicative dynamical systems has been examined using a multiplicative version of the Lorenz system. The classical Lorenz system contains three nonlinear equations with three positive real parameters:

$$\begin{aligned} \frac{dx}{dt} &= \sigma y - \sigma x, \\ \frac{dy}{dt} &= rx - y - xz, \\ \frac{dz}{dt} &= xy - bz. \end{aligned} \tag{13}$$

The multiplicative Lorenz system was formed by replacing both sides of (13) with their multiplicative equivalent quantities:

$$\begin{aligned} \frac{\pi x}{\pi t} &= y^{\sigma} x^{-\sigma}, \\ \frac{\pi y}{\pi t} &= x^{r} y^{-1} x^{-\ln z}, \\ \frac{\pi z}{\pi t} &= x^{\ln y} z^{-b}, \end{aligned} \tag{14}$$

which means that additive derivatives were replaced by multiplicative ones, addition was replaced by multiplication, and multiplication by raising to suitable power according to rule that $a^{\ln b} = b^{\ln a}$. Finally, $\ln \sigma$, $\ln r$, and $\ln b$ were replaced by σ, r, and b.

Equilibria of the multiplicative Lorenz system (14) calculated from:

$$\begin{aligned} \frac{\pi x}{\pi t} &= 1 \to y^{\sigma} x^{-\sigma} = 1, \\ \frac{\pi y}{\pi t} &= 1 \to x^{r} y^{-1} x^{-\ln z} = 1, \\ \frac{\pi z}{\pi t} &= 1 \to x^{\ln y} z^{-b} = 1 \end{aligned} \tag{15}$$

are equal:

$$\begin{aligned} C_1 &= (1, 1, 1), \\ C_2 &= \big(\exp\big(-\sqrt{b(r-1)}\big), \\ &\qquad \exp\big(-\sqrt{b(r-1)}\big), \exp(r-1)\big), \\ C_3 &= \big(\exp\big(\sqrt{b(r-1)}\big), \exp\big(\sqrt{b(r-1)}\big), \\ &\qquad \exp(r-1)\big). \end{aligned} \tag{16}$$

For the multiplicative Lorenz system matrix containing logarithms of Jacobian matrix elements is equal:

$$\begin{aligned} Df &= \begin{bmatrix} \ln\big(\frac{\pi f_1}{\pi x}\big) & \ln\big(\frac{\pi f_1}{\pi y}\big) & \ln\big(\frac{\pi f_1}{\pi z}\big) \\ \ln\big(\frac{\pi f_2}{\pi x}\big) & \ln\big(\frac{\pi f_2}{\pi y}\big) & \ln\big(\frac{\pi f_2}{\pi z}\big) \\ \ln\big(\frac{\pi f_3}{\pi x}\big) & \ln\big(\frac{\pi f_3}{\pi y}\big) & \ln\big(\frac{\pi f_3}{\pi z}\big) \end{bmatrix} \\ &= \begin{bmatrix} -\sigma & \sigma & 0 \\ -\ln z + r & -1 & -\ln x \\ \ln y & \ln x & -b \end{bmatrix}. \end{aligned} \tag{17}$$

For fixed point C_1, we obtain:

$$Df(C_1) = \begin{bmatrix} -\sigma & \sigma & 0 \\ r & -1 & 0 \\ 0 & 0 & -b \end{bmatrix}, \tag{18}$$

and for fixed points $C_{2,3}$, we have:

$$Df(C_{2,3}) = \begin{bmatrix} -\sigma & \sigma & 0 \\ 1 & -1 & -c \\ c & c & -b \end{bmatrix}, \tag{19}$$

where $c = \pm\sqrt{b(r-1)}$. Both matrices (18) and (19) are equal to corresponding matrices calculated for the classical Lorenz system according to [7]. Therefore, stability analysis for the multiplicative Lorenz system gives the same results as for the classical one. For $r < 1$, fixed point C_1 is stable and for $r < r_c = \sigma\frac{\sigma+b+3}{\sigma-b-1}$ fixed points $C_{2,3}$ are stable. For $r > r_c$, equilibria $C_{2,3}$ are unstable.

The relation (9) between classical derivative $f'(x)$ and multiplicative derivative $\frac{\pi f(x)}{\pi x}$ of function allows us to describe the multiplicative Lorenz system (14) with the additive derivatives:

$$\begin{aligned} \frac{dx}{dt} &= \frac{x}{t}\sigma(\ln y - \ln x), \\ \frac{dy}{dt} &= \frac{y}{t}(r\ln x - \ln y - \ln x \ln z), \\ \frac{dz}{dt} &= \frac{z}{t}(\ln x \ln y - b\ln z). \end{aligned} \tag{20}$$

It was examined with analytical and numerical methods and results were presented in [6]. Similarly, as in the case of classical Lorenz system (13), chaos appears when $r > r_c$. In logarithmic scale, we may observe the butterfly shape attractor created by trajectory alternatingly spiraling out one of the equilibria. Both systems (14) and (20) contain values from $\mathbb{R}_+$.

