Nonlinear Dyn (2008) 53: 117–127
DOI 10.1007/s11071-007-9300-8

ORIGINAL PAPER

# Analysis of forced vibrations by nonlinear modes

**K.V. Avramov**

Received: 22 March 2007 / Accepted: 28 August 2007 / Published online: 7 October 2007


**Abstract** The combination of Rausher method and nonlinear modes is suggested to analyze the forced vibrations of nonlinear discrete systems. The basis of the Rausher method is iterative procedure. In this case, the analysis of a nonautonomous dynamical system reduces to the multiple solutions of the autonomous ones. As an example, the forced vibrations of shallow arch close to equilibrium position are considered in this paper. The results of the analysis are shown on the frequency response.

**Keywords** Rausher method · Nonlinear modes · Forced vibrations · Shallow arch

## 1 Introduction

Many methods exist to analyze forced vibrations of discrete nonlinear systems. The asymptotic methods (multiple scales method, Van der Pol transformations, Melnikov method) are used for such analysis [1–4]. Forced vibrations of essential nonlinear systems are analyzed by the harmonic balance method, continuation technique and nonlinear modes [5–7].

In this paper, the method of forced vibrations analysis based on the combination of the Rausher method and nonlinear modes is suggested. Note, that this method can be used easily to analyze the forced vibrations in the engineering systems with many degree-of-freedoms.

Rausher method is an effective tool to study forced vibrations. This method has been suggested for analysis of single-DOF nonautonomous systems [8]. Let $q$ be the general coordinate of such oscillator. At first, the solution of the corresponding autonomous system is obtained. Let $q(t)$ be this solution. Then this solution is inversed into a form: $t = t(q)$. Using this function, the nonautonomous system is transformed into the autonomous one, which approximates nonautonomous dynamical system. The generalization of the Rausher method has been suggested in the book [9]. Chebyshev polynomials were used to obtain the approximate functions $t = t(q)$ in [10]. The inverse functions $t = t(q)$ were applied to prove the existence of subharmonic vibrations. The existence of a function $t(q)$ for a wide class of dynamical systems was proved in the paper [11]. Rosenberg [11] used the Rausher method for qualitative analysis of one-degree-of-freedom dynamical system. Manevitch, Mikhlin, Pilipchuk [7] suggested the combination of

K.V. Avramov (✉)
Department of Nonstationary Vibrations, A.N. Podgornogo Institute for Problems of Engineering Mechanical NAS of Ukraine, Dm. Pogarski St. 2/10, Kharkov 61046, Ukraine
e-mail: kvavr@kharkov.ua

K.V. Avramov
Department of Gas and Fluid Mechanics, National Technical University "KhPI", Frunze St.21, Kharkov 61002, Ukraine

the Rausher method and the NNMs to analyze discrete systems with arbitrary number of DOF. The forced vibrations close to rectilinear NNMs of 2DOF system were studied by Rausher method in [6]. Nonlinear 2DOF system, describing interaction of the linear subsystem and the snap-through truss has been investigated by Rausher approach [12].

Note, that all above-presented publications considered the Rausher method jointly with the Kauderer–Rosenberg nonlinear modes, which present the motions in the configuration space. In this paper, the Rausher method combines with the nonlinear modes, which are two dimensional invariant manifolds. These normal modes are suggested by Shaw, Pierre [13, 14]. This is the novelty of the method presented in this paper.

Nonlinear modes are effective tools to solve engineering problems. They are used to analyze the problems of absorption of mechanical vibrations in the papers [15, 16]. Nonlinear modes are used to analyze the nonlinear vibrations of rotating pretwisted beams [17].

## 2 Combination of Rausher approach and nonlinear modes

The forced vibrations of a discrete dynamical system are considered in this paper. These systems can be presented in the following form:

$$\ddot{q}_i + \omega_i^2 q_i + R_i(q_1, \dots, q_m) = h_i \cos(\Omega t);$$
$$i = \overrightarrow{1, m}. \tag{1}$$

Note that damping is not taken into account in this paper. The vector-function $R_i(q_1, \dots, q_m)$ describes the nonlinear terms, which contains the second, the third and others degrees of the general coordinates. It is assumed in this paper, that the eigenfrequencies $\omega_i$ do not satisfy the conditions of internal resonances.

