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Singularly perturbed feedback linearization with linear attitude deviation dynamics realization

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Abstract A new approach for feedback linearization of attitude dynamics for rigid gas jet-actuated spacecraft control is introduced. The approach is aimed at providing global feedback linearization of the spacecraft dynamics while realizing a prescribed linear attitude deviation dynamics. The methodology is based on nonuniqueness representation of underdetermined linear algebraic equations solution via nullspace parametrization using generalized inversion. The procedure is to prespecify a stable second-order linear time-invariant differential equation in a norm measure of the spacecraft attitude variables deviations from their desired values. The evaluation of this equation along the trajectories defined by the spacecraft equations of motion yields a linear relation in the control variables. These control variables can be solved by utilizing the Moore–Penrose generalized inverse of the involved *controls coefficient* row vector. The resulting control law consists of auxiliary and particular parts, residing in the nullspace of the controls coefficient and the range space of its generalized inverse, respectively. The free *null-control vector* in the auxiliary part is projected onto the controls coefficient nullspace by a nullprojection matrix, and is designed to yield exponentially stable spacecraft internal dynamics, and *singularly perturbed feedback linearization* of the spacecraft attitude dynamics. The feedback control design utilizes the concept of *damped generalized inverse* to limit the growth of the Moore–Penrose generalized inverse, in addition to the concepts of *singularly perturbed controls coefficient nullprojection* and *damped controls coefficient nullprojection* to disencumber the nullprojection matrix from its rank deficiency, and to enhance the closed loop control system performance. The methodology yields desired linear attitude deviation dynamics realization with globally uniformly ultimately bounded trajectory tracking errors, and reveals a tradeoff between trajectory tracking accuracy and damped generalized inverse stability. The paper bridges a gap between the nonlinear control problem applied to spacecraft dynamics and some of the basic generalized inversion-related analytical dynamics principles.

Keywords Spacecraft attitude control · Singularly perturbed feedback linearization · Linear attitude deviation dynamics realization · Control authority redundancy · Nullspace parametrization · Controls coefficient · Moore–Penrose generalized inverse · Damped generalized inverse · Singularly perturbed nullprojection · Damped nullprojection · Null-control vector

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1 Introduction

The history of spacecraft attitude control goes back to the beginning of the second half of the twentieth century [1, 2]. Throughout this history, various control methodologies have been applied to this problem, benefiting from the rapid development in system and control theory. Similar to the case with other control system applications, there have been continuous efforts to create linear equivalencies to the nonlinear spacecraft control problem to facilitate designing spacecraft control laws.

The first attempts to create linear equivalency to the spacecraft attitude control problem were through Jacobian linearizations of spacecraft dynamics about design operating points via Taylor series expansions. The approach suffers from locality, which causes the control system performance to deteriorate as the highly nonlinear spacecraft dynamics deviates from the nominal design points. Accordingly, the approach loses validity with the increasing needs for high accuracy pointing and trajectory tracking.

Beyond Jacobian linearization, linear equivalization to the nonlinear spacecraft attitude control problem has passed through two stages. The first stage is global feedback linearizing transformations [3, 4]. The approach is the most systematic among those applied to the control problem of rigid spacecraft. Despite the simplicity and richness of linear control theory gained from feedback linearization, the approach has its disadvantages, among which are the need to apply large control forces in order to cancel nonlinear terms that are frequently beneficial for spacecraft maneuvering control, and the need to invert the mathematical model of the spacecraft to obtain the required control forces, which requires high fidelity mathematical modeling of the spacecraft dynamics.

The second stage of linear equivalization to the nonlinear spacecraft attitude control problem is at the level of attitude error dynamics, first introduced in Ref. [5]. The approach is aimed at imposing a predetermined linear dynamics on the errors of Euler's attitude variables from some desired attitude trajectories. The errors in the attitude variables are stacked together to form an attitude error vector. The approach was enhanced later in Ref. [6] by considering the minimal nonsingular modified Rodrigues parameters (MRPs) [7] to be the attitude variables, with an extension of adaptivity to uncertain spacecraft inertia and to unknown external disturbances.

This paper presents a novel control system design approach that benefits from the elegant linear control theory while preserving the nonlinear nature of spacecraft dynamics. The approach creates linear equivalization to the attitude control problem by combining feedback linearization of errors in the attitude parameters from desired attitude trajectories with global feedback linearization of spacecraft state space mathematical model. This is made by casting the nonlinear spacecraft control problem in a pointwise-linear form, and utilizing simple linear algebra to tackle the problem. The primary tools used are the Moore–Penrose generalized matrix inverse [8, 9], and the Greville formula for general solution of linear equations [10]. The approach enforces linear attitude error dynamics by considering the attitude error in a norm sense rather than in a vectorial sense. This consideration is more justifiable from a physical perspective, because the attitude error is really a scalar.

The procedure begins by defining a norm measure function of the spacecraft's attitude variables deviations from their desired values, and prespecifying a stable second-order linear differential equation in the measure function, resembling the desired attitude deviation dynamics. The differential equation is then transformed to a relation that is linear in the control vector by differentiating the norm measure function along the trajectories defined by the solution of the spacecraft's state space mathematical model. The Moore–Penrose generalized inverse is utilized thereafter to invert this relation for the control law required to realize the desired stable linear attitude deviation dynamics.

The derived control law has a special structure. It consists of auxiliary and particular parts, residing in the nullspace of the *controls coefficient* row vector and the range space of its generalized inverse, respectively. The auxiliary part contains a free nullvector, named the *null-control vector*, and is being projected onto the controls coefficient nullspace by means of a null-projection matrix. Therefore, the choice of the null-control vector does not affect the dynamics of the deviation measure function, and it parameterizes *all* control laws that are capable of realizing that desired dynamics.

The above mentioned structure has been utilized extensively in engineering, science, and their applications for the purpose of mathematical representation of solution nonuniqueness to problems where the requirements can be satisfied in more than one course of action. This is possible because of the generalized inverse geometric property that provides a parametrization of the solution's coefficient matrix nullspace via the free nullvector that appears explicitly in the solution expression. Furthermore, this explicit appearance makes the solution expression readable for further analysis, synthesis, and optimization.

Remarkable utilization of the generalized inverse-based Greville formula in the field of analytical dynamics was made by deriving the Udwadia–Kalaba equations of motion for constrained dynamical systems [11]. The corresponding free nullvector was chosen in order to optimize acceleration energy of the system, yielding its natural accelerations, i.e., those obeying Gauss' principle of least constraints [12], or equivalently yielding constraint forces that satisfy D'Alembert's principle of virtual work [13].

Similar to several other formulations in analytical dynamics that turned out to be the roots of some well-known control system design methodologies, the focus of the Udwadia–Kalaba formulation in viewing constrained motion has later shifted in Ref. [14] from the context of passive constraints, i.e., constraints imposed by the environment of the dynamical system to the context of program, or servo-constraints, i.e., constraints generated actively by control forces in order to alter the acceleration of the dynamical system, causing it to behave in prescribed desired manner. This is analogous to the natural attempt of the dynamical system in course of motion to minimize its own acceleration energy according to Gauss' principle. While this is an indirect application of the principle, direct applications in the arena of control system design are found in Refs. [15, 16].

Nevertheless, the control problem is a problem of nonuniqueness; that is, if a dynamical system is controllable then there exists no unique strategy to control it. In particular, a set of acceleration variables for a system need not to minimize the system acceleration energy or any other function of the motion variables for the system to satisfy a servo-constraint. More importantly, restricting the generalized inversion nullvector to perform a pointwise function minimization does not take into account the behavior of the internal states of the system, and is likely to cause system internal instability in despite of satisfying the servo-constraint, in contrast with the case of a passively constrained dynamical system where this is not exhibited in the internal system behavior.

To utilize the power of generalized inversion in parameterization of solution nonuniqueness, the Moore–Penrose generalized inverse was reintroduced in Ref. [17] to parameterize *redundancy in control authority*. An observation is made that the choice of the null-control vector substantially affects the inner system states. Therefore, it provides a design freedom that can be utilized in different manners depending on the control objectives to be achieved. In particular, it can be utilized to subdue internal instability of the closed loop control system.

The procedure of Ref. [17] is generalized in this work and is applied to the gas jet-actuated spacecraft control problem by considering nulling the deviation from desired spacecraft kinematics to be a desired servo-constraint that is to be realized. The corresponding generalized inversion of the controls coefficient guarantees outer kinematics tracking stability. To fulfill internal stability requirement, and inspired by the control law's affinity in the null-control vector, the later is chosen to be linear in the angular velocity vector, and the control law is shown to guarantee exponential stability of the internal spacecraft dynamics and *singularly perturbed feedback linearization* of the spacecraft dynamics.

Nevertheless, using the Moore–Penrose generalized inverse is known for an undesirable characteristic that can spoil the advantage of this tool. Although well defined for any matrix, regardless of its size or rank, the Moore–Penrose generalized inverse mapping of a matrix that is continuous in its variable elements suffers from a discontinuity. This appears as a divergence of the generalized inverse matrix elements to infinite values as the mapped matrix changes rank.

Several remedies for the generalized inverse instability problem have been offered in the literature of robotics and control moment gyroscopic devices, in what has become known as the singularity avoidance problem. Remedies are either nullspace parametrization-based, made by proper choices of the nullvector in the auxiliary part of the solution, e.g., [18–20], or approximation-based, made by modifying the definition of the generalized inverse itself in the particular part of the solution, e.g., [21–23].

The present development of spacecraft control system design is not an exception from the generalized inversion singularity problem. Singularity treatment is actually more critical in the present development because stabilizing the norm kinematic deviation measure implies nullifying the controls coefficient elements, leading to *singular steady state trajectories*. In other words, approaching steady state response implies destabilizing the controls coefficient generalized inverse. Singularity is, therefore, inherent in the present approach, and robustness against the corresponding potential for closed loop control system instability is necessary.

Generalized inversion stability robustness in this work is achieved by modifying the structure of the controls coefficient Moore–Penrose generalized inverse by means of a damping factor that limits its growth as steady state response is approached. Depending on the amount of modification, this *damped controls coefficient generalized inverse* results in a tradeoff between trajectory tracking accuracy and generalized inversion stability.

Modifying the definition of the controls coefficient Moore–Penrose generalized inverse results in approximate realization of the desired spacecraft attitude deviation dynamics. It is shown that the closed loop attitude trajectories tracking errors resulting from applying the proposed generalized inversion-based control law are globally uniformly ultimately bounded, and that the ultimate bound is inversely proportional to the damping factor by which the generalized inverse is modified.

Following a brief summary of the linear algebra and linear system theory tools employed and a description of the spacecraft mathematical model used, the paper proceeds with introducing the linear attitude deviation norm measure dynamics as the building block of the feedback linearizing transformation made by means of the proposed trajectory tracking control law. The infinite set of control laws that globally realize the desired attitude deviation norm measure dynamics is shown to be parameterizable by a single Moore–Penrose generalized inversion-based control expression that contains the free null-control vector. The special structure of this set is utilized to design a subset of control laws that additionally guarantee exponentially stable internal spacecraft dynamics and singularly perturbed global linear spacecraft dynamics. This subset is modified by altering the definition of the generalized inverse to avoid generalized inversion instability, yielding globally uniformly ultimately bounded spacecraft trajectory tracking errors.

The contribution of the article is twofold. First, new generalized inverse control system design tools are developed, namely the controls coefficient, its damped generalized inverse, and the corresponding damped and singularly perturbed nullprojectors. Second, the Greville formula is utilized to model control authority redundancy, and the associated free null-control vector is utilized to design a spacecraft attitude tracking control law with simultaneous global linear dynamics realization up to a singular perturbation from the corresponding nullprojector, and linear attitude deviation norm measure dynamics linearization up to globally uniformly ultimately bounded trajectory tracking errors.

2 Mathematical preliminaries

For the purpose of self-containment, and to emphasize that linear algebra tools are solely employed in the present nonlinear spacecraft control system design, this section summarizes the basic linear algebra and linear system theory tools used through the paper.

2.1 Moore–Penrose generalized matrix inverse

For any matrix $A \in \mathbb{R}^{m\times n}$, where m and n are positive integers, there exist a unique matrix $A^+ \in \mathbb{R}^{n\times m}$, called the Moore–Penrose generalized inverse of A, satisfying the following four properties [8, 9]

1. $AA^+A = A$
2. $A^+AA^+ = A^+$
3. $(AA^+)^T = AA^+$
4. $(A^+A)^T = A^+A$

2.2 General solution of consistent linear equations

Consider the linear matrix system

$$Ax = b \tag{1}$$

where $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$, m and n are positive integers. If the system is consistent, i.e., b is in the range space of A, then the general solution for x is given by [10]

$$x = A^+b + (I_{n\times n} - A^+A)y \tag{2}$$

where $I_{n\times n}$ is the n dimensional identity matrix, and $y \in \mathbb{R}^n$ is arbitrary.

The expression for x given by (2) is composed of two parts. The first part A^+b is called the particular solution (also called minimum norm solution), and it resides in the range space of A^+. The second part $(I_{n\times n} - A^+A)y$ is called the auxiliary solution (also called homogeneous solution), and it resides in the orthogonal complement subspace, i.e., the nullspace of A, where the nullvector y is projected to this subspace by means of the nullprojection matrix $P := I_{n\times n} - A^+A$. If A is not left invertible, i.e., $A^+A \neq I_{n\times n}$, then the nullspace of A is nontrivial. However, P is always rank deficient unless $A = \mathbf{0}_{m\times n}$.