3.1 Numerical simulations

Similarly, as in the case of classical nonlinear dynamical systems described with additive derivatives, the solutions of nonlinear multiplicative dynamical systems can be calculated using numerical methods. The Runge–Kutta methods belong to the most popular numerical methods of solving ordinary differential equations. Therefore, the multiplicative fourth order Runge–Kutta method was derived, tested, and presented in [8]. The general formula of this method written for multiplicative differential equation $\frac{\pi x}{\pi t} = f(x,t)$ has a form:

$$\begin{aligned} &x_{t(1+\epsilon)} = x_t \prod_{i=1}^{p} {k_i}^{c_i} O\big(\epsilon^{p+1}\big), \\ &i = 1 \;\Rightarrow\; k_i = f(t_t, x_t)^{\epsilon}, \\ &i > 1 \;\Rightarrow\; k_i = \left(f\left(t(1+a_i\epsilon), x_t \prod_{j=1}^{i-1} k_j^{b_{ij}} \right) \right)^{\epsilon}, \end{aligned} \tag{21}$$

which contains information about midpoints position expressed by coefficients a_i, multiplicative shift ϵ, and method order p.

For the fourth order multiplicative Runge–Kutta method, the following system of equations presented in [8] allows to indicate values of b_{ij} and c_i:

$$\begin{aligned} &c_1 + c_2 + c_3 + c_4 = 1 - \frac{1}{2}\epsilon + \frac{1}{3}\epsilon^2 - \frac{1}{4}\epsilon^3, \\ &a_2c_2 + a_3c_3 + a_4c_4 = \frac{\frac{1}{2}\epsilon - \frac{1}{2}\epsilon^2 + \frac{11}{24}\epsilon^3}{\epsilon - \frac{1}{2}\epsilon^2 + \frac{1}{3}\epsilon^3}, \\ &{a_2}^2c_2 + {a_3}^2c_3 + {a_4}^2c_4 = \frac{\frac{1}{6}\epsilon^2 - \frac{1}{4}\epsilon^3}{\frac{1}{2}\epsilon^2 - \frac{1}{2}\epsilon^3}, \\ &a_2b_{32}c_3 + a_2b_{42}c_4 + a_3b_{43}c_4 = \frac{1}{2}\frac{\frac{1}{6}\epsilon^2 - \frac{1}{4}\epsilon^3}{\frac{1}{2}\epsilon^2 - \frac{1}{2}\epsilon^3}, \\ &{a_2}^3c_2 + {a_3}^3c_3 + {a_4}^3c_4 = \frac{1}{4}, \\ &{a_2}^2b_{32}c_3 + {a_2}^2b_{42}c_4 + {a_3}^2b_{43}c_4 = \frac{1}{12}, \\ &a_2b_{32}b_{43}c_4 = \frac{1}{24}, \\ &a_2a_3b_{32}c_3 + a_2a_4b_{42}c_4 + a_3a_4b_{43}c_4 = \frac{1}{8}. \end{aligned} \tag{22}$$

Coefficients b_{ij}, where $a_i = \sum_j b_{ij}$, and c_i depend on multiplicative shift ϵ value, therefore, they should be calculated every time when it is changed.

Computer simulations of multiplicative Lorenz system were performed for $a_2 = \frac{1}{3}$, $a_3 = \frac{2}{3}$, $a_4 = 1$, and $10^{-10} \leqslant \epsilon \leqslant 10^{-3}$ to provide relative error value less then 1%.

Numerical solutions of the multiplicative Lorenz system (14) have confirmed analysis of stability. Chaos appears for $r > r_c$. Trajectory spirals out two equilibria C_2 and C_3 and creates butterfly shape attractor, which is visible in logarithmic scale. Exemplary trajectories for $\sigma = 10$, $b = \frac{8}{3}$ and various r values are presented in Fig. 1. Bifurcation diagram is presented in Fig. 2.

4 Lyapunov exponent for multiplicative dynamical systems

Stability analysis (11) indicates that divergence or convergence of trajectories with close starting points proceeds according to power of time:

$$\epsilon_k \approx \epsilon_0 t^{\lambda}, \tag{23}$$

where ϵ_k is multiplicative shift of argument in k-step of time. Therefore, the Lyapunov exponent for the multiplicative dynamical system can be defined as:

$$\lambda = \lim_{t\to\infty} \frac{1}{\ln t} \ln \frac{\epsilon_k}{\epsilon_0} \tag{24}$$

and if it is positive, trajectory behavior is chaotic.