Moreover, in this paper, the values of the frequency $\Omega$ are considered close to the eigenfrequency $\omega_l$. Then in the region of the main resonance, the vibrations amplitudes of the general coordinate $q_l$ is higher than the vibrations amplitudes of the rest general coordinates due to the absence of the internal resonances. The iterative procedure is suggested to analyze the forced resonance vibrations in this paper. On the first iteration of this procedure, it is assumed that $q_l \neq 0$ and the rest general coordinates are equal to zero: $q_\mu = 0$; $\mu = \overrightarrow{1, m}$; $\mu \neq l$. Then the following equation follows from the system (1):

$$\ddot{q}_l + \omega_l^2 q_l + \tilde{R}_l(q_l) = h_l \cos(\Omega t), \tag{2}$$

where $\tilde{R}_l(q_l) = R_l(0, \dots, 0, q_l, 0, \dots, 0)$. Following the Rausher approach, the solution is presented as

$$\cos \Omega t = r(q_l) = \alpha_0 + \alpha_1 q_l + \alpha_2 q_l^2 + \cdots. \tag{3}$$

The main idea of the combination of the nonlinear modes and the Rausher approach is the following. If the solution (3) is obtained, this solution is substituted into the system (1). As a result, pseudo-autonomous dynamical system is obtained [6, 12]. Then the nonlinear modes of free vibrations are investigated in the pseudo-autonomous dynamical system [13]. Using the nonlinear modes, the system of $m$ equations (1) is reduced to one equation, which is solved approximately.

Now the solution of (2) is derived in the form (3). The harmonic balance method is used and the solution of the (2) is presented as

$$q_l = A_0 + A_1 \cos \Omega t + A_2 \cos 2\Omega t + A_3 \cos 3\Omega t. \tag{4}$$

Our analysis is limited by three harmonics of the Fourier series. Now the solution (4) is substituted into (2) and the harmonic balance is carried out. As a result, the system of four nonlinear algebraic equations with respect to $(\Omega, A_0, A_1, A_2, A_3)$ is derived. In general, this system can be presented as

$$\Phi_\nu(\Omega, A_0, A_1, A_2, A_3) = 0; \quad \nu = \overrightarrow{1, 4}. \tag{5}$$

The final aim of this paper is calculation of the frequency response in the region of the main resonance $\Omega \approx \omega_l$. Therefore, the values $A_1$ are set to calculate the frequency response with some step. At every value $A_1$, the system of the nonlinear algebraic equations (5) is solved. The parameters $(\Omega, A_0, A_2, A_3)$ are obtained from this system. The coefficients of the expansion (3) $\alpha_0, \alpha_1, \alpha_2, \dots$ are obtained by the values $\Omega, A_0, A_2, A_3$.

Now the coefficients of the expansion (3) are determined. Then the solution (4) is presented in the following form:

$$q_l = A_0 - A_2 + z(A_1 - 3A_3) + z^2 2A_2$$
$$+ z^3 4A_3 + \cdots, \tag{6}$$

where $z = \cos \Omega t$. Now the expansion (3) is substituted into (6) and the coefficients of the same orders of $q_l$ are equated. As a result, the cubic equation with respect to $\alpha_0$ is derived:

$$4A_3\alpha_0^3 + 2A_2\alpha_0^2 + (A_1 - 3A_3)\alpha_0 + A_0 - A_2 = 0. \tag{7}$$