2.3 Stability of time varying systems in semi-linear structure

Consider the system given in the following semi-linear state space form

$$\dot{x} = f(x,t)x, \qquad x(0) = x_0 \tag{3}$$

where $x \in \mathbb{R}^n$ is the state vector, and its equilibrium point $x = \mathbf{0}_{n\times 1}$ is exponentially stable over a domain of attraction D, where $D \subseteq \mathbb{R}^n$.

2.3.1 *Krasovskii's theorem for stability*

For any positive definite constant matrix $Q \in \mathbb{R}^{n\times n}$, there exists a positive definite state-time dependent matrix $P(x,t)$ and a corresponding Lyapunov function

$$V(x,t) = x^T P(x,t)x \tag{4}$$

such that the time derivative of $V(x,t)$ along the trajectories of the system given by (3) satisfies

$$\begin{aligned}\dot{V}(x,t) &= x^T\big[P(x,t)f(x,t) + f^T(x,t)P(x,t)\big]x \\ &\leq -x^T Qx\end{aligned} \tag{5}$$

for all $x \in D$. One particular solution for the matrix $P(x,t)$ is a result of imposing the equality part of inequalities (5) such that the following Krasovskii equations are obtained ([24], p. 224)

$$P(x,t)f(x,t) + f^T(x,t)P(x,t) = -Q. \tag{6}$$

Vectorizing the individual terms by stacking their columns above each others and employing the relation between the matrix vectorizing operation and the Kronecker product of matrices yields ([25], p. 251)

$$\begin{aligned}&\big\{f^T(x,t) \otimes I_{n\times n}\big\} \operatorname{vec} P(x,t) \\ &\quad + \big\{I_{n\times n} \otimes f(x,t)\big\} \operatorname{vec} P(x,t) = -\operatorname{vec}\{Q\}.\end{aligned} \tag{7}$$

Lyapunov indirect method for stability analysis implies that the linear system

$$\dot{x} = A(t)x \tag{8}$$

is asymptotically stable, where $A(t)$ is the system Jacobian evaluated at $x = \mathbf{0}_{n\times 1}$ given by

$$A(t) := \left.\frac{\partial[f(x,t)x]}{\partial x}\right|_{x=\mathbf{0}_{n\times 1}}. \tag{9}$$

If the system Jacobian is invertible over a subset of D in the neighborhood of the system equilibrium point $x = \mathbf{0}_{n\times 1}$, then (7) have a unique solution for the matrix $P(x,t)$ over this subset, and is given by

$$\begin{aligned}P(x,t) &= -\operatorname{vec}^{-1}\Big\{\big[f^T(x,t) \otimes I_{n\times n} \\ &\qquad + I_{n\times n} \otimes f(x,t)\big]^{-1} \operatorname{vec}\{Q\}\Big\} \\ &= -\operatorname{vec}^{-1}\Big\{\big[f^T(x,t) \oplus f(x,t)\big]^{-1} \operatorname{vec}\{Q\}\Big\}.\end{aligned} \tag{10}$$

2.3.2 *Exponential stability of vanishingly-perturbed systems*

Consider a perturbation from the system given by (3) in the form

$$\dot{x} = f(x,t)x + \mu(x,t), \qquad x(0) = x_0 \tag{11}$$

where $\mu(x,t) : \mathbb{R}^{n+1} \to \mathbb{R}^n$ is a vanishing perturbation function, i.e., if $x = \mathbf{0}_{n\times 1}$ then $\mu(x,t) = \mathbf{0}_{n\times 1}$. If the time derivative of the Lyapunov $V(x)$ given by (4) along the trajectories of the system given by (11) for all $t \geq 0$ satisfies the following inequality ([26], p. 340)

$$\begin{aligned}\dot{V}(x,t) &= 2x^T P(x,t)\big[f(x,t)x + \mu(x,t)\big] \\ &\leq -\lambda_{\min}(Q_{x_v})\|x\|^2 \quad \forall\|x\| \geq 0\end{aligned} \tag{12}$$

where $Q_{x_v} \in \mathbb{R}^{n\times n}$ is a positive definite constant matrix, and $\lambda_{\min}(Q_{x_v})$ is its minimum eigenvalue, then the system given by (11) is exponentially stable over the domain D.

2.3.3 *Boundedness of nonvanishingly-perturbed systems*

Consider a perturbation from the system given by (3) in the form

$$\dot{x} = f(x,t)x + \nu(x,t), \qquad x(0) = x_0 \tag{13}$$

where $\nu(x,t) : \mathbb{R}^{n+1} \to \mathbb{R}^n$ is a nonvanishing perturbation function such that its vector norm $\|\nu(x,t)\| < \bar{\nu}$ $\forall x \in \mathbb{R}^n$, $\forall t \geq 0$, where $\bar{\nu}$ is a positive scalar. If the time derivative of the Lyapunov $V(x)$ given by (4) along the trajectories of the system given by (13) for all $t \geq 0$ satisfies the following inequality ([26], p. 347)

$$\dot{V}(x,t) = 2x^T P(x,t)\big[f(x)x + \nu(x,t)\big] \leq -\lambda_{\min}(Q_x)\|x\|^2 \quad \forall \|x\| \geq c > 0 \tag{14}$$

where $Q_x \in \mathbb{R}^{n\times n}$ is a positive definite constant matrix, and $\lambda_{\min}(Q_x)$ is its minimum eigenvalue, then the condition given by inequality (14) implies that there exists for all $x(0) \in \mathbb{R}^n$ a finite time $T > 0$ such that $x(t)$ satisfies [26]

$$\big\|x(t)\big\| \leq \alpha e^{-\gamma t}\big\|x(0)\big\| \quad \forall 0 \leq t < T \tag{15}$$

and

$$\big\|x(t)\big\| \leq b \quad \forall t \geq T, \tag{16}$$

where

$$\alpha = \sqrt{\frac{\lambda_{\max}(P(x,t))}{\lambda_{\min}(P(x,t))}}, \qquad \gamma = \frac{\lambda_{\min}(Q_x)}{2\lambda_{\max}(P(x,t))}, \qquad b = \sqrt{\frac{\lambda_{\max}(P(x,t))}{\lambda_{\min}(P(x,t))}}c \tag{17}$$

and $\lambda_{\min}(\cdot)$, $\lambda_{\max}(\cdot)$ are respectively the minimum and maximum eigenvalues of the corresponding matrices.

3 Spacecraft mathematical model

The spacecraft mathematical model is given by the following system of kinematical and dynamical differential equations

$$\dot{\rho} = G(\rho)\omega, \qquad \rho(0) = \rho_0, \tag{18}$$

$$\dot{\omega} = J^{-1}\omega^\times J\omega + \tau, \qquad \omega(0) = \omega_0 \tag{19}$$

where $\rho \in \mathbb{R}^{3\times 1}$ is the spacecraft vector of modified Rodrigues attitude parameters [7], $\omega \in \mathbb{R}^{3\times 1}$ is the vector of spacecraft angular velocity components in its body reference frame, $J \in \mathbb{R}^{3\times 3}$ is a diagonal matrix containing the spacecraft's body principal moments of inertia, and $\tau := J^{-1}u \in \mathbb{R}^{3\times 1}$ is the vector of scaled control torques, where $u \in \mathbb{R}^{3\times 1}$ contains the applied jet actuator torque components about the spacecraft's principal axes. The cross product matrix $x^\times$ which corresponds to a vector $x \in \mathbb{R}^{3\times 1}$ is skew symmetric of the form

$$x^\times = \begin{bmatrix} 0 & x_3 & -x_2 \\ -x_3 & 0 & x_1 \\ x_2 & -x_1 & 0 \end{bmatrix}$$

and the matrix valued function $G(\rho) : \mathbb{R}^{3\times 1} \to \mathbb{R}^{3\times 3}$ is given by

$$G(\rho) = \frac{1}{2}\left(\frac{1-\rho^T\rho}{2}I_{3\times 3} - \rho^\times + \rho\rho^T\right). \tag{20}$$

The modified Rodrigues parameters are used as the attitude state variables because of their validity in describing any angular displacement about the spacecraft's body axes up to 2π rad, such that $G(\rho)$ remains finite and invertible for any value of ρ that corresponds to such spacecraft angular displacement.

4 Attitude deviation norm measure dynamics

Let $\rho_d(t) \in \mathbb{R}^{3\times 1}$ be a prescribed desired spacecraft attitude vector. The spacecraft attitude deviation vector from $\rho_d(t)$ is defined as

$$z(\rho,t) := \rho - \rho_d(t). \tag{21}$$

Assumption 1 (Smoothness of desired spacecraft attitude trajectories) *$\rho_d(t)$ is at least twice continuously differentiable in t.*

We define the scalar attitude deviation norm measure function $\phi : \mathbb{R}^{4\times 1} \to \mathbb{R}$ to be half the squared Euclidean norm of $z(\rho,t)$

$$\phi = \frac{1}{2}\big\|z(\rho,t)\big\|^2 = \frac{1}{2}\big\|\rho - \rho_d(t)\big\|^2. \tag{22}$$

For the purpose of forthcoming development in this paper, there is no loss of generality in specifying the attitude deviation norm measure function to be the Euclidean norm. This is due to the fact that all vector p norms are equivalent, in the sense that any vector p norm is bounded from above and below by two other scaled vector p norms [26].

The first two time derivatives of ϕ along the spacecraft trajectories given by the solution of (18) and (19) are

$$\dot{\phi} = \frac{\partial \phi}{\partial \rho} G(\rho)\omega + \frac{\partial \phi}{\partial t} \tag{23}$$

$$= z^T(\rho, t)\big[G(\rho)\omega - \dot{\rho}_d(t)\big] \tag{24}$$

and

$$\begin{aligned}\ddot{\phi} &= \big[G(\rho)\omega - \dot{\rho}_d(t)\big]^T \big[G(\rho)\omega - \dot{\rho}_d(t)\big] \\ &\quad + z^T(\rho, t)\big[\dot{G}(\rho, \omega)\omega + G(\rho)\big[J^{-1}\omega^\times J\omega + \tau\big] \\ &\quad - \ddot{\rho}_d(t)\big]\end{aligned} \tag{25}$$

where $\dot{G}(\rho, \omega)$ is the time derivative of $G(\rho)$ obtained by differentiating the individual elements of $G(\rho)$ along the kinematical subsystem given by (18).

The procedure is to prespecify a desired stable linear second-order dynamics of ϕ in the form

$$\ddot{\phi} + c_1\dot{\phi} + c_2\phi = 0, \quad c_1, c_2 > 0. \tag{26}$$

With ϕ, $\dot{\phi}$, and $\ddot{\phi}$ given by (22), (24), and (25), it is possible to write (26) in the pointwise-linear form

$$\mathcal{A}(\rho, t)\tau = \mathcal{B}(\rho, \omega, t), \tag{27}$$

where the vector valued function $\mathcal{A}(\rho, t) : \mathbb{R}^{4\times 1} \to \mathbb{R}^{1\times 3}$ is given by

$$\mathcal{A}(\rho, t) = z^T(\rho, t)G(\rho) \tag{28}$$

and the scalar valued function $\mathcal{B}(\rho, \omega, t) : \mathbb{R}^{7\times 1} \to \mathbb{R}$ is

$$\begin{aligned}&\mathcal{B}(\rho, \omega, t) \\ &\quad = -\big[G(\rho)\omega - \dot{\rho}_d(t)\big]^T \big[G(\rho)\omega - \dot{\rho}_d(t)\big] \\ &\qquad - z^T(\rho, t)\big[\dot{G}(\rho, \omega)\omega + G(\rho)J^{-1}\omega^\times J\omega \\ &\qquad - \ddot{\rho}_d(t)\big] \\ &\qquad - c_1 z^T(\rho, t)\big[G(\rho)\omega - \dot{\rho}_d(t)\big] - \frac{c_2}{2}\big\|z(\rho, t)\big\|^2.\end{aligned} \tag{29}$$

We name the row vector function $\mathcal{A}(\rho, t)$ the *controls coefficient* of the attitude deviation norm measure dynamics given by (26) along the spacecraft trajectories, and the scalar function $\mathcal{B}(\rho, \omega, t)$ the corresponding *controls load*.

Definition 1 (Realizability of linear attitude deviation norm measure dynamics) For a given desired spacecraft attitude vector $\rho_d(t)$ satisfying Assumption 1, the linear attitude deviation norm measure dynamics given by (26) is said to be realizable by the spacecraft equations of motion (18) and (19) at specific values of ρ and t if there exists a control vector τ that solves (27) for these values of ρ and t. If this is true for all ρ and t such that $z(\rho, t) \neq \mathbf{0}_{3\times 1}$, then the linear attitude deviation norm measure dynamics is said to be globally realizable by the spacecraft equations of motion.

The quasi-linear form given by (27) makes it feasible to assess realizability of the linear attitude deviation norm measure dynamics given by (26) in a pointwise manner.