4.1 Lyapunov exponent for logistic equation

The Lyapunov exponent has been calculated for simple one-dimensional nonlinear system: multiplicative version of logistic equation, which we introduce in the form:

$$x_{2n} = \exp\big(r x_n (1 - x_n)\big), \tag{25}$$

where n is a step number. In case of the multiplicative system, shift of step must be multiplicative: $n \to n(1+\Delta n)$, where $\Delta n = 1$ for discrete system. Therefore, following numbers of steps calculated as $2n$ create geometric sequence with a constant ratio equal 2.

There are possible, other than (25), multiplicative versions of the classical logistic equation known as

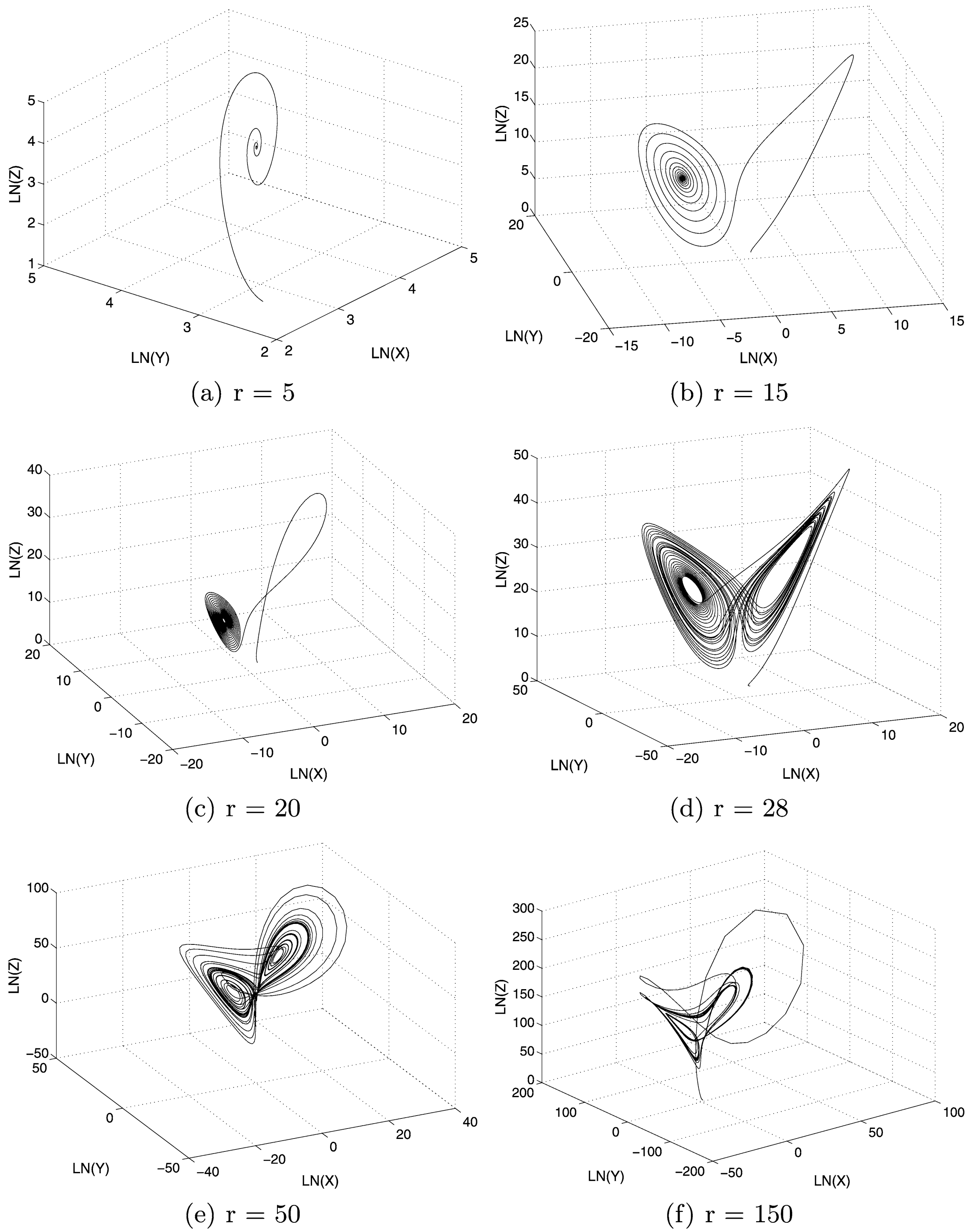

Fig. 1 Trajectories of the multiplicative Lorenz system (14) for $\sigma = 10$, $b = \frac{8}{3}$, and various r values