Then the parameters $\alpha_1, \alpha_2, \alpha_3$ are determined in the following way:

$$\begin{aligned}
\alpha_1 &= \frac{1}{A_1 - 3A_3 + 4A_2\alpha_0 + 12A_3\alpha_0^2};\\
\alpha_2 &= -\frac{\alpha_1^2(12A_3\alpha_0 + 2A_2)}{A_1 - 3A_3 + 4A_2\alpha_0 + 12A_3\alpha_0^2};\\
\alpha_3 &= -\frac{4\alpha_1(A_2\alpha_2 + A_3\alpha_1^2 + 6A_3\alpha_0\alpha_2)}{A_1 - 3A_3 + 4A_2\alpha_0 + 12A_3\alpha_0^2}.
\end{aligned} \tag{8}$$

Now the expansion (3) is substituted into the nonautonomous dynamical system (2). As a result, the pseudo-autonomous dynamical system is derived. This system has the following form:

$$\begin{aligned}
&\ddot{q}_i + \omega_i^2 q_i + R_i(q_1, \ldots, q_m)\\
&\quad = h_i\big(\alpha_0 + \alpha_1 q_l + \alpha_2 q_l^2 + \cdots\big); \quad i = \overrightarrow{1, m}.
\end{aligned} \tag{9}$$

The change of the variables is considered to analyze the pseudo-autonomous dynamical system (9):

$$q_i = q_{0,i} + \eta_i(t), \tag{10}$$

where $q_{0,i}$; $i = \overrightarrow{1, m}$ are coordinates of the fixed points of the dynamical system (9). The equilibrium is denoted from the system of $m$ nonlinear algebraic equations:

$$\begin{aligned}
&\omega_i^2 q_{0,i} + R_i(q_{0,1}, \ldots, q_{0,m})\\
&\quad = h_i\big(\alpha_0 + \alpha_1 q_{0,l} + \alpha_2 q_{0,l}^2 + \cdots\big); \quad i = \overrightarrow{1, m}.
\end{aligned} \tag{11}$$

Then the dynamical system (9) with respect to the variables $\eta_1, \ldots, \eta_m$ has the following form:

$$\begin{aligned}
&\ddot{\eta}_i + \sum_{j=1}^{m} \alpha_{ij}\eta_j + F_i(\eta_1, \ldots, \eta_m) = 0;\\
&\quad i = \overrightarrow{1, m},
\end{aligned} \tag{12}$$

where $F_i(\eta_1, \ldots, \eta_m)$ are nonlinear functions.

The matrix $\Lambda = \|\alpha_{ij}\|_{j=\overrightarrow{1,m}}^{i=\overrightarrow{1,m}}$ is presented in the following form:

$$\Lambda = U A U^{-1}, \tag{13}$$

where $U$ is the matrix of eigenvectors of $\Lambda$; $A = \operatorname{diag}(\nu_1^2, \ldots, \nu_m^2)$. Then the new variables are introduced:

$$\xi = U^{-1}\eta, \tag{14}$$

where $\xi = [\xi_1, \ldots, \xi_m]^T$; $\eta = [\eta_1, \ldots, \eta_m]^T$. Then the dynamical system (12) has the following form:

$$\ddot{\xi}_i + \nu_i^2 \xi_i + L_i(\xi_1, \ldots, \xi_m) = 0; \quad i = \overrightarrow{1, m}, \tag{15}$$

where $L_i(\xi_1, \ldots, \xi_m)$; $i = \overrightarrow{1, m}$ are nonlinear part of the dynamical system.

The nonlinear modes [13] are used to analyze the system (15). Then all general coordinates $\xi_i$ and velocities $v_i = \dot{\xi}_i$ are expressed in terms of one general coordinates $\xi_l$ and one velocity $v_l$:

$$\begin{aligned}
&\xi_k = \xi_k(\xi_l, v_l) = F_1^{(1)}\xi_l^2 + F_2^{(1)}\xi_l v_l + F_3^{(1)} v_l^2 + \cdots;\\
&v_k = v_k(\xi_l, v_l)\\
&\quad = F_1^{(2)}\xi_l^2 + F_2^{(2)}\xi_l v_l + F_3^{(2)} v_l^2 + \cdots,\\
&\quad k = \overrightarrow{1, m};\ k \neq l,
\end{aligned} \tag{16}$$