5 Linearly parameterized attitude control laws

Proposition 1 Linearly parameterized nonlinear control laws *For any desired spacecraft attitude vector $\rho_d(t)$ satisfying Assumption 1, the linear attitude deviation norm measure dynamics given by (26) is globally realizable by the spacecraft equations of motion (18) and (19). Furthermore, the infinite set of all control laws realizing that dynamics by the spacecraft equations of motion is parameterized by an arbitrarily chosen nullvector $y \in \mathbb{R}^{3\times 1}$ as*

$$\tau = \mathcal{A}^+(\rho, t)\mathcal{B}(\rho, \omega, t) + \mathcal{P}(\rho, t)y \tag{30}$$

where "$\mathcal{A}^+$" stands for the Moore–Penrose generalized inverse of the controls coefficient given by

$$\mathcal{A}^+(\rho, t) = \frac{\mathcal{A}^T(\rho, t)}{\|\mathcal{A}(\rho, t)\|^2}, \qquad \mathcal{A}(\rho, t) \neq \mathbf{0}_{1\times 3} \tag{31}$$

and $\mathcal{P}(\rho, t) \in \mathbb{R}^{3\times 3}$ is the corresponding nullprojector given by

$$\mathcal{P}(\rho, t) = I_{3\times 3} - \mathcal{A}^+(\rho, t)\mathcal{A}(\rho, t). \tag{32}$$

Proof A necessary and sufficient condition for the existence of a control vector τ that solves (27) at specific values of ρ and t is the consistency of the equation at these values, i.e., $\mathcal{B}(\rho, \omega, t)$ is in the range space of $\mathcal{A}(\rho, t)$. This is guaranteed for all values of $\omega \in \mathbb{R}^{3\times 1}$, provided that $\mathcal{A}(\rho, t)$ does not vanish at the specified values of ρ and t, at which the linear attitude deviation norm measure dynamics given by (26) is realizable by the spacecraft equations of motion (18) and (19) according to definition (1). Since the matrix $G(\rho)$ is invertible for all values of ρ, it has a trivial nullspace, which implies from (28) that $\mathcal{A}(\rho,t)$ vanishes if and only if $z(\rho,t)$ does. Therefore, (27) is consistent at all ρ and t such that $z(\rho,t)\neq \mathbf{0}_{3\times 1}$, and the linear attitude deviation norm measure dynamics is globally realizable by the spacecraft equations of motion according to Definition 1. Consequently, the argument of Sect. 2.2 implies that the infinite set of *all* control laws that realize the linear attitude deviation norm measure dynamics by the spacecraft equations of motion at all ρ and t such that $\mathcal{A}(\rho,t)\neq \mathbf{0}_{1\times 3}$ is given by (30), where the expression of $\mathcal{A}^+(\rho,t)$ given by (31) is easily verified to satisfy the four conditions defining the Moore–Penrose generalized matrix inverse provided in Sect. 2.1. □

Remark 1 The control law τ given by the expression (30) is composed of particular and auxiliary parts residing in two orthogonal subspaces. The particular part $\mathcal{A}^+(\rho,t)\mathcal{B}(\rho,\omega,t)$ resides in the range space of $\mathcal{A}^+(\rho,t)$, and the auxiliary part $\mathcal{P}(\rho,t)y$ resides in the nullspace of $\mathcal{A}(\rho,t)$, where the free nullvector y is projected to this subspace by means of the projector $\mathcal{P}(\rho,t)$.

Remark 2 Global realizability of the attitude deviation norm measure dynamics guarantees global uniform convergence of the attitude vector ρ to its desired attitude vector $\rho_d(t)$, but it does not guarantee internal stability of the spacecraft dynamics, i.e., stability of angular velocity vector ω.

Since any choice of the nullvector y in the control law expression given by (30) yields a solution to (27), y does not affect realizability of the linear attitude deviation norm measure dynamics given by (26). Nevertheless, y substantially affects the spacecraft transient state response [17]. In particular, an inadequate choice of y can destabilize the spacecraft internal dynamics given by (19) or causes unsatisfactory closed loop performance. Because of its importance in the present development as a control vector by itself, we name the nullvector y the *null-control vector*.

Corollary 1 (Parameterized set of spacecraft closed loop control equations) *The infinite set of spacecraft closed loop systems equations realizing the linear attitude deviation norm measure dynamics given by* (26) *is parameterized by the null-control vector y as*

$$\dot{\rho}=G(\rho)\omega,\qquad \rho(0)=\rho_0 \tag{33}$$

$$\dot{\omega}=J^{-1}\omega^{\times}J\omega+\mathcal{A}^+(\rho,t)\mathcal{B}(\rho,\omega,t)+\mathcal{P}(\rho,t)y,\quad \omega(0)=\omega_0. \tag{34}$$

Proof Equations (33) and (34) are obtained by substituting the control laws expressions given by (30) in the spacecraft's mathematical model given by (18) and (19). □

6 Null-control vector design

The choice of the null-control vector y affects neither realizability of the attitude deviation norm measure dynamics given by (26) nor steady state spacecraft response. However, the choice of the null-control vector y affects both of spacecraft internal dynamics and spacecraft transient response. Hence, it provides a freedom that can be utilized to stabilize internal states of the spacecraft. Internal dynamics stability and stability robustness against controls coefficient singularity are the most important factors to be considered in designing the null-control vector y.

The structure of the control law τ given by (30) has a special feature, namely the affinity of its auxiliary part in y, which provides a pointwise-linear parametrization to the nonlinear control law. Hence, let y be chosen as

$$y=K\omega \tag{35}$$

where $K\in\mathbb{R}^{3\times 3}$ is to be determined. With this choice of y, a class of control laws that globally realize the attitude deviation dynamics given by (26) is given by

$$\tau=\mathcal{A}^+(\rho,t)\mathcal{B}(\rho,\omega,t)+\mathcal{P}(\rho,t)K\omega \tag{36}$$

$$=\big[\mathcal{H}_1(\rho,\omega,t)+\mathcal{P}(\rho,t)K\big]\omega+\mathcal{H}_2(\rho,t) \tag{37}$$

where

$$\mathcal{H}_1(\rho,\omega,t)=-\mathcal{A}^+(\rho,t)z^T(\rho,t)\big[\dot{G}(\rho,\omega)+G(\rho)J^{-1}\omega^{\times}J+c_1G(\rho)\big]-\mathcal{A}^+(\rho,t)\big[G(\rho)\omega-\dot{\rho}_d(t)\big]^TG(\rho) \tag{38}$$

and

$$\begin{aligned}\mathcal{H}_2(\rho,t) &= -\frac{c_2}{2}\mathcal{A}^+(\rho,t)z^T(\rho,t)z(\rho,t)\\ &\quad+\mathcal{A}^+(\rho,t)z^T(\rho,t)\big[\ddot{\rho}_d(t)+c_1\dot{\rho}_d(t)\big]\\ &\quad-\mathcal{A}^+(\rho,t)\big\|\dot{\rho}_d(t)\big\|_2^2. \end{aligned}\tag{39}$$

Hence, a class of closed loop dynamical subsystems realizing the dynamics given by (26) is obtained by substituting the control law given by (37) in (19), and it takes the form

$$\begin{aligned}\dot{\omega} &= \big[J^{-1}\omega^{\times}J+\mathcal{H}_1(\rho,\omega,t)+\mathcal{P}(\rho,t)K\big]\omega\\ &\quad+\mathcal{H}_2(\rho,t).\end{aligned}\tag{40}$$

The term $\mathcal{H}_2(\rho,t)$ in the above equation can be viewed as a forcing term that drives the internal dynamics of the spacecraft to realize the desired attitude deviation dynamics.

7 Controls coefficient singularity analysis

If the controls coefficient $\mathcal{A}(\rho,t)$ is singular at specific values of ρ and t, i.e., has zero elements, then its Moore–Penrose generalized inverse $\mathcal{A}^+(\rho,t)$ given by (31) is infinite. The following proposition relates global realizability of linear attitude deviation norm measure dynamics to controls coefficient singularity.

Proposition 2 (Controls coefficient singularity) *Given any desired spacecraft attitude vector $\rho_d(t)$ satisfying Assumption 1, a control law τ given by (37) globally realizes the linear attitude deviation norm measure dynamics given by (26) by the spacecraft equations of motion (18) and (19) only if*

$$\lim_{t\to\infty}\mathcal{A}(\rho,t)=\mathbf{0}_{1\times 3}.\tag{41}$$

Proof Because of the equivalency of linear attitude deviation norm measure dynamics given by (26) and its quasi-linear form given by (27), global realizability of the first implies the existence of a control law that drives ϕ according to the dynamics given by (26) at all ρ and t such that $z(\rho,t)\neq\mathbf{0}_{3\times 1}$. The norm property of ϕ implies that $z(\rho,t)=\mathbf{0}_{3\times 1}$ if and only if $\phi=0$. Therefore, global realizability of the stable dynamics given by (26) implies that

$$\lim_{t\to\infty}\phi=0,\quad\text{and}\quad\lim_{t\to\infty}z(\rho,t)=\mathbf{0}_{3\times 1}.\tag{42}$$

Since the matrix $G(\rho)$ is nonsingular for all finite values of ρ, (28) implies that

$$\begin{aligned}&\lim_{t\to\infty}z(\rho,t)=\mathbf{0}_{3\times 1}\quad\text{if and only if}\\ &\lim_{t\to\infty}\mathcal{A}(\rho,t)=\mathbf{0}_{1\times 3}.\end{aligned}\tag{43}$$

□

Proposition 2 tells that the controls coefficient $\mathcal{A}(\rho,t)$ must approach singularity in order for the stable linear attitude deviation norm measure dynamics given by (26) to be globally realized.

With the expression of $\mathcal{A}(\rho,t)$ given by (28), the Moore–Penrose generalized inverse controls coefficient given by (31) can be written as

$$\mathcal{A}^+(\rho,t)=\frac{G^T(\rho)z(\rho,t)}{\|G^T(\rho)z(\rho,t)\|^2}.\tag{44}$$

Therefore,

$$\big\|\mathcal{A}^+(\rho,t)\big\|=\frac{\|G^T(\rho)z(\rho,t)\|}{\|G^T(\rho)z(\rho,t)\|^2}=\frac{1}{\|G^T(\rho)z(\rho,t)\|}.\tag{45}$$

An interesting property of $G(\rho)$ is [27]

$$G^T(\rho)G(\rho)=\left(\frac{1+\rho^T\rho}{4}\right)^2 I_{3\times 3}\tag{46}$$

implying that

$$\sigma(G(\rho))=\big[\lambda\big(G^T(\rho)G(\rho)\big)\big]^{1/2}\tag{47}$$

$$=\frac{1+\rho^T\rho}{4}\geq\frac{1}{4}\tag{48}$$

where $\sigma(G(\rho))$ and $\lambda(G^T(\rho)G(\rho))$ refer to the three times-repeated singular value of $G(\rho)$ and the three times-repeated eigenvalue of $G^T(\rho)G(\rho)$, respectively. Since

$$\big\|G^T(\rho)z(\rho,t)\big\|$$

$$=\big[z^T(\rho,t)G(\rho)G^T(\rho)z(\rho,t)\big]^{\frac{1}{2}}\tag{49}$$

$$\leq\big[\lambda_{\max}\big(G^T(\rho)G(\rho)\big)\big]^{\frac{1}{2}}\big\|z(\rho,t)\big\|\tag{50}$$

$$=\sigma_{\max}\big(G(\rho)\big)\big\|z(\rho,t)\big\|\tag{51}$$

and

$$\begin{aligned}
&\left\|G^T(\rho)z(\rho,t)\right\| \\
&\quad \geq \left[\lambda_{\min}\left(G^T(\rho)G(\rho)\right)\right]^{\frac{1}{2}}\left\|z(\rho,t)\right\| && (52)\\
&\quad = \sigma_{\min}\left(G(\rho)\right)\left\|z(\rho,t)\right\| && (53)
\end{aligned}$$

and since

$$\sigma_{\min}\left(G(\rho)\right) = \sigma_{\max}\left(G(\rho)\right) = \sigma\left(G(\rho)\right) \tag{54}$$

it follows that

$$\left\|G^T(\rho)z(\rho,t)\right\| = \sigma\left(G(\rho)\right)\left\|z(\rho,t)\right\|. \tag{55}$$

Hence, (45) implies that

$$\left\|\mathcal{A}^+(\rho,t)\right\| = \frac{1}{\sigma(G(\rho))\|z(\rho,t)\|} \tag{56}$$

and

$$\begin{aligned}
&\left\|\mathcal{A}^+(\rho,t)z^T(\rho,t)\right\| \\
&\quad \leq \left\|\mathcal{A}^+(\rho,t)\right\|\left\|z(\rho,t)\right\| && (57)\\
&\quad = \frac{1}{\sigma(G(\rho))\|z(\rho,t)\|}\left\|z(\rho,t)\right\| && (58)\\
&\quad = \frac{1}{\sigma(G(\rho))} \leq 4 && (59)
\end{aligned}$$

and

$$\begin{aligned}
&\left\|\mathcal{A}^+(\rho,t)z^T(\rho,t)z(\rho,t)\right\| \\
&\quad \leq \left\|\mathcal{A}^+(\rho,t)z^T(\rho,t)\right\|\left\|z(\rho,t)\right\| && (60)\\
&\quad \leq 4\left\|z(\rho,t)\right\|. && (61)
\end{aligned}$$

Since $G(\rho)$ is finite for all finite values of ρ, (56) implies that

$$\lim_{z(\rho,t)\to \mathbf{0}_{3\times 1}} \left\|\mathcal{A}^+(\rho,t)\right\| = \infty. \tag{62}$$

However, inequalities (59) and (61) imply that

$$\lim_{z(\rho,t)\to \mathbf{0}_{3\times 1}} \left\|\mathcal{A}^+(\rho,t)z^T(\rho,t)\right\| \leq 4 \tag{63}$$

and

$$\lim_{z(\rho,t)\to \mathbf{0}_{3\times 1}} \left\|\mathcal{A}^+(\rho,t)z^T(\rho,t)z(\rho,t)\right\| = 0. \tag{64}$$

Equation (62) implies that unbounded controls coefficient generalized inverse $\mathcal{A}^+(\rho,t)$ in a control law given by (30) is indispensable to globally realize the desired attitude deviation dynamics given by (26). If the growth of the vector fields given by the parameterized set of closed loop control systems (33) and (34) due to the unbounded growth of $\mathcal{A}^+(\rho,t)$ is not controlled, then this yields internal dynamics instability.