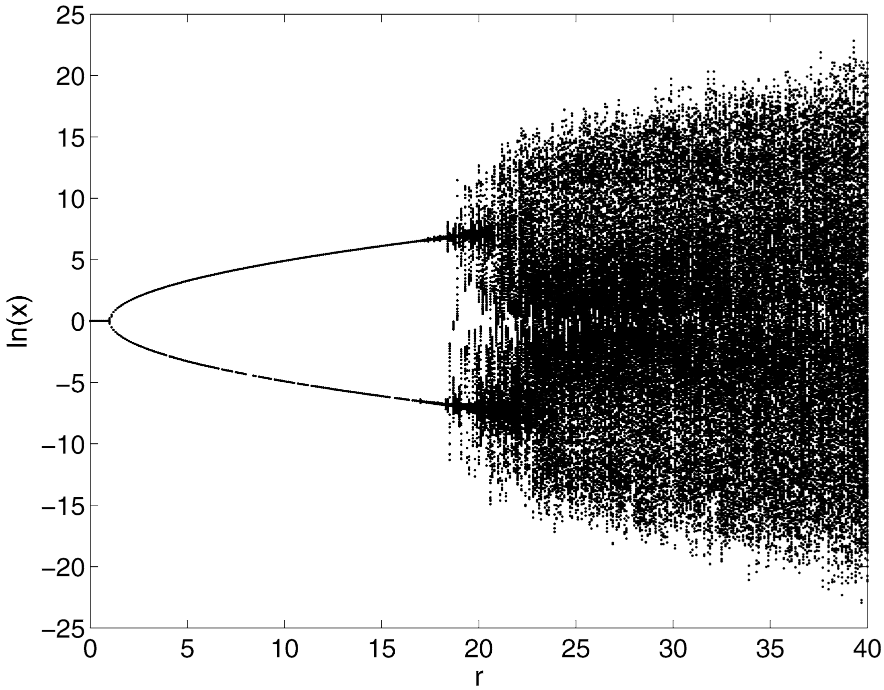

Fig. 2 Bifurcation diagram of the multiplicative Lorenz system

$x_{n+1} = rx_n(1 - x_n)$. In this case, replacing addition by multiplication and multiplication by raising to a suitable power leads to the following form of the multiplicative system: $rx - rx^2 \to \frac{x^{\ln x}}{x^{\ln(rx)}} = x^{-\ln x}$, which does not depend on r parameter. However, we can propose several other multiplicative versions, which are featured with similar chaotic behavior as a classical logistic equation. For example, for $x_{2n} = rx^{1-x}$ and for $x_{2n} = (rx)^{1-x}$ chaos appears for certain r parameter values, although the range of r value differs in both cases. In this paper, version (25) is considered as a simple example of the one-dimensional nonlinear system and tested in detail.

Because (25) is a simple transformation of a classical logistic equation making use of exponential function, we expect that its behavior is similar to the classical one and the solution depends on parameter r value. Numerical simulations were performed and their results prove that chaos manifests itself for certain r values. The bifurcation diagram for the multiplicative logistic equation shown in Fig. 3 presents limiting trajectory behavior for parameter r values in range $0.01 < r < 4$.

Phase diagrams and step series for various r values are presented in Fig. 4. For $r = 1.5$, there are only two solutions and for $r = 4$, chaotic behavior appears.

The general principle of calculating Lyapunov exponent for the multiplicative nonlinear system is the same as for classical dynamical system described in [9]. Two trajectories starts from nearby points with distance determined by a very small multiplicative shift ϵ_0 and in each kth step, multiplicative shifts ϵ_k between points are calculated. For many steps, we obtain value of Lyapunov exponent:

$$\lambda = \frac{1}{\ln n} \sum_{k=1}^{n} \ln \frac{\epsilon_k}{\epsilon_0}, \tag{26}$$

where n informs how many iterations were calculated. In every iteration, readjustments of distance between two trajectories are applied, therefore, in each step starting distance is determined by a small multiplicative shift ϵ_0.

For multiplicative logistic equation (25), the Lyapunov exponent has been calculated also in a different way: using distance d_k between points of two trajectories:

$$\lambda = \frac{1}{\ln n} \sum_{k=1}^{n} \ln \frac{d_k}{d_0}, \tag{27}$$

where d_0 is the initial small distance, d_k is distance in kth step of iteration.