where $F_1^{(1)}, F_2^{(1)}, \ldots$ are unknown coefficients, which will be determined. Equation (16) is differentiated with respect to time and (15) is used. As a result, the following system of partial differential equations is derived:

$$\begin{aligned}
&v_k(\xi_l, v_l)\\
&\quad = \frac{\partial \xi_k}{\partial \xi_l} v_l - \frac{\partial \xi_k}{\partial v_l}\big\{\nu_l^2 \xi_l\\
&\qquad + L_l\big[\xi_1(\xi_l, v_l), \ldots, \xi_l, \ldots, \xi_m(\xi_l, v_l)\big]\big\};\\
&\nu_k^2 \xi_k(\xi_l, v_l)\\
&\qquad + L_k\big[\xi_1(\xi_l, v_l), \ldots, \xi_l, \ldots, \xi_m(\xi_l, v_l)\big]\\
&\quad = -\frac{\partial v_k}{\partial \xi_l} v_l + \frac{\partial v_k}{\partial v_l}\big\{\nu_l^2 \xi_l\\
&\qquad + L_l\big[\xi_1(\xi_l, v_l), \ldots, \xi_l, \ldots, \xi_m(\xi_l, v_l)\big]\big\};\\
&\quad k = \overrightarrow{1, m};\ k \neq l.
\end{aligned} \tag{17}$$

Following the paper [13], (16) are substituted into the system (17) and matching the coefficients at the same summands $\xi_l^{j_1} v_l^{j_2}$; $j_1 = 0, 1, 2, \ldots$; $j_2 = 0, 1, 2, \ldots$ is performed. As a result, the system of linear algebraic equations with respect to $F_1^{(1)}, F_2^{(1)}, \ldots$ is obtained. Solving this system, the functions (16) are determined.

Now the function (16) is presented with respect to the variables $\eta_1, \ldots, \eta_m$, using (14). Then the functions (16) take the following form:

$$\begin{aligned} \eta_k &= \eta_k(\eta_l, s_l) \\ &= C_1^{(1)}\eta_l + C_2^{(1)} s_l + C_3^{(1)}\eta_l^2 + C_4^{(1)}\eta_l s_l \\ &\quad + C_5^{(1)} s_l^2 + \cdots; \\ s_k &= s_k(\eta_l, s_l) \\ &= C_1^{(2)}\eta_l + C_2^{(2)} s_l + C_3^{(2)}\eta_l^2 + C_4^{(2)}\eta_l s_l \\ &\quad + C_5^{(2)} s_l^2 + \cdots, \end{aligned} \tag{18}$$

where $s_k = \dot{\eta}_k$; $C_1^{(1)}, C_2^{(1)}, \ldots$—known parameters.

Equation (18) is substituted into (10) and the obtained functions are substituted into the equation with the number $l$ of the system (1). As a result, one DOF dynamical system is derived. This system can be presented in the form (2). The harmonic balance method is used to analyze this equation. The solutions are presented in the form (4). Then the system of four nonlinear algebraic equations with respect to $\Omega$, $A_0$, $A_2$, $A_3$ is derived. This system is presented in the form (5).

The iterative loop is used to analyze he forced vibrations. One (2) is solved on the first iteration. Note, that the contribution of $q_1, q_2, \ldots, q_{l-1}, q_{l+1}, \ldots, q_m$ is not taken into account on the first iteration. On the first iteration, the nonlinear modes (18) is used to take into account the contribution of all general coordinates. If the values of $\Omega$, $A_0$, $A_2$, $A_3$ on the first iteration are close to the same parameters on the second iteration, the iterative loop is stopped. If these parameters are not enough close, the iterative loop is continued. When the iterative loop is finished, the new value $A_1$ is set and the new point of the frequency response is calculated.