Generalized inversion instability of variable elements matrices is a well-known problem in mathematics and engineering applications, and it has been investigated thoroughly in the arena of robotics, e.g., Ref. [28]. Nevertheless, its treatment is more critical in the present methodology for spacecraft control system design because approaching steady state response of the closed loop control systems given by (33) and (34) implies singularity of the controls coefficient, leading to *singular steady state trajectories*. Singularity is, therefore, inherent in the present methodology, and it is an aim rather than an escape. Robustness against its potential to cause closed loop control system instability is what is necessary. For that purpose, a limited-growth modified controls coefficient generalized inverse is introduced next.

7.1 Damped controls coefficient generalized inverse

Definition 2 Damped controls coefficient generalized inverse The damped controls coefficient generalized inverse $\mathcal{A}_d^+(\rho,\beta,t)$ is defined as

$$\mathcal{A}_d^+(\rho,\beta,t) := \begin{cases} \frac{\mathcal{A}^T(\rho,t)}{\|\mathcal{A}(\rho,t)\|^2}: & \|\mathcal{A}(\rho,t)\| \geq \beta, \\ \frac{\mathcal{A}^T(\rho,t)}{\beta^2}: & \|\mathcal{A}(\rho,t)\| < \beta \end{cases} \tag{65}$$

where the scalar β is a positive *generalized inverse damping factor*.

Therefore,

$$\begin{aligned}
&\left\|\mathcal{A}_d^+(\rho,\beta,t)\right\| \\
&\quad = \begin{cases} \frac{1}{\|G^T(\rho)z(\rho,t)\|}: & \|G^T(\rho)z(\rho,t)\| \geq \beta, \\ \frac{1}{\beta^2}\|G^T(\rho)z(\rho,t)\|: & \|G^T(\rho)z(\rho,t)\| < \beta \end{cases}
\end{aligned} \tag{66}$$

which implies that

$$\left\|\mathcal{A}_d^+(\rho,\beta,t)\right\| < \frac{1}{\beta} \tag{67}$$

and

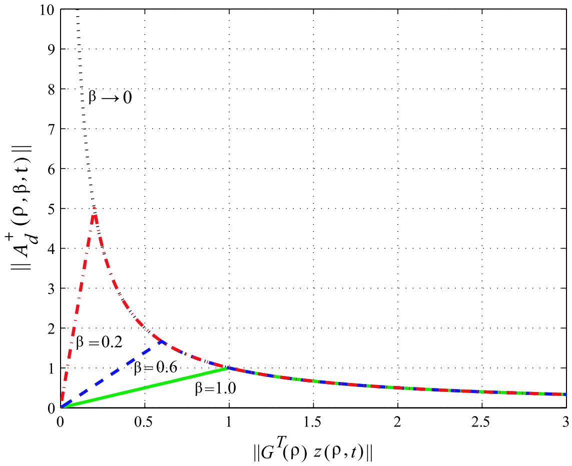

Fig. 1 Damped controls coefficient generalized inverse

$$\lim_{z(\rho,t)\to\mathbf{0}_{3\times1}} \left\|\mathcal{A}_d^+(\rho,\beta,t)\right\| = \left\|\mathcal{A}_d^+(\rho,\beta,t)\right\|_{z(\rho,t)=\mathbf{0}_{3\times1}} = 0 \tag{68}$$

and that $\mathcal{A}_d^+(\rho,\beta,t)$ pointwise converges to $\mathcal{A}^+(\rho,t)$ as β vanishes (see Fig. 1). Accordingly, we define $\mathcal{H}_{1d}(\rho,\omega,\beta,t)$ and $\mathcal{H}_{2d}(\rho,\beta,t)$ by replacing the controls coefficient generalized inverse $\mathcal{A}^+(\rho,t)$ in the last terms of the $\mathcal{H}_1(\rho,\omega,t)$ and $\mathcal{H}_2(\rho,t)$ expressions given by (38) and (39) with the damped controls coefficient generalized inverse $\mathcal{A}_d^+(\rho,\beta,t)$

$$\begin{aligned}\mathcal{H}_{1d}(\rho,\omega,\beta,t) &= -\mathcal{A}^+(\rho,t)z^T(\rho,t)\\ &\quad\times\left[\dot G(\rho,\omega)+G(\rho)J^{-1}\omega^\times J+c_1G(\rho)\right]\\ &\quad-\mathcal{A}_d^+(\rho,\beta,t)\left[G(\rho)\omega-\dot\rho_d(t)\right]^TG(\rho)\end{aligned} \tag{69}$$

and

$$\begin{aligned}\mathcal{H}_{2d}(\rho,\beta,t) = &-\frac{c_2}{2}\mathcal{A}^+(\rho,t)z^T(\rho,t)z(\rho,t)\\ &+\mathcal{A}^+(\rho,t)z^T(\rho,t)\left[\ddot\rho_d(t)+c_1\dot\rho_d(t)\right]\\ &-\mathcal{A}_d^+(\rho,\beta,t)\left\|\dot\rho_d(t)\right\|^2.\end{aligned} \tag{70}$$

7.2 Damped controls coefficient nullprojector

Similar to other nullprojection matrices, a fundamental property of the controls coefficient nullprojector is that it is a nonexpansive operator, i.e., it works to contract or at most preserve the length of the null-control vector. Nevertheless, the elements of the controls coefficient nullprojector are dependent on the spacecraft attitude parameters even during steady state response, which may cause undesirable closed loop performance. For this reason, a modified controls coefficient nullprojector with vanishing dependency on the steady state attitude variables is defined based on the damped controls coefficient generalized inverse.

Definition 3 (Damped controls coefficient nullprojector) The damped controls coefficient nullprojector $\mathcal{P}_d(\rho,\beta,t)$ is defined as

$$\mathcal{P}_d(\rho,\beta,t) := I_{3\times3} - \mathcal{A}_d^+(\rho,\beta,t)\mathcal{A}(\rho,t) \tag{71}$$

where $\mathcal{A}_d^+(\rho,\beta,t)$ is given by (65).

The above definition implies that

$$\mathcal{P}_d(\rho,\beta,t) = \begin{cases} I_{3\times3} - \frac{G^T(\rho)z(\rho,t)z^T(\rho,t)G(\rho)}{\|G^T(\rho)z(\rho,t)\|^2}: \\ \quad \|G^T(\rho)z(\rho,t)\| \ge \beta, \\ I_{3\times3} - \frac{G^T(\rho)z(\rho,t)z^T(\rho,t)G(\rho)}{\beta^2}: \\ \quad \|G^T(\rho)z(\rho,t)\| < \beta \end{cases} \tag{72}$$

and consequently,

$$\lim_{z(\rho,t)\to\mathbf{0}_{3\times1}} \mathcal{P}_d(\rho,\beta,t) = I_{3\times3}. \tag{73}$$

Hence, the damped nullprojector maps the null-control vector to itself in a steady state phase of response during which the auxiliary part of the control law converges to the null-control vector. The independency of nullprojection on the attitude state of the spacecraft substantially reduces unnecessary abrupt behavior of the control vector.

8 Singularly perturbed controls coefficient nullprojector

The concept of singularly perturbed controls coefficient nullprojector facilitates the present development of the generalized inversion-based feedback linearizing spacecraft control law.

Definition 4 (Perturbed controls coefficient nullprojector) The perturbed controls coefficient nullprojector $\widetilde{\mathcal{P}}(\rho,\delta,t)$ is defined as

$$\widetilde{\mathcal{P}}(\rho,\delta,t) := I_{3\times 3} - h(\delta)\mathcal{A}^{+}(\rho,t)\mathcal{A}(\rho,t) \tag{74}$$

where $h(\delta):\mathbb{R}^{1\times 1}\rightarrow\mathbb{R}^{1\times 1}$ is any continuous function such that

$$h(\delta)=1 \quad \text{if and only if} \quad \delta=0. \tag{75}$$

Proposition 3 (Controls coefficient nullprojector invertibility) *The perturbed controls coefficient nullprojector $\widetilde{\mathcal{P}}(\rho,\delta,t)$ is of full rank for all $\delta\neq 0$.*

Proof The singular value decomposition of $\mathcal{A}(\rho,t)$ is given by

$$\mathcal{A}(\rho,t)=\Sigma(\rho,t)\mathcal{V}^{T}(\rho,t) \tag{76}$$

where

$$\Sigma(\rho,t)=\begin{bmatrix}\|\mathcal{A}(\rho,t)\| & 0 & 0\end{bmatrix} \tag{77}$$

and $\mathcal{V}(\rho,t)\in\mathbb{R}^{3\times 3}$ is orthonormal, i.e.,

$$\mathcal{V}^{-1}(\rho,t)=\mathcal{V}^{T}(\rho,t), \quad \text{and} \quad \det\mathcal{V}(\rho,t)=1. \tag{78}$$

By inspecting the four conditions in Sect. 2.1, it can be easily verified that the Moore–Penrose generalized inverse of $\mathcal{A}(\rho,t)$ is given by

$$\mathcal{A}^{+}(\rho,t)=\mathcal{V}(\rho,t)\Sigma^{+}(\rho,t) \tag{79}$$

where $\Sigma^{+}(\rho,t)$ is the Moore–Penrose generalized inverse of $\Sigma(\rho,t)$

$$\Sigma^{+}(\rho,t)=\begin{bmatrix}\frac{1}{\|\mathcal{A}(\rho,t)\|} & 0 & 0\end{bmatrix}^{T}. \tag{80}$$

Therefore,

$$\mathcal{A}^{+}(\rho,t)\mathcal{A}(\rho,t)=\mathcal{V}(\rho,t)\Sigma^{+}(\rho,t)\Sigma(\rho,t)\mathcal{V}^{T}(\rho,t). \tag{81}$$

The right-hand side of (81) is a singular value decomposition of $\mathcal{A}^{+}(\rho,t)\mathcal{A}(\rho,t)$, where the diagonal matrix $\Sigma^{+}(\rho,t)\Sigma(\rho,t)$ contains the singular values of $\mathcal{A}^{+}(\rho,t)\mathcal{A}(\rho,t)$ as its diagonal elements

$$\Sigma^{+}(\rho,t)\Sigma(\rho,t)=\begin{bmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}. \tag{82}$$

Consequently, the perturbed controls coefficient nullprojector $\widetilde{\mathcal{P}}(\rho,\delta,t)$ is

$$\begin{aligned}
&\widetilde{\mathcal{P}}(\rho,\delta,t)\\
&\quad = I_{3\times 3}-h(\delta)\mathcal{A}^{+}(\rho,t)\mathcal{A}(\rho,t) && (83)\\
&\quad = I_{3\times 3}-h(\delta)\mathcal{V}(\rho,t)\Sigma^{+}(\rho,t)\Sigma(\rho,t)\\
&\qquad \times\mathcal{V}^{T}(\rho,t) && (84)\\
&\quad = \mathcal{V}(\rho,t)\big[I_{3\times 3}-h(\delta)\Sigma^{+}(\rho,t)\Sigma(\rho,t)\big]\\
&\qquad \times\mathcal{V}^{T}(\rho,t) && (85)\\
&\quad = \mathcal{V}(\rho,t)\begin{bmatrix}1-h(\delta) & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\mathcal{V}^{T}(\rho,t) && (86)
\end{aligned}$$

which is of full rank for all $\delta\neq 0$. □

Proposition 4 (Nullprojector commutativity with its inverted perturbation) *The controls coefficient nullprojector $\mathcal{P}(\rho,t)$ commutes with its inverted perturbation $\widetilde{\mathcal{P}}^{-1}(\rho,\delta,t)$ for all $\delta\neq 0$. Furthermore, their matrix multiplication equals to the controls coefficient nullprojector itself, i.e.,*

$$\widetilde{\mathcal{P}}^{-1}(\rho,\delta,t)\mathcal{P}(\rho,t)=\mathcal{P}(\rho,t)\widetilde{\mathcal{P}}^{-1}(\rho,\delta,t)=\mathcal{P}(\rho,t). \tag{87}$$