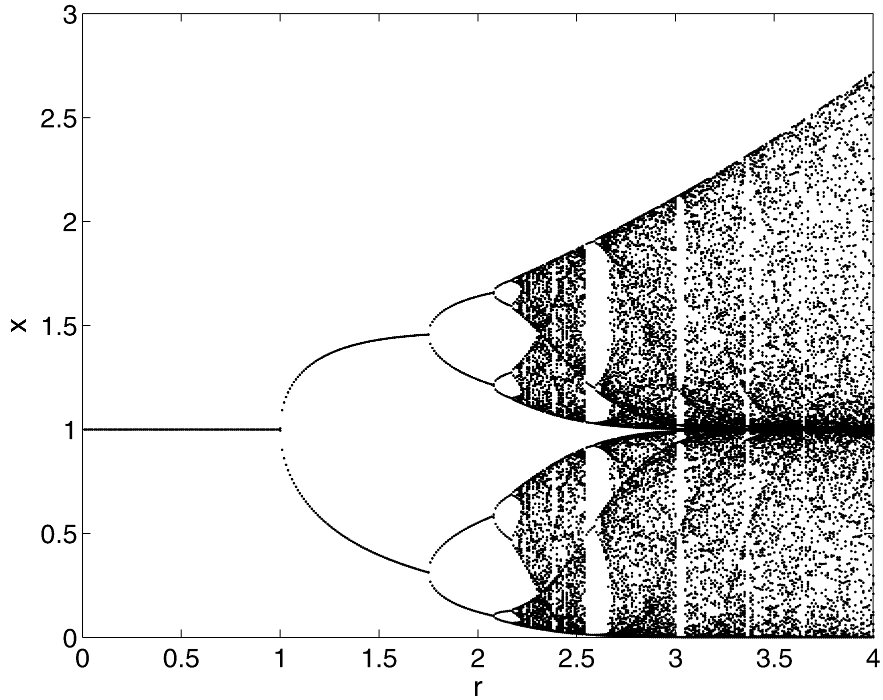

Fig. 3 Bifurcation diagram of the multiplicative logistic equation

Relationship between values of Lyapunov exponents and r parameter is presented in Fig. 5; it is compatible with the bifurcation diagram and it also indicates when behavior of the multiplicative logistic equation becomes chaotic.

Values of the Lyapunov exponent for $r = 1.5$ and $r = 4$ calculated according to method (26) and (27) are presented respectively in Tables 1 and 2. Because there are not significant differences between results from these two methods, we assume that also for multidimensional multiplicative systems, it is appropriate to calculate Lyapunov exponents tracking distance, which means not multiplicative, but an additive shift between close trajectories.

4.2 Lyapunov exponent for multiplicative Lorenz system

In case multidimensional dynamical systems calculating Lyapunov exponents is more complicated, local behavior of close trajectories may vary with the direction. The small sphere centered at starting point evolves in time into ellipsoid, which orientation in space is changing. On the base of ellipsoid linear extent $L(t)$, the largest Lyapunov exponent introduced in [9] is equal:

$$\lambda = \lim_{t\to\infty} \frac{1}{t} \ln \frac{L(t)}{L(0)} \tag{28}$$

and for multiplicative dynamical systems, the largest Lyapunov exponent should be calculated as:

$$\lambda = \lim_{t\to\infty} \frac{1}{\ln t} \ln \frac{L(t)}{L(0)}, \tag{29}$$

because time shift is multiplicative.

Results of the largest Lyapunov exponent computation for the multiplicative Lorenz system are given in Table 3.

In Fig. 6, values of the largest Lyapunov exponent for the multiplicative Lorenz system for $r = 5$ and $r = 28$ are presented. Value n in plots in Fig. 6 is a step number: for $n = 4 \times 10^5$ steps, time starts from 0.01 with multiplicative shift 0.01 is equal $1.31031535943794 \times 10^{1726}$.

Comparing results obtained for multiplicative Lorenz system described with multiplicative derivatives (14) and the multiplicative Lorenz system described with additive derivatives (20) presented in [6], we may conclude that this is the same system expressed in various calculus. Stability analysis gives the same results, which means that chaos appears for