Now the stability of the considered periodic motions is studied. As a result of nonlinear mode analysis, the periodic motions $q_*(t)$ are obtained. The small perturbations from these trajectories $\Delta q_i(t)$; $i = \overrightarrow{1, m}$ are considered. The variables $\Delta q_i(t)$ are described by the following system of linear differential equations:

$$\Delta\ddot{q}_i + \omega_i^2 \Delta q_i + \sum_{\nu=1}^{m} \frac{\partial R_i}{\partial q_\nu} \Delta q_\nu = 0; \quad i = \overrightarrow{1, m}. \tag{19}$$

The system (19) is rewritten in the following form:

$$\Delta\dot{v} = G(t)\Delta v, \tag{20}$$

where $\Delta v = (\Delta q_1, \ldots, \Delta q_m, \Delta\dot{q}_1, \ldots, \Delta\dot{q}_m)$; $G(t) = G(t+T)$; $T = 2\pi\Omega^{-1}$. The dynamical system (20) is analyzed numerically to study stability. The numerical procedure from the book [5] is used. According to this procedure, the fundamental matrix and multipliers are calculated. The stability is analyzed by the values of multipliers.

## 3 Nonlinear vibrations of shallow arches

In this section, the method for forced vibrations analysis, which is treated in Sect. 2, is used to analyze the nonlinear vibrations of the shallow arch (Fig. 1). The nonlinear vibrations of arch are described by the following system of integro-differential equations [18, 19]:

$$\begin{aligned} &A\rho y_{tt} + EI(y - y_0)_{xxxx} - H y_{xx} \\ &\quad = \delta\left(x - \frac{L}{n}\right) F\cos(\Omega t); \\ &H = \frac{EA}{2L}\int_0^L \left(y_x^2 - y_{0,x}^2\right) dx, \end{aligned} \tag{21}$$

where $A$, $I$ are area and the second moment of the cross section; $y(x,t)$ is a flexure of the arch; $E$, $\rho$ are Young's modulus and arch material density; $y_0(x)$ is an initial flexure of the arch; $\delta(x - \frac{L}{n})$ is delta function; $n$ is arbitrary value; $F\cos(\Omega t)$ is concentrated force.

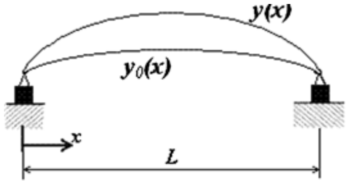


**Fig. 1** The shallow arch

The following dimensionless parameters and variables are considered:

$$x^* = \frac{x}{L}; \qquad y^* = \frac{y}{r}; \qquad t^* = t\sqrt{\frac{EI}{\rho A l^4}};$$
$$y_0 = r y_0^*; \qquad \delta^*\left(x^* - \frac{1}{n}\right) = L\delta\left(x - \frac{L}{n}\right); \tag{22}$$
$$f = \frac{FL^3}{EIr},$$

where $r$ is gyration radius of a cross section. Dropping the asterisk in the notation, (21) with respect dimensionless variables and parameters is

$$y_{tt} + (y - y_0)_{xxxx} - \frac{y_{xx}}{2}\int_0^1 \left(y_x^2 - y_{0,x}^2\right) dx$$
$$= \delta\left(x - \frac{1}{n}\right) f \cos(\Omega t). \tag{23}$$

The initial flexure of the arch has the following form:

$$y_0 = \lambda_1 \sin(\pi x) + \lambda_2 \sin(2\pi x). \tag{24}$$

Then the arch vibrations are presented as

$$y(x,t) = \left(\lambda_1 + q_1(t)\right)\sin(\pi x)$$
$$+ \left(\lambda_2 + q_2(t)\right)\sin(2\pi x). \tag{25}$$