Proof Using the Morrison–Sherman–Woodbery matrix inversion lemma [29]

$$\begin{aligned}
&(A+BCD)^{-1}\\
&\quad = A^{-1}-A^{-1}B(C^{-1}+DA^{-1}B)^{-1}DA^{-1} && (88)
\end{aligned}$$

with $A=I_{3\times 3}$, $B=-h(\delta)I_{3\times 3}$, $C=I_{3\times 3}$, $D=\mathcal{A}^{+}(\rho,t)\mathcal{A}(\rho,t)$ implies that

$$\begin{aligned}
&\widetilde{\mathcal{P}}^{-1}(\rho,\delta,t)\\
&\quad = I_{3\times 3}-h(\delta)\big[I_{3\times 3}-h(\delta)\mathcal{A}^{+}(\rho,t)\mathcal{A}(\rho,t)\big]^{-1}\\
&\qquad \times\mathcal{A}^{+}(\rho,t)\mathcal{A}(\rho,t)\\
&\quad = I_{3\times 3}-h(\delta)\widetilde{\mathcal{P}}^{-1}(\rho,\delta,t)\mathcal{A}^{+}(\rho,t)\mathcal{A}(\rho,t), && (89)
\end{aligned}$$

so that

$$\begin{aligned}
&\widetilde{\mathcal{P}}^{-1}(\rho,\delta,t)\mathcal{P}(\rho,t)\\
&\quad = \big[I_{3\times 3}-h(\delta)\widetilde{\mathcal{P}}^{-1}(\rho,\delta,t)\mathcal{A}^{+}(\rho,t)\mathcal{A}(\rho,t)\big]\\
&\qquad \times\mathcal{P}(\rho,t)
\end{aligned}$$

$$
\begin{aligned}
&= \mathcal{P}(\rho, t) \\
&\quad - h(\delta)\widetilde{\mathcal{P}}^{-1}(\rho, \delta, t)\mathcal{A}^{+}(\rho, t)\mathcal{A}(\rho, t)\mathcal{P}(\rho, t) \\
&= \mathcal{P}(\rho, t) - h(\delta)\widetilde{\mathcal{P}}^{-1}(\rho, \delta, t)\mathcal{A}^{+}(\rho, t)\mathcal{A}(\rho, t) \\
&\quad \times \left[I_{3\times 3} - \mathcal{A}^{+}(\rho, t)\mathcal{A}(\rho, t)\right] \\
&= \mathcal{P}(\rho, t) - h(\delta)\widetilde{\mathcal{P}}^{-1}(\rho, \delta, t) \\
&\quad \times \left[\mathcal{A}^{+}(\rho, t)\mathcal{A}(\rho, t) - \mathcal{A}^{+}(\rho, t)\mathcal{A}(\rho, t)\right] \\
&= \mathcal{P}(\rho, t).
\end{aligned}
$$

The second part of the identities (87) is obtained by interchanging the definitions of B and D in the lemma and proceeding in the same manner. □

9 Spacecraft internal stability

Theorem 1 (Spacecraft internal stability) *Let the matrix gain K be*

$$
K = -J^{-1}\omega^{\times} J - \mathcal{H}_{1d}(\rho, \omega, t) + k \tag{90}
$$

where the matrix gain $k \in \mathbb{R}^{3\times 3}$ is constant and has strictly negative-real part eigenvalues. Then k can be further chosen such that the control law

$$
\tau_1 = \left[\mathcal{H}_{1d}(\rho, \omega, t) + \mathcal{P}(\rho, t)K\right]\omega \tag{91}
$$

renders the equilibrium point $\omega = \mathbf{0}_{3\times 1}$ of the spacecraft dynamical subsystem given by (19) locally exponentially stable.

Proof Substituting the control law τ_1 given by (91) with matrix gain K given by (90) into (19) gives the closed loop internal subsystem

$$
\begin{aligned}
\dot{\omega} &= \left[J^{-1}\omega^{\times} J + \mathcal{H}_{1d}(\rho, \omega, t)\right. \\
&\quad + \mathcal{P}(\rho, t)\left[-J^{-1}\omega^{\times} J - \mathcal{H}_{1d}(\rho, \omega, t)\right. \\
&\quad \left.\left. + k\right]\right]\omega \qquad (92) \\
&= \left[\mathcal{A}^{+}(\rho, t)\mathcal{A}(\rho, t)\left[J^{-1}\omega^{\times} J + \mathcal{H}_{1d}(\rho, \omega, t)\right]\right. \\
&\quad \left. + \mathcal{P}(\rho, t)k\right]\omega \qquad (93) \\
&= \left[k + \eta_1(\rho, \omega, t)\right]\omega + \eta_2(\rho, t)\omega \\
&\quad - \mathcal{A}^{+}(\rho, t)\mathcal{A}(\rho, t)k\omega \qquad (94)
\end{aligned}
$$

where

$$
\begin{aligned}
&\eta_1(\rho, \omega, t) \\
&\quad = \mathcal{A}^{+}(\rho, t)\mathcal{A}(\rho, t)J^{-1}\omega^{\times} J \\
&\qquad - \mathcal{A}^{+}(\rho, t)z^{T}(\rho, t)\left[\dot{G}(\rho, \omega) + G(\rho)J^{-1}\omega^{\times} J\right] \\
&\qquad - \mathcal{A}^{+}(\rho, t)\mathcal{A}(\rho, t)\mathcal{A}_d^{+}(\rho, \beta, t)\left[G(\rho)\omega\right]^{T} G(\rho)
\end{aligned} \tag{95}
$$

and

$$
\begin{aligned}
\eta_2(\rho, t) &= -\mathcal{A}^{+}(\rho, t)z^{T}(\rho, t)c_1 G(\rho) \\
&\quad + \mathcal{A}^{+}(\rho, t)\mathcal{A}(\rho, t)\mathcal{A}_d^{+}(\rho, \beta, t) \\
&\quad \times \dot{\rho}_d^{T}(t)G(\rho).
\end{aligned} \tag{96}
$$

Stability of the first part of the system equations (94) given by

$$
\dot{\omega} = \left[k + \eta_1(\rho, \omega, t)\right]\omega \tag{97}
$$

can be analyzed by Lyapunov indirect method by verifying that its Jacobian at $\omega = \mathbf{0}_{3\times 1}$ is given by

$$
\left.\frac{\partial[k + \eta_1(\rho, \omega, t)]\omega}{\partial\omega}\right|_{\omega = \mathbf{0}_{3\times 1}} = k + \eta_1(\rho, \mathbf{0}_{3\times 1}, t) = k, \tag{98}
$$

which is strictly stable, implying that the system given by (97) is locally exponentially stable. Therefore, the argument of Sect. 2.3.1 implies that for any strictly stable $k \in \mathbb{R}^{3\times 1}$, for any positive definite constant matrix $Q_\omega \in \mathbb{R}^{3\times 1}$, for any bounded $\rho \in \mathbb{R}^3$ and $t > 0$, and for all $\omega \in \mathbb{R}^{3\times 1}$ in the domain of attraction D_ω of $\omega = \mathbf{0}_{3\times 1}$, there exists a control Lyapunov function

$$
V_\omega(\rho, \omega, t) = \omega^{T} P_\omega(\rho, \omega, t)\omega \tag{99}
$$

where $P_\omega(\rho, \omega, t) \in \mathbb{R}^{3\times 1}$ is positive definite, such that

$$
\begin{aligned}
&\left[k + \eta_1(\rho, \omega, t)\right]^{T} P_\omega(\rho, \omega, t) \\
&\quad + P_\omega(\rho, \omega, t)\left[k + \eta_1(\rho, \omega, t)\right] \le -Q_\omega
\end{aligned} \tag{100}
$$

for all $\omega \in D_\omega$. On the other hand,

$$
\begin{aligned}
\left\|\eta_2(\rho, t)\omega\right\| &\le \left\|\eta_2(\rho, t)\right\| \|\omega\| \\
&\le \left[4c_1 + \frac{1}{\beta}\left\|\dot{\rho}_d(t)\right\|\right] \\
&\quad \times \sigma\left(G(\rho)\right)\|\omega\|
\end{aligned} \tag{101}
$$

and

$$\left\|\mathcal{A}^{+}(\rho,t)\mathcal{A}(\rho,t)k\omega\right\| \leq \left\|\mathcal{A}^{+}(\rho,t)\mathcal{A}(\rho,t)\right\|\|k\omega\| = \|k\omega\|. \tag{102}$$

The time derivative of $V_\omega(\rho,\omega,t)$ along the trajectories of the system given by (94) is

$$\dot{V}_\omega(\rho,\omega,t) = 2\omega^T P_\omega(\rho,\omega,t)\big[\big[k+\eta_1(\rho,\omega,t)\big]\omega + \eta_2(\rho,t)\omega - \mathcal{A}^{+}(\rho,t)\mathcal{A}(\rho,t)k\omega\big] \tag{103}$$

which can be written as

$$\begin{aligned}\dot{V}_\omega(\rho,\omega,t) &= \omega^T\big[\big[k+\eta_1(\rho,\omega,t)\big]^T P_\omega(\rho,\omega,t) \\ &\quad + P_\omega(\rho,\omega,t)\big[k+\eta_1(\rho,\omega,t)\big]\big]\omega \\ &\quad + 2\omega^T P_\omega(\rho,\omega,t) \\ &\quad \times \big[\eta_2(\rho,t)\omega - \mathcal{A}^{+}(\rho,t)\mathcal{A}(\rho,t)k\omega\big].\end{aligned} \tag{104}$$

Therefore, inequalities (100), (101), and (102) imply that $\dot{V}_\omega(\rho,\omega,t)$ satisfies

$$\begin{aligned}\dot{V}_\omega(\rho,\omega,t) &\leq -\lambda_{\min}(Q_\omega)\|\omega\|^2 \\ &\quad + 2\lambda_{\max}\big(P_\omega(\rho,\omega,t)\big) \\ &\quad \times \left\|\eta_2(\rho,t) - \mathcal{A}^{+}(\rho,t)\mathcal{A}(\rho,t)k\right\|\|\omega\|^2 \\ &\leq -\lambda_{\min}(Q_\omega)\|\omega\|^2 \\ &\quad + 2\lambda_{\max}\big(P_\omega(\rho,\omega,t)\big)\left[4c_1 + \frac{1}{\beta}\left\|\dot{\rho}_d(t)\right\|\right] \\ &\quad \times \sigma\big(G(\rho)\big)\|\omega\|^2 \\ &\quad + 2\lambda_{\max}\big(P_\omega(\rho,\omega,t)\big)\lambda_{\max}(k)\|\omega\|^2 \\ &= \epsilon_\omega\|\omega\|^2\end{aligned} \tag{105}$$

where

$$\epsilon_\omega = -\lambda_{\min}(Q_\omega) + 2\lambda_{\max}\big(P_\omega(\rho,\omega,t)\big) \times \left(\left[4c_1 + \frac{1}{\beta}\left\|\dot{\rho}_d(t)\right\|\right]\sigma\big(G(\rho)\big) + \lambda_{\max}(k)\right). \tag{106}$$

Choosing k such that

$$\lambda_{\max}(k) < \frac{\lambda_{\min}(Q_\omega)}{2\lambda_{\max}(P_\omega(\rho,\omega,t))} - \left[4c_1 + \frac{1}{\beta}\left\|\dot{\rho}_d(t)\right\|\right]\sigma\big(G(\rho)\big) \tag{107}$$

implies that ϵ_ω is negative. By the argument of Sect. 2.3.2, local exponential stability of the system given by (97) together with the condition given by inequality (107) imply that the equilibrium point $\omega = \mathbf{0}_{3\times1}$ of the dynamical subsystem given by (94) is locally exponentially stable on the set D_ω given by the values of ω that satisfy inequalities (100). □

An upper bound estimate of the matrix $P_\omega(\rho,\omega,t)$ is obtained from the argument of Sect. 2.3.1 by solving inequalities (100), resulting in

$$\begin{aligned}P_\omega(\rho,\omega,t) &\leq -\operatorname{vec}^{-1}\big\{\big[\big[k+\eta_1(\rho,\omega,t)\big]^T \\ &\quad \oplus \big[k+\eta_1(\rho,\omega,t)\big]\big]^{-1}\operatorname{vec}\{Q_\omega\}\big\}.\end{aligned} \tag{108}$$

Remark 3 The identities given by (87) imply that the closed loop internal subsystem given by (92) can be written as

$$\begin{aligned}\dot{\omega} &= \big[J^{-1}\omega^\times J + \mathcal{H}_{1d}(\rho,\omega,t) \\ &\quad + \mathcal{P}(\rho,t)\widetilde{\mathcal{P}}^{-1}(\rho,\delta,t) \\ &\quad \times \big[-J^{-1}\omega^\times J - \mathcal{H}_{1d}(\rho,\omega,t) + k\big]\big]\omega.\end{aligned} \tag{109}$$

Therefore, the closed loop subsystem given by (92) is a singular perturbation from the system

$$\dot{\omega} = k\omega \tag{110}$$

obtained by replacing the controls coefficient nullprojector $\mathcal{P}(\rho,t)$ in (109) by the perturbed controls coefficient nullprojector $\widetilde{\mathcal{P}}(\rho,\delta,t)$.

Remark 4 The set D_ω can be brought arbitrarily large via increasing the magnitude of the closed loop gain k. This can be shown by rewriting inequalities (100) as

$$\begin{aligned}&\big[k^T P_\omega(\rho,\omega,t) + P_\omega(\rho,\omega,t)k + Q_\omega\big] \\ &\quad + \big[\eta_1^T(\rho,\omega,t)P_\omega(\rho,\omega,t) \\ &\quad + P_\omega(\rho,\omega,t)\eta_1(\rho,\omega,t)\big] \leq \mathbf{0}_{3\times3}.\end{aligned} \tag{111}$$

For a specific spacecraft state value, the first among the two terms composing the left side of inequalities

(111) can be arbitrarily decreased by bringing the set of eigenvalues of k further to the left in the complex plane, for which inequalities (111) are guaranteed to hold true for a larger subset of ω in the neighborhood of the origin at specific values of ρ, t, $P_\omega(\rho,\omega,t)$, and Q_ω.