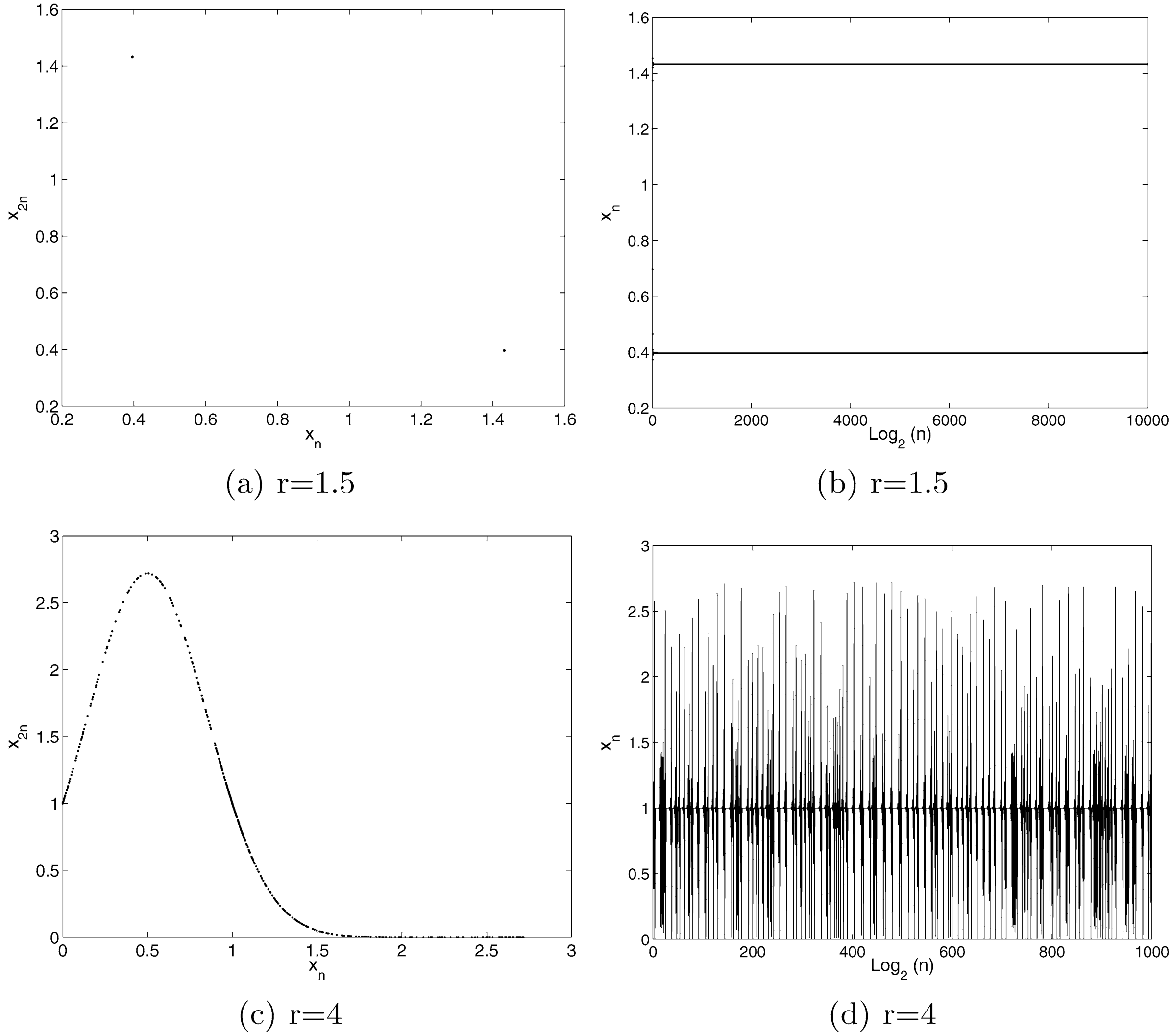

Fig. 4 Phase diagrams and step series for the multiplicative logistic equation

the same values of r parameter. Attractors in both systems have the same shape. There are differences following that system (20) is nonautonomous and time t is in denominator. In numerical simulations of system (20), construction of the attractor is delayed, which is manifested in the bifurcation diagram presented in [6]. Another difference occurs in Lyapunov exponent values because dependence on time has significant influence on decreasing or increasing exponent values, which converge to zero for a long time.

According to [5], in additive unstable chaotic systems, trajectories disperse in an exponential way. In the same time, instability or chaos in multiplicative dynamical systems entails power form of dispersion of trajectories. Nevertheless, comparison of two kinds of the Lorenz system suggests that chaotic behavior is such a general property that it is not too sensitive to this type of calculus, additive, or multiplicative one.