10 Singularly perturbed feedback linearization

The control law τ_d is defined as

$$\tau_d(\rho,\omega,\beta,t) = \big[\mathcal{H}_{1d}(\rho,\omega,\beta,t) + \mathcal{P}(\rho,t)K\big]\omega + \mathcal{H}_{2d}(\rho,\beta,t). \tag{112}$$

Using τ_d in the dynamical subsystem given by (19) yields the closed loop dynamical subsystem

$$\dot{\omega} = \big[J^{-1}\omega^\times J + \mathcal{H}_{1d}(\rho,\omega,\beta,t) + \mathcal{P}(\rho,t)K\big]\omega + \mathcal{H}_{2d}(\rho,\beta,t). \tag{113}$$

Theorem 2 (Singularly perturbed feedback linearization) *Let the matrix gain K be given by* (90). *Then the constant matrix gain k can be chosen such that the control law given by* (112) *renders the spacecraft attitude deviation vector $z(\rho,t)$ given by* (21), *and the spacecraft angular velocity components globally uniformly ultimately bounded.*

Proof Substituting the matrix gain K given by (90) into the control law given by (112) yields

$$\begin{aligned}\tau_d&(\rho,\omega,\beta,t)\\ &= \big[\mathcal{H}_{1d}(\rho,\omega,\beta,t)\\ &\quad + \mathcal{P}(\rho,t)\big[-J^{-1}\omega^\times J - \mathcal{H}_{1d}(\rho,\omega,\beta,t) + k\big]\big]\omega\\ &\quad + \mathcal{H}_{2d}(\rho,\beta,t).\end{aligned} \tag{114}$$

Let ϕ_d be a norm measure function of the attitude deviation obtained by applying the control law given by (112) to the spacecraft equations of motion (18) and (19), and let $\dot{\phi}_d$, $\ddot{\phi}_d$ be its first two time derivatives. Hence,

$$\phi_d := \phi_d(\rho,t) = \phi(\rho,t), \tag{115}$$

$$\dot{\phi}_d := \dot{\phi}_d(\rho,\omega,t) = \dot{\phi}(\rho,\omega,t), \tag{116}$$

$$\begin{aligned}\ddot{\phi}_d &:= \ddot{\phi}_d(\rho,\omega,\tau_d,t)\\ &= \ddot{\phi}(\rho,\omega,\tau,t) + \mathcal{A}(\rho,t)\tau_d - \mathcal{A}(\rho,t)\tau\end{aligned} \tag{117}$$

where τ is obtained by substituting the matrix gain K given by (90) in the desired attitude deviation dynamics globally realizing control law given by (37), so that

$$\begin{aligned}\tau &= \big[\mathcal{H}_1(\rho,\omega,t)\\ &\quad + \mathcal{P}(\rho,t)\big[-J^{-1}\omega^\times J - \mathcal{H}_1(\rho,\omega,t) + k\big]\big]\omega\\ &\quad + \mathcal{H}_2(\rho,t).\end{aligned} \tag{118}$$

Adding $c_1\dot{\phi}_d + c_2\phi_d$ to both sides of (117) yields

$$\ddot{\phi}_d + c_1\dot{\phi}_d + c_2\phi_d$$
$$= \ddot{\phi} + c_1\dot{\phi} + c_2\phi + \mathcal{A}(\rho,t)\tau_d - \mathcal{A}(\rho,t)\tau \tag{119}$$
$$= \mathcal{A}(\rho,t)[\tau_d - \tau]. \tag{120}$$

Let the state vector $\Phi_d \in \mathbb{R}^{2\times 1}$ be defined as

$$\Phi_d = [\phi_d \quad \dot{\phi}_d]^T. \tag{121}$$

The attitude deviation norm measure closed loop dynamics can be written in the state space form

$$\dot{\Phi}_d = \Lambda_{11}\Phi_d + \Lambda_{12}(\rho,\omega,\beta,t)\omega + \Delta_1(\rho,\beta,t) \tag{122}$$

where the strictly stable system matrix $\Lambda_{11} \in \mathbb{R}^{2\times 2}$ is

$$\Lambda_{11} = \begin{bmatrix} 0 & 1 \\ -c_2 & -c_1 \end{bmatrix} \tag{123}$$

and the matrix valued function $\Lambda_{12}(\rho,\omega,\beta,t) : \mathbb{R}^{7\times 1} \to \mathbb{R}^{2\times 3}$ is

$$\Lambda_{12}(\rho,\omega,\beta,t) = \begin{bmatrix} \mathbf{0}_{1\times 3} \\ \mathcal{A}(\rho,t)[\mathcal{H}_{1d}(\rho,\omega,\beta,t) - \mathcal{H}_1(\rho,\omega,t)] \end{bmatrix} \tag{124}$$

and the matrix valued function $\Delta_1(\rho,\beta,t) : \mathbb{R}^{5\times 1} \to \mathbb{R}^{2\times 1}$ is

$$\Delta_1(\rho,\beta,t) = \begin{bmatrix} 0 \\ \mathcal{A}(\rho,t)[\mathcal{H}_{2d}(\rho,\beta,t) - \mathcal{H}_2(\rho,t)] \end{bmatrix}. \tag{125}$$

The definitions of $\mathcal{A}(\rho,t)$ and $\mathcal{A}_d^+(\rho,\beta,t)$ given by (31) and (65) are identical over the domain defined by the condition

$$\big\|G^T(\rho)z(\rho,t)\big\| \geq \beta. \tag{126}$$

Therefore, the definitions of $\mathcal{H}_1(\rho,\omega,t)$ and $\mathcal{H}_2(\rho,t)$ given by (38) and (39) are also identical to the definitions of $\mathcal{H}_{1d}(\rho,\omega,\beta,t)$ and $\mathcal{H}_{2d}(\rho,\beta,t)$ given by

(69) and (70) over the domain given by inequality (126), for which $\Lambda_{12}(\rho,\omega,\beta,t)=\mathbf{0}_{2\times 3}$ and $\Delta_1(\rho,\beta,t)=\mathbf{0}_{2\times 1}$. Hence, the definition of the attitude deviation norm measure function ϕ given by (22) implies that if

$$\phi \geq \frac{\beta^2}{2\sigma(G(\rho))} \tag{127}$$

then the attitude deviation norm measure closed loop dynamics given by the state space equations (122) reduces to the globally exponentially stable system given by the state space model

$$\dot{\Phi}_d = \Lambda_{11}\Phi_d. \tag{128}$$

Consider the Lyapunov function V_{Φ_d} given by

$$V_{\Phi_d}(\Phi_d) = \Phi_d^T P_{\Phi_d}\Phi_d \tag{129}$$

where $P_{\Phi_d}\in\mathbb{R}^{2\times 2}$ is a positive definite constant matrix. Then, the time derivative of V_{Φ_d} along the system equations (122) satisfies

$$\begin{aligned}\dot{V}(\Phi_d,\rho,\omega,\beta,t) &= \Phi_d^T\big[P_{\Phi_d}\Lambda_{11}+\Lambda_{11}^T P_{\Phi_d}\big]\Phi_d \\ &\quad + 2\Phi_d^T P_{\Phi_d}\big[\Lambda_{12}(\rho,\omega,\beta,t)\omega \\ &\quad + \Delta_1(\rho,\beta,t)\big] \qquad (130)\\ &\leq \Phi_d^T Q_{\Phi_d}\Phi_d \leq \lambda_{\max}(Q_{\Phi_d})\|\Phi_d\|^2 \qquad (131)\end{aligned}$$

for all values of ϕ in the domain given by inequality (127), where $Q_{\Phi_d}\in\mathbb{R}^{2\times 2}$ is any positive definite constant matrix such that

$$P_{\Phi_d}\Lambda_{11}+\Lambda_{11}^T P_{\Phi_d} \leq Q_{\Phi_d}. \tag{132}$$

Therefor, the argument of Sect. 2.3.3 implies that the trajectories of Φ_d (equivalently the trajectories of ϕ and $\dot{\phi}$) and the trajectories of $z(\rho,t)$ are globally uniformly ultimately bounded. On the other hand, substituting the matrix gain K given by (90) into (113) yields the driven closed loop dynamical subsystem

$$\begin{aligned}\dot{\omega} &= \big[J^{-1}\omega^\times J + \mathcal{H}_{1d}(\rho,\omega,\beta,t) \\ &\quad + \mathcal{P}(\rho,t)\big[-J^{-1}\omega^\times J - \mathcal{H}_{1d}(\rho,\omega,\beta,t)+k\big]\big]\omega \\ &\quad + \mathcal{H}_{2d}(\rho,\beta,t).\end{aligned} \tag{133}$$

Defining the matrix $\Lambda_{22}(\rho,\omega,t):\mathbb{R}^{8\times 1}\to\mathbb{R}^{3\times 3}$ as

$$\Lambda_{22}(\rho,\omega,t) = \mathcal{P}(\rho,t)k + \eta_1(\rho,\omega,t) + \eta_2(\rho,t) \tag{134}$$

where $\eta_1(\rho,\omega,t)$ and $\eta_2(\rho,t)$ are given by (95) and (96), and defining the perturbation vector $\Delta_2(\rho,\beta,t):\mathbb{R}^{5\times 1}\to\mathbb{R}^{3\times 1}$ as

$$\Delta_2(\rho,\beta,t) = \mathcal{H}_{2d}(\rho,\beta,t) \tag{135}$$

then the system of (133) can be written as

$$\dot{\omega} = \Lambda_{22}(\rho,\omega,t)\omega + \Delta_2(\rho,\beta,t). \tag{136}$$

The argument of Theorem 1 implies that the unforced part of the closed loop internal subsystem given by (136) obtained by setting $\Delta_2(\rho,\beta,t)=\mathbf{0}_{3\times 1}$ is exponentially stable if the constant matrix k satisfies the conditions given by inequalities (107). Defining the augmented state vector ξ to be

$$\xi = [\Phi_d^T \quad \omega^T]^T, \tag{137}$$

(122) and (136) form the augmented state space model

$$\dot{\xi} = \Lambda(\rho,\omega,\beta,t)\xi + \Delta(\rho,\beta,t) \tag{138}$$

where

$$\begin{aligned}\Lambda(\rho,\omega,\beta,t) &= \begin{bmatrix}\Lambda_{11} & \Lambda_{12}(\rho,\omega,\beta,t)\\ \mathbf{0}_{3\times 2} & \Lambda_{22}(\rho,\omega,t)\end{bmatrix},\\ \Delta(\rho,\beta,t) &= \begin{bmatrix}\Delta_1(\rho,\beta,t)\\ \Delta_2(\rho,\beta,t)\end{bmatrix}.\end{aligned} \tag{139}$$

The nominal part of the spacecraft closed loop state space model is obtained by setting $\Delta(\rho,\beta,t)=\mathbf{0}_{5\times 1}$ in (138) such that

$$\dot{\xi} = \Lambda(\rho,\omega,\beta,t)\xi. \tag{140}$$

The Jacobian of the system given by (140) at $\xi=\mathbf{0}_{5\times 1}$ is

$$\begin{aligned}&\frac{\partial[\Lambda(\rho,\omega,\beta,t)\xi]}{\partial\xi}\bigg|_{\xi=\mathbf{0}_{5\times 1}} \\ &\quad = \Lambda(\rho,\omega,\beta,t)|_{\xi=\mathbf{0}_{5\times 1}} \\ &\quad = \Lambda_{l1}(\rho,t) + \Lambda_{l2}(\rho,\beta,t)\end{aligned} \tag{141}$$

where

$$\Lambda_{l1}(\rho,t) = \begin{bmatrix}\Lambda_{11} & \mathbf{0}_{2\times 3}\\ \mathbf{0}_{3\times 2} & \mathcal{P}(\rho,t)k+\eta_2(\rho,t)\end{bmatrix} \tag{142}$$

and

$$\Lambda_{l2}(\rho, \beta, t) = \begin{bmatrix} \mathbf{0}_{2\times 2} & \Lambda_{12}(\rho, \omega = \mathbf{0}_{3\times 1}, \beta, t) \\ \mathbf{0}_{3\times 2} & \mathbf{0}_{3\times 3} \end{bmatrix}. \tag{143}$$

Lyapunov indirect method of stability can be used to analyze local stability of the system given by (140) by investigating stability of the linearized system

$$\dot{\xi}_l = \Lambda_{l1}(\rho, t)\xi_l + \Lambda_{l2}(\rho, \beta, t)\xi_l. \tag{144}$$

Strict stability of the matrix Λ_{11} together with the guaranteed spacecraft internal exponential stability for values of k satisfying inequalities (107) imply that the system

$$\dot{\xi}_l = \Lambda_{l1}(\rho, t)\xi_l \tag{145}$$

obtained by setting $\Lambda_{l2}(\rho, \beta, t) = \mathbf{0}_{5\times 5}$ has an exponentially stable equilibrium point $\xi_l = \mathbf{0}_{5\times 1}$. Therefore, for all values of ρ and for all $t > 0$, and for any positive definite constant matrix Q_{ξ_l}, there exists a Lyapunov function

$$V(\xi, \rho, t) = \xi_l^T P_{\xi_l}(\rho, t)\xi_l \tag{146}$$

where $P_{\xi_l}(\rho, t)$ is a positive definite matrix, such that

$$\Lambda_{l1}^T(\rho, t)P_{\xi_l}(\rho, t) + P_{\xi_l}(\rho, t)\Lambda_{l1} \le -Q_{\xi_l}. \tag{147}$$

The time derivative of $V(\xi, \rho, t)$ along the trajectories of the system given by (144) is

$$\begin{aligned} &\dot{V}(\xi, \rho, \beta, t) \\ &\quad = \xi_l^T\left[\Lambda_{l1}^T(\rho, t)P_{\xi_l}(\rho, t) + P_{\xi_l}(\rho, t)\Lambda_{l1}\right]\xi_l \\ &\qquad + 2\xi^T P_{\xi_l}(\rho, t)\Lambda_{l2}(\rho, \beta, t)\xi_l \end{aligned} \tag{148}$$

which implies from inequalities (147) that

$$\begin{aligned} \dot{V}(\xi, \rho, \beta, t) \le &-\xi_l^T Q_{\xi_l}\xi_l \\ &+ 2\xi^T P_{\xi_l}(\rho, t)\Lambda_{l2}(\rho, \beta, t)\xi_l. \end{aligned} \tag{149}$$

Moreover, the matrix $\Lambda_{12}(\rho, \omega = \mathbf{0}_{3\times 1}, \beta, t)$ is given by

$$\begin{aligned} &\Lambda_{12}(\rho, \omega = \mathbf{0}_{3\times 1}, \beta, t) \\ &\quad = \begin{bmatrix} \mathbf{0}_{1\times 3} \\ \mathcal{A}(\rho, t)[\mathcal{H}_{1d}(\rho, \omega = \mathbf{0}_{3\times 1}, \beta, t) - \mathcal{H}_1(\rho, \omega = \mathbf{0}_{3\times 1}, t)] \end{bmatrix} \\ &\quad = \begin{bmatrix} \mathbf{0}_{1\times 3} \\ (\mathcal{A}(\rho, t)\mathcal{A}_d^+(\rho, \beta, t) - 1)\dot{\rho}_d^T(t)G(\rho) \end{bmatrix} \end{aligned} \tag{150}$$

which is bounded for all bounded values of ρ. Therefore, a norm bound on $\Lambda_{l2}(\rho, \beta, t)$ is given by

$$\begin{aligned} &\left\|\Lambda_{l2}(\rho, \beta, t)\right\| \\ &\quad = \left\|\Lambda_{12}(\rho, \omega = \mathbf{0}_{3\times 1}, \beta, t)\right\| \qquad (151) \\ &\quad \le \left\|\left(\mathcal{A}(\rho, t)\mathcal{A}_d^+(\rho, \beta, t) - 1\right)\dot{\rho}_d^T(t)G(\rho)\right\| \qquad (152) \end{aligned}$$

implying from (55) and inequality (67) that

$$\begin{aligned} &\left\|\Lambda_{l2}(\rho, \beta, t)\right\| \\ &\quad \le \left(\frac{\sigma(G(\rho))\|z(\rho, t)\|}{\beta} + 1\right)\left\|\dot{\rho}_d(t)\right\|\sigma\left(G(\rho)\right). \end{aligned} \tag{153}$$

Hence, inequality (149) becomes

$$\begin{aligned} &\dot{V}(\xi, \rho, \beta, t) \\ &\quad \le \left[-\lambda_{\min}(Q_{\xi_l}) \right. \\ &\qquad \left. + 2\lambda_{\max}\left(P_{\xi_l}(\rho, t)\right)\left\|\Lambda_{l2}(\rho, \beta, t)\right\|\right]\|\xi_l\|^2 \qquad (154) \\ &\quad \le \epsilon_{\xi_l}\|\xi_l\|^2 \qquad (155) \end{aligned}$$

where

$$\begin{aligned} \epsilon_{\xi_l} = &-\lambda_{\min}(Q_{\xi_l}) \\ &+ 2\lambda_{\max}\left(P_{\xi_l}(\rho, t)\right)\left(\frac{\sigma(G(\rho))\|z(\rho, t)\|}{\beta} + 1\right) \\ &\times \left\|\dot{\rho}_d(t)\right\|\sigma\left(G(\rho)\right). \end{aligned} \tag{156}$$

Choosing Q_{ξ_l} such that

$$\begin{aligned} &\lambda_{\min}(Q_{\xi_l}) \\ &\quad > 2\lambda_{\max}\left(P_{\xi_l}(\rho, t)\right)\left(\frac{\sigma(G(\rho))\|z(\rho, t)\|}{\beta} + 1\right) \\ &\qquad \times \left\|\dot{\rho}_d(t)\right\|\sigma\left(G(\rho)\right) \end{aligned} \tag{157}$$

implies that ϵ_{ξ_l} is negative, which together with the global exponential stability of the system given by (145) imply from Sect. 2.3.2 that $\xi_l = \mathbf{0}_{5\times 1}$ is a globally exponentially stable equilibrium point of the linear system given by (144). Therefore, it is inferred from Lyapunov indirect method that the system given by (140) has a locally exponentially stable equilibrium point $\xi = \mathbf{0}_{5\times 1}$. Hence, for any bounded $\rho \in \mathbb{R}^3$

and $t > 0$, for any positive definite constant matrix $Q_\xi \in \mathbb{R}^{5\times 1}$ and for all $\xi \in \mathbb{R}^{5\times 1}$ in the domain of attraction D_ξ of $\xi = \mathbf{0}_{5\times 1}$, there exists a control Lyapunov function ([26], p. 167)

$$V_\xi(\rho,\xi,t) = \xi^T P_\xi(\rho,\xi,t)\xi \tag{158}$$

where $P_\xi(\rho,\xi,t) \in \mathbb{R}^{5\times 1}$ is positive definite, such that the time derivative of $V_\xi(\rho,\xi,t)$ along the trajectories of the system given by (140) satisfies

$$\dot V_\xi(\rho,\xi,t) \le -\xi^T Q_\xi \xi \tag{159}$$

resulting in

$$\begin{aligned}&\Lambda^T(\rho,\omega,\beta,t)P_\xi(\rho,\xi,t) + P_\xi(\rho,\xi,t)\Lambda(\rho,\omega,\beta,t)\\ &\quad\le -Q_\xi\end{aligned} \tag{160}$$

for all $\xi \in D_\xi$. To show global uniform ultimate boundedness of the dynamical system given by (138), we evaluate the time derivative of $V(\xi) = \xi^T P_\xi \xi$ along the trajectories of the system, resulting in

$$\begin{aligned}&\dot V(\rho,\omega,\beta,t)\\ &\quad= 2\xi^T P_\xi \Lambda(\rho,\omega,\beta,t)\xi + 2\xi^T P_\xi \Delta(\rho,\beta,t)\\ &\quad= \xi^T\big[P_\xi\Lambda(\rho,\omega,\beta,t) + \Lambda^T(\rho,\omega,\beta,t)P_\xi\big]\xi\\ &\qquad+ 2\xi^T P_\xi\Delta(\rho,\beta,t)\\ &\quad= -\xi^T Q_\xi(\rho,\omega,\beta,t)\xi + 2\xi^T P_\xi \Delta(\rho,\beta,t).\end{aligned} \tag{161}$$

Therefore,

$$\begin{aligned}&\dot V(\rho,\omega,\beta,t)\\ &\quad\le -\lambda_{\min}\big[Q_\xi(\rho,\omega,\beta,t)\big]\|\xi\|^2 + 2\xi^T P_\xi\Delta(\rho,\beta,t)\\ &\quad\le -\lambda_{\min}\big(Q_\xi(\rho,\omega,\beta,t)\big)\|\xi\|^2\\ &\qquad+ 2\lambda_{\max}(P_\xi)\big\|\Delta(\rho,\beta,t)\big\|\|\xi\|\\ &\quad= -[1-\theta]\lambda_{\min}\big(Q_\xi(\rho,\omega,\beta,t)\big)\|\xi\|^2\\ &\qquad- \theta\lambda_{\min}\big(Q_\xi(\rho,\omega,\beta,t)\big)\|\xi\|^2\\ &\qquad+ 2\lambda_{\max}(P_\xi)\big\|\Delta(\rho,\beta,t)\big\|\|\xi\|, \quad 0<\theta<1.\end{aligned} \tag{162}$$

Hence, for all ξ such that

$$\|\xi\| > \frac{2\lambda_{\max}(P_\xi)}{\theta\lambda_{\min}(Q_\xi(\rho,\omega,\beta,t))}\big\|\Delta(\rho,\beta,t)\big\| \tag{163}$$

the following inequality holds

$$\dot V(\rho,\omega,\beta,t) \le -[1-\theta]\lambda_{\min}\big(Q_\xi(\rho,\omega,\beta,t)\big)\|\xi\|^2 \tag{164}$$

and the argument of Sect. 2.3.3 implies that for any initial condition $\xi(0) \in \mathbb{R}^{5\times 1}$, the solution of the dynamical system given by (138) satisfies

$$\big\|\xi(t)\big\| \le \alpha e^{-\gamma t}\big\|\xi(0)\big\|, \tag{165}$$

for all $0 \le t \le T$ for some finite time T, and

$$\big\|\xi(t)\big\| \le b, \tag{166}$$

for all $t \ge T$, where

$$\begin{aligned}&\alpha = \sqrt{\frac{a_2}{a_1}}, \qquad \gamma = \frac{(1-\theta)a_3}{2a_2},\\ &b = \frac{a_4}{a_3}\sqrt{\frac{a_2}{a_1}}\frac{\|\Delta(\rho,\beta,t)\|}{\theta}.\end{aligned} \tag{167}$$

The constants a_i, $i = 1,\dots,4$ are

$$a_1 = \lambda_{\min}(P_\xi), \tag{168}$$

$$a_2 = \lambda_{\max}(P_\xi), \tag{169}$$

$$a_3 = \lambda_{\min}\big(Q_\xi(\rho,\omega,\beta,t)\big), \tag{170}$$

$$a_4 = 2\lambda_{\max}(P_\xi). \tag{171}$$

□

A norm bound on the expression of $\Delta_1(\rho,\beta,t)$ can be obtained as

$$\begin{aligned}&\big\|\Delta_1(\rho,\beta,t)\big\|\\ &\quad= \big|-\mathcal{A}(\rho,t)\mathcal{A}_d^+(\rho,\beta,t)+1\big|\big\|\dot\rho_d(t)\big\|^2 &(172)\\ &\quad\le \left[\frac{1}{\beta}\big\|G^T(\rho)z(\rho,t)\big\|_2 + 1\right]\big\|\dot\rho_d(t)\big\|^2 &(173)\\ &\quad\le \left[\frac{1}{\beta}\big[\sigma\big(G^T(\rho)\big)\big]\big\|z(\rho,t)\big\|_2 + 1\right]\big\|\dot\rho_d(t)\big\|^2 &(174)\end{aligned}$$

and a norm bound on the expression of $\Delta_2(\rho,\beta,t)$ given by (135) can be obtained from (70), (61), (59), and (67) as

$$\begin{aligned}&\big\|\Delta_2(\rho,\beta,t)\big\|\\ &\quad= \left\|-\frac{c_2}{2}\mathcal{A}^+(\rho,t)z^T(\rho,t)z(\rho,t)\right.\end{aligned}$$

$$+\mathcal{A}^{+}(\rho,t)z^{T}(\rho,t)\big[\ddot{\rho}_d(t)+c_1\dot{\rho}_d(t)\big]$$
$$-\mathcal{A}_d^{+}(\rho,\beta,t)\big\|\dot{\rho}_d(t)\big\|^2\Big\|$$
$$\leq 2c_2\big\|z(\rho,t)\big\|+4(\ddot{\rho}_d+c_1\dot{\rho}_d)+\frac{1}{\beta}\|\dot{\rho}_d\|^2. \quad (175)$$

Inequalities (147) imply that

$$\lambda_{\min}(Q_{\xi_l})\leq 2\big\|\Lambda_{l1}(\rho,t)\big\|\big\|P_{\xi_l}(\rho,t)\big\|$$
$$\leq 2\big\|\Lambda_{l1}(\rho,t)\big\|\lambda_{\max}P_{\xi_l}(\rho,t) \quad (176)$$

which together with inequality (157) imply that a sufficient condition for the system given by (140) to have a locally exponentially stable equilibrium point $\xi=\mathbf{0}_{5\times 1}$ is that the matrix norm $\|\Lambda_{l1}(\rho,t)\|$ is bounded from below as

$$\big\|\Lambda_{l1}(\rho,t)\big\|>\left(\frac{\sigma(G(\rho))\|z(\rho,t)\|}{\beta}+1\right)$$
$$\times\big\|\dot{\rho}_d(t)\big\|\sigma\big(G(\rho)\big). \quad (177)$$

The block-diagonal structure of the matrix $\Lambda_{l1}(\rho,t)$ given by (142) implies that

$$\big\|\Lambda_{l1}(\rho,t)\big\|$$
$$<\|\Lambda_{11}\|+\big\|\mathcal{P}(\rho,t)k\big\|+\big\|\eta_2(\rho,t)\big\| \quad (178)$$
$$\leq\sigma_{\max}(\Lambda_{11})+\sigma_{\max}(k)+\big\|\eta_2(\rho,t)\big\|. \quad (179)$$

Therefore, inequality (177) is satisfied if

$$\sigma_{\max}(k)>\left(\frac{\sigma(G(\rho))\|z(\rho,t)\|}{\beta}+1\right)\big\|\dot{\rho}_d(t)\big\|\sigma\big(G(\rho)\big)$$
$$-\sigma_{\max}(\Lambda_{11})-\big\|\eta_2(\rho,t)\big\|. \quad (180)$$

Remark 5 The identities given by (87) imply that the driven closed loop internal subsystem given by (133) can be written as

$$\dot{\omega}=\big[J^{-1}\omega^{\times}J+\mathcal{H}_{1d}(\rho,\omega,\beta,t)$$
$$+\mathcal{P}(\rho,t)\widetilde{\mathcal{P}}^{-1}(\rho,\delta,t)$$
$$\times\big[-J^{-1}\omega^{\times}J-\mathcal{H}_{1d}(\rho,\omega,\beta,t)+k\big]\big]\omega$$
$$+\mathcal{H}_{2d}(\rho,\beta,t). \quad (181)$$

Therefore, the closed loop subsystem given by (133) is a singular perturbation from the system

$$\dot{\omega}=k\omega+\mathcal{H}_{2d}(\rho,t) \quad (182)$$

obtained by replacing the controls coefficient nullprojector $\mathcal{P}(\rho,t)$ in (181) by the perturbed controls coefficient nullprojector $\widetilde{\mathcal{P}}(\rho,\delta,t)$. Moreover, the definitions of $\mathcal{A}(\rho,t)$ and $\mathcal{A}_d^{+}(\rho,\beta,t)$ given by (31) and (65) are identical over the domain defined by inequality (126). Therefore, the spacecraft closed loop dynamics in that domain is a singular perturbation from the driven linear system

$$\ddot{\phi}+\dot{\phi}+\phi=0, \qquad \dot{\omega}=k\omega+\mathcal{H}_2(\rho,t) \quad (183)$$

obtained by replacing τ_d by τ in (120), and $\mathcal{H}_{2d}(\rho,t)$ by $\mathcal{H}_2(\rho,t)$ in (182).