5 Discussion

The paper presents method of the Lyapunov type stability examination for the multiplicative dynamical systems described with multiplicative derivatives. Because of proportional increments in biological and

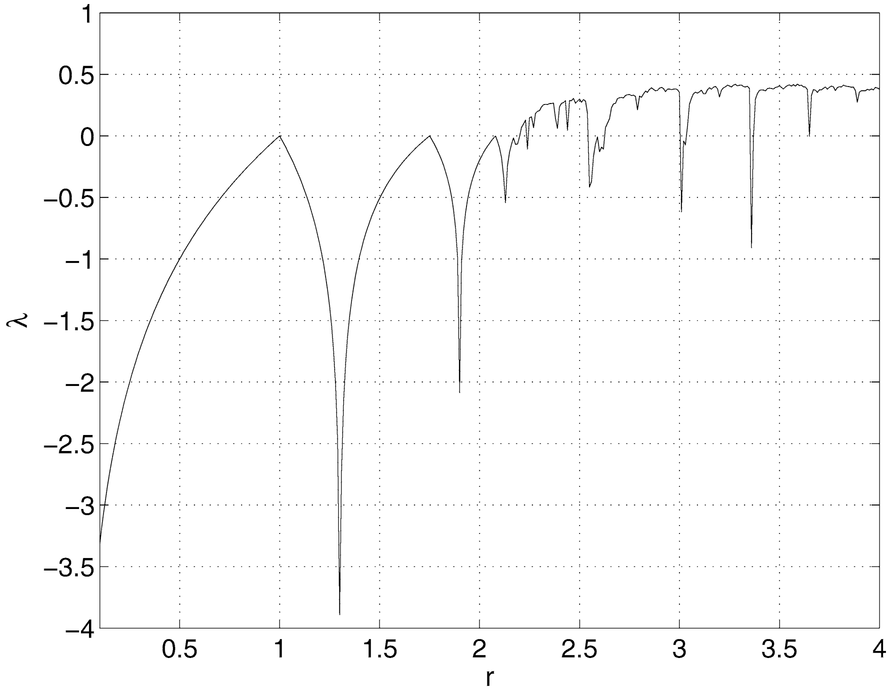

Fig. 5 Relationship between Lyapunov exponent and parameter r values for the multiplicative logistic equation

Table 1 Lyapunov exponent of the multiplicative logistic equation for $r = 1.5$, starting point $x_1 = 1.1$ and initial small distance $d_0 = 10^{-8}$

2^n	λ based on distance d	λ based on shifts ϵ
2^{10000}	−0.507521	−0.507559
2^{20000}	−0.507638	−0.507657
2^{30000}	−0.507678	−0.50769
2^{40000}	−0.507697	−0.507707
2^{50000}	−0.507709	−0.507716
2^{60000}	−0.507717	−0.507723
2^{70000}	−0.507722	−0.507728
2^{80000}	−0.507726	−0.507731
2^{90000}	−0.50773	−0.507734
2^{100000}	−0.507726	−0.507711

Table 2 Lyapunov exponent of the multiplicative logistic equation for $r = 4$, starting point $x_1 = 0.1$ and initial small distance $d_0 = 10^{-8}$

2^n	λ based on distance d	λ based on shifts ϵ
2^{10000}	0.392464	0.392224
2^{20000}	0.385317	0.385151
2^{30000}	0.385328	0.385217
2^{40000}	0.384345	0.384645
2^{50000}	0.384408	0.384343
2^{60000}	0.383494	0.383786
2^{70000}	0.383787	0.383774
2^{80000}	0.384259	0.384434
2^{90000}	0.38452	0.384601
2^{100000}	0.386334	0.386302

physical systems, such as population growth or material defects evolution, the multiplicative differential calculus seems to be appropriate to describe their dynamics. Chaos in multiplicative dynamical systems manifest itself under the same key requirement: nonlinearity. Presented in this paper are numerical methods of the Lyapunov exponent calculation that have been successfully tested for exemplary multiplicative dynamical systems: multiplicative logistic equation and multiplicative Lorenz system, but we expect that they are applicable to a wide variety of multiplicative dynamical systems.

Table 3 The largest Lyapunov exponent of the multiplicative Lorenz system for $\sigma = 10$, $b = \frac{8}{3}$, 40 000 steps, starting time 0.01 and time step 0.01