Remark 6 Inequalities (174) and (175) imply that the upper bound on $\|\Delta(\rho,\beta,t)\|$ is inversely proportional to the generalized inverse damping factor β. Since the global uniform ultimate bound b is directly proportional to $\|\Delta(\rho,\beta,t)\|$ as implied by (167), this indicates a tradeoff between trajectory tracking accuracy and damped generalized inverse stability.

11 Damped singularly perturbed feedback linearization

Global uniform ultimate boundedness of the augmented state space model given by (138) is shown in the previous section by utilizing the function given by (158) as a Lyapanov-like function. If the controls coefficient nullprojection matrix $\mathcal{P}(\rho,t)$ in the control law given by (114) is replaced by the damped controls coefficient nullprojection matrix $\mathcal{P}_d(\rho,\beta,t)$, then the resulting control law becomes

$$\tau_{dp}(\rho,\omega,\beta,t)$$
$$=\big[\mathcal{H}_{1d}(\rho,\omega,\beta,t)$$
$$+\mathcal{P}_d(\rho,\beta,t)\big[-J^{-1}\omega^{\times}J-\mathcal{H}_{1d}(\rho,\omega,\beta,t)$$
$$+k\big]\big]\omega+\mathcal{H}_{2d}(\rho,\beta,t). \quad (184)$$

The same function $V(\xi)$ can be used to show global uniform ultimate boundedness of the resulting dynamical system.

Corollary 2 (Damped singularly perturbed feedback linearization) *The closed loop spacecraft control system states resulting from applying the control law given by* (184) *to the spacecraft equations of motion* (18) *and* (19) *are globally uniformly ultimately* *bounded, and any resulting closed loop spacecraft attitude control trajectory with initial condition* $\rho(0) \in \mathbb{R}^3$ *enters the domain defined by*

$$\|z(\rho, t)\| < \frac{\beta}{\sigma(G(\rho))} \tag{185}$$

in finite time, and remains in it for all future time.

Proof According to (72), the definitions of $\mathcal{P}(\rho, t)$ and $\mathcal{P}_d(\rho, \beta, t)$ are identical in the domain defined by $\|G^T(\rho)z(\rho, t)\| \geq \beta$. Therefore, exponential stability of the nominal part of the augmented state space model given by (138) and the corresponding Lyapanov function $V(\xi) := \xi^T P \xi$ imply that the time derivative of $V(\xi)$ along the closed loop trajectories resulting from applying the control law τ_{dp} given by (184) is negative definite in that domain. According to Lyapanov theory, the trajectories of the closed loop system moves in the direction of decreasing $V(\xi)$. Therefore, the trajectories must cross in finite time the boundary of the domain to its open complement domain defines by $\|G^T(\rho)z(\rho, t)\| < \beta$, which becomes an invariant set.

Moreover, since the term $\Delta(\rho, \beta, t)$ in the augmented state space model given by (138) renders the corresponding trajectories of dynamical system globally uniformly ultimately bounded, the same term renders a similar augmented state space model that is obtained by applying the control law τ_{dp} given by (184) globally uniformly ultimately bounded. Since the control law τ given by (118) globally realizes the desired attitude deviation dynamics given by (26), replacing $\mathcal{H}_1(\rho, \omega, t)$, $\mathcal{H}_2(\rho, t)$, and $\mathcal{P}(\rho, t)$ in the control law τ given by (118) by $\mathcal{H}_{1d}(\rho, \omega, \beta, t)$, $\mathcal{H}_{2d}(\rho, \beta, t)$, and $\mathcal{P}_d(\rho, \beta, t)$ to obtain τ_{dp} realizes the desired attitude deviation dynamics given by (26) for all ρ and t in the domain defined by $\|G^T(\rho)z(\rho, t)\| \geq \beta$. Finally, since

$$\|G^T(\rho)z(\rho, t)\| = \sigma(G(\rho))\|z(\rho, t)\| \tag{186}$$

the bound estimate of the attitude deviation vector norm given by (185) follows. □

12 Control system design procedure

The procedures for designing controls coefficient generalized inverse-based singularly perturbed feedback linearization spacecraft control systems are summarized in the following steps

1. The attitude deviation norm measure equation coefficients c_1 and c_2 are chosen such that ϕ is stable. This implies that both c_1 and c_2 are strictly positive.
2. The expressions given by (28) and (29) for $\mathcal{A}(\rho, t)$ and $\mathcal{B}(\rho, \omega, t)$ are obtained, where $G(\rho)$ and $z(\rho, t)$ are given by (20) and (21), respectively, and $\rho_d(t)$ satisfies Assumption 1.
3. The controls coefficient generalized inverse $\mathcal{A}^+(\rho, t)$ given by (31) is modified in the manner of (65), and $\mathcal{A}_d^+(\rho, \beta, t)$ is used to define the expressions of $\mathcal{H}_{1d}(\rho, \omega, \beta, t)$, $\mathcal{H}_{2d}(\rho, \beta, t)$, and $\mathcal{P}_d(\rho, \beta, t)$ according to (69), (70), and (71), respectively.
4. The control law τ_{dp} given by (184) is applied, where the constant matrix gain $k \in \mathbb{R}^{3\times 3}$ is strictly stable and satisfies inequalities (107) and (180). The constant matrix Q_ω involved in inequality (107) is arbitrary but positive definite, and the matrix $P_\omega(\rho, \omega, t)$ satisfies inequality (108).
5. Integrate (18) and (19) to obtain the trajectories of ρ and ω, where $u = J\tau$. The resulting trajectory tracking errors are globally uniformly ultimately bounded according to inequality (185).

13 Numerical simulations

The spacecraft model used for numerical simulations has inertia parameters $I_1 = 100$ kg m^2, $I_2 = 150$ kg m^2, $I_3 = 85$ kg m^2. The desired MRPs trajectories are chosen to be $\rho_{d_i} = \cos 0.1t$, $i = 1, 2, 3$. Their initial values are given by the vector $\rho(0) = [0.75\ -1.42\ -0.26]^T$, and the initial spacecraft body angular velocity vector is $\omega(0) = [0.43\ -0.61\ 1.50]^T$. Values of $c_1 = 3$, $c_2 = 1.5$, $k = -2I_{3\times 3}$, and $\beta = 0.1$ are chosen, and fourth-order Runge–Kutta numerical integration scheme is used to integrate (18) and (19). Figures 2, 3, and 4 show plots of the MRP ρ_1, the angular velocity component ω_1, and the scaled control variable τ_1 versus time t. Similar plots for the remaining state and control variables are obtained, but are not shown. MRP ρ_1 tracking errors and the corresponding control variables time histories for different values of β are shown in Figs. 5 and 6, revealing the tradeoff between trajectory tracking accuracy and damped generalized inverse stability.

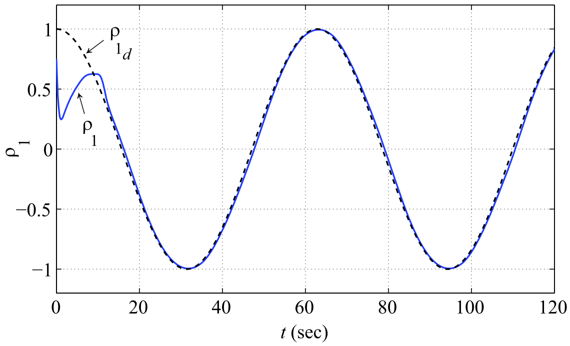

Fig. 2 Modified Rodriguez attitude parameter ρ_1 vs. t ($\beta = 0.1$)

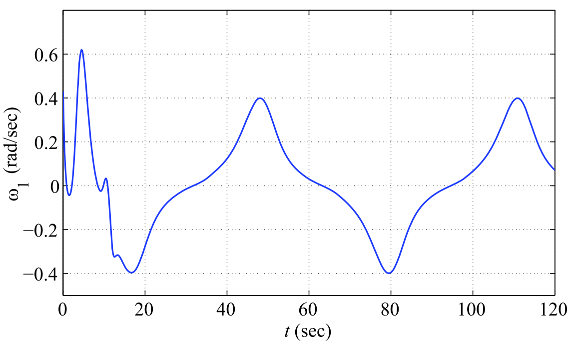

Fig. 3 Angular velocity component ω_1 vs. t ($\beta = 0.1$)

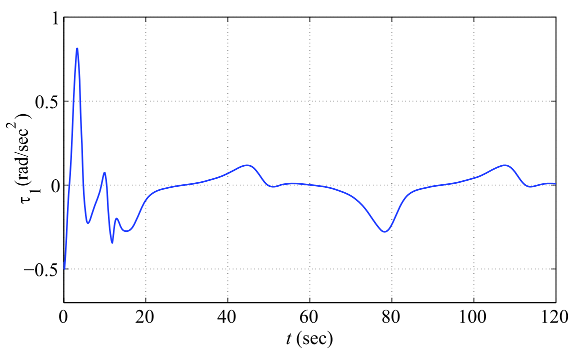

Fig. 4 Scaled control variable τ_1 vs. t ($\beta = 0.1$)

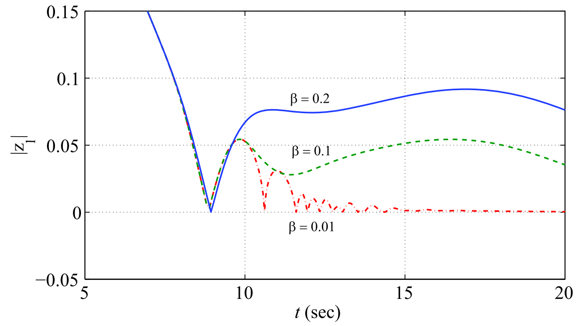

Fig. 5 Attitude parameter error $|z_1| = |\rho_1 - \rho_{1_d}(t)|$ for different values of β

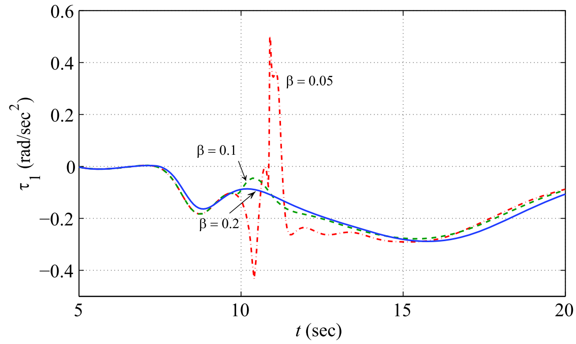

Fig. 6 Scaled control variable τ_1 for different values of β

14 Conclusion and potential applications

A generalized inversion-based design paradigm for nonlinear spacecraft control is introduced, using the Greville formula for general solution of linear system of equations. The main feature of the approach is to parameterize solution nonuniqueness of the control problem while preserving a prescribed design constraint that is imposed by a linear differential equation in the kinematic variables.

The key factor in the design is constructing a free control law's auxiliary part null-control vector. Affinity of the control law in the null-control vector is utilized to yield exponentially stable spacecraft inner dynamics and a singularly perturbed linear spacecraft global dynamics, which adds to the predetermined linear design constraint on the attitude deviation dynamics.

The controls coefficient Moore–Penrose generalized inverse definition is modified in the neighborhood of the desired trajectory in order to limit the generalized inverse growth as steady state closed loop response is approached. This implies approximate tracking with globally ultimately bounded trajectory tracking errors that can be brought arbitrarily small by decreasing the damping factor by which the Moore–Penrose generalized inverse is modified, at the attending cost of decreasing the gained stability of the generalized inverse.

The domain of attraction of the closed loop system can be increased arbitrarily by increasing the closed loop gain k in the sense of spectral radius and in the sense of magnitude, i.e., by removing the least stable eigenvalue of k further to the left in the complex plane according to inequality (107), and by increasing the maximum singular value of k according to inequality (180).

Although closed loop stability analysis in this paper is not affected by the type of vector p norm used to define the attitude deviation norm measure function ϕ, exploring the effect of using vector p norms other than the Euclidean norm on the closed loop control system performance is important for the purpose of optimizing the methodology. For instance, considering the role of Euler principal direction about which Euler principal angle of rotation is measured in choosing the type of p norm used may lead to better control system designs, since the proposed norm-based control design methodology simultaneously considers all three degrees of rotational freedom as a feedback control criterion.

The methodology solely employs linear design tools, and it has the potential to solve attitude control problems for spacecraft equipped with actuating devices other than gas jet thrusters, where there are needs to enhance existing control methodologies. In particular, the methodology can be utilized in deriving steering feedback control laws for spacecraft equipped with internal control moment gyros (CMGs) [23, 30], for the purpose of avoiding singularities and enhancing performance.

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