r	5	10	15	20	25	28
λ	−0.93	−0.59	−0.34	−0.12	0.82	0.91

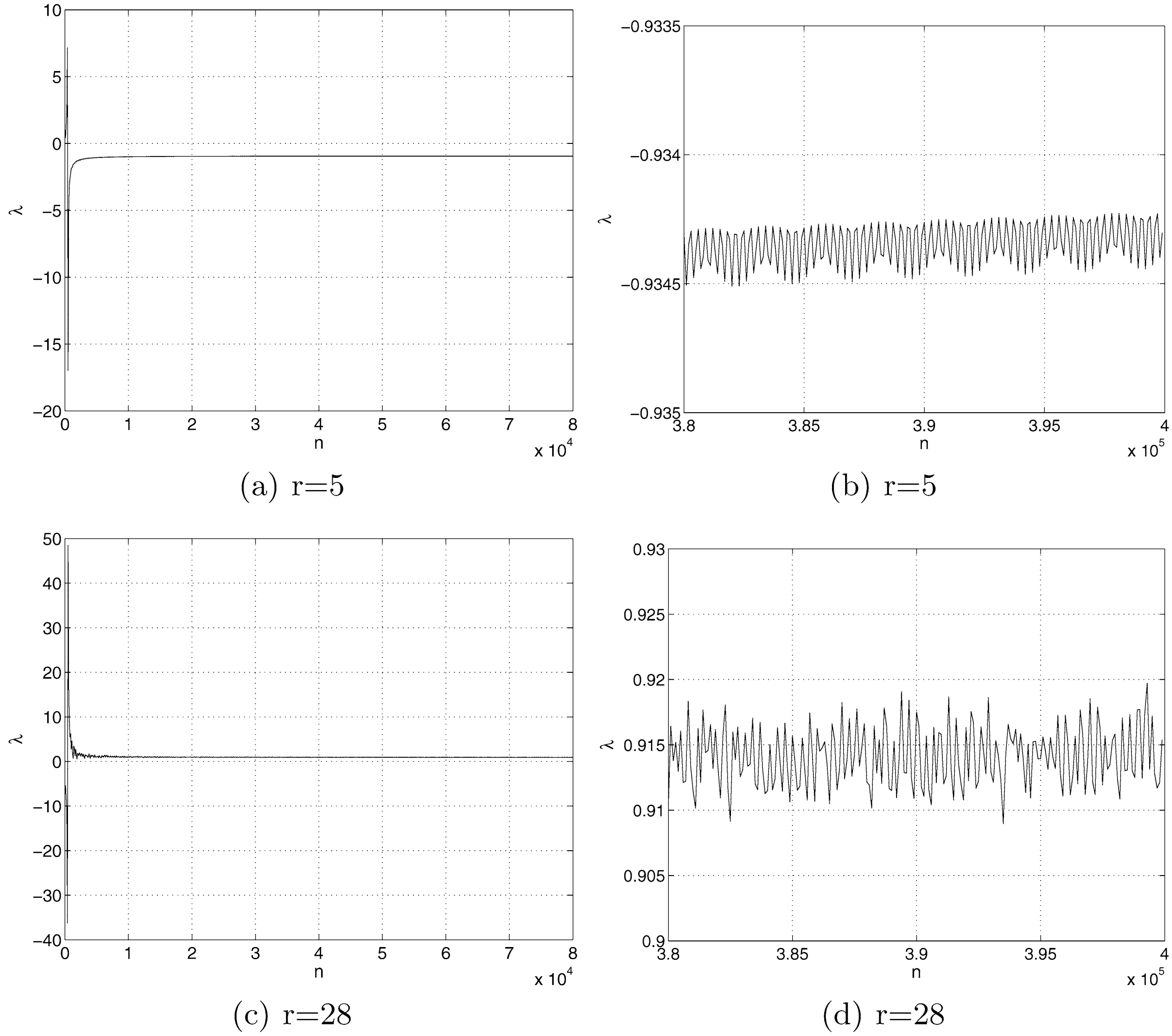

Fig. 6 The largest Lyapunov exponent of the multiplicative Lorenz system for $r = 5$ and $r = 28$

References

1. Rybaczuk, M., Kedzia, A., Zielinski, W.: The concept of physical and fractal dimension, II: the differential calculus in dimensional spaces. Chaos Solitons Fractals **12**, 2537–2552 (2001)
2. Rybaczuk, M., Stoppel, P.: The fractal growth of fatigue defects in materials. Int. J. Fract. **103**, 71–94 (2000)
3. Kasprzak, W., Lysik, B., Rybaczuk, M.: Measurements, Dimensions, Invariants Models and Fractals. Ukrainian Society on Fracture Mechanics Publishing House/SPOLOM, Lviv/Wroclaw (2004)
4. Volterra, V., Hostinsky, B.: Operations Infinitesimales Lineares. Herman, Paris (1938)
5. Lyapunov, A.M.: Stability of Motion. Academic Press, New York (1966)
6. Aniszewska, D., Rybaczuk, M.: Analysis of the multiplicative Lorenz system. Chaos Solitons Fractals **25**, 79–90 (2005)
7. Sparrow, C.: The Lorenz Equations: Bifurcations, Chaos and Strange Attractors. Springer, New York (1982)
8. Aniszewska, D.: Multiplicative Runge–Kutta method. Nonlinear Dyn. **50**, 265–272 (2007)
9. Wolf, A., Swift, J.B., Swinney, H.L., Vastano, J.A.: Determining Lyapunov exponents from a time series. Physica D **16**, 285–317 (1985)