In future analysis, it is assumed that $n = 2$. Equations (24, 25) are substituted into (23) and the Bubnov–Galerkin procedure is carried out. As a result, the following dynamical system is derived:

$$\ddot{q}_1 + \pi^4 q_1 + \frac{\pi^4}{4}(q_1 + \lambda_1)\mu(q_1, q_2) = 2f\cos(\Omega t);$$
$$\ddot{q}_2 + 16\pi^4 q_2 + \pi^4(q_2 + \lambda_2)\mu(q_1, q_2) = 0; \tag{26}$$
$$\mu(q_1, q_2) = q_1(q_1 + 2\lambda_1) + 4q_2(q_2 + 2\lambda_2).$$

Normal coordinates of linear part of the system (26) are considered. These coordinates are determined from the equation:

$$\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} r_1 & r_2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \tag{27}$$

where

$$r_j = \frac{p_j^2 - 8\lambda_2^2 - 16}{2\lambda_1\lambda_2}; \quad j = 1, 2;$$

$$2p_{1,2}^2 = 17 + 0.5\lambda_1^2 + 8\lambda_2^2$$
$$\mp \left[225 - 15\lambda_1^2 + 240\lambda_2^2 + 0.25\lambda_1^4\right.$$
$$\left. + 64\lambda_2^4 + 8\lambda_1^2\lambda_2^2\right]^{1/2}.$$

Then the dynamical system (26) with respect to the variables $(x_1, x_2)$ takes the following form:

$$\ddot{x}_i + p_i^2 x_i$$
$$= \sigma\left[\theta_i f_1 \cos(\Omega_1 t) + A_1^{(i)} x_1^2 + A_2^{(i)} x_2^2 + A_3^{(i)} x_1 x_2\right.$$
$$\left. + A_4^{(i)} x_1^3 + A_5^{(i)} x_1^2 x_2 + A_6^{(i)} x_1 x_2^2 + A_7^{(i)} x_2^3\right];$$
$$i = 1, 2, \tag{28}$$

where $\theta_1 = 1$; $\theta_2 = -1$; $\sigma = \frac{2\lambda_1\lambda_2}{p_1^2 - p_2^2}$; $f_1 = \frac{2f}{\pi^4}$. The parameters $A_1^{(1)}, A_2^{(1)}, \ldots, A_7^{(2)}$ are presented in Appendix 1.

The forced vibrations in the region of the main resonance are considered. In this case, the frequency of the excitation $\Omega$ is close to $p_1$. At the first iteration, it is assumed that $x_2 = 0$. Then the following system is derived:

$$\ddot{x}_1 + p_1^2 x_1 = \sigma\left[f_1 \cos(\Omega_1 t) + A_1^{(1)} x_1^2 + A_4^{(1)} x_1^3\right]. \tag{29}$$

The solutions of this equation have the form (4). Then, using harmonic balance method, the system of nonlinear algebraic equations (5) is obtained. The functions $\Phi_\nu(\nu = \overline{1,4})$ for this case are presented in Appendix 2. If the system of the nonlinear algebraic equations (5) is solved, (29) solutions in the form (3) can be obtained. This solution is substituted into (26) and the following pseudo-autonomous dynamical system is derived:

$$\ddot{x}_i + p_i^2 x_i$$
$$= \sigma\left[\theta_i f_1 r(x_1) + A_1^{(i)} x_1^2 + A_2^{(i)} x_2^2 + A_3^{(i)} x_1 x_2\right.$$
$$\left. + A_4^{(i)} x_1^3 + A_5^{(i)} x_1^2 x_2 + A_6^{(i)} x_1 x_2^2 + A_7^{(i)} x_2^3\right];$$
$$i = 1, 2. \tag{30}$$

For future analysis, the change of the variables (10) is used. For the system (30), the equations (10) have the form:

$$x_i = x_{0,i} + \eta_i(t); \quad i = 1, 2. \tag{31}$$

Then parameters $x_{0,1}$; $x_{0,2}$ are determined from the system of the nonlinear algebraic equations, which is similar to (11). Now the system (30) is presented with respect to the variables $(\eta_1, \eta_2)$:

$$\begin{aligned}
&\ddot{\eta}_i + p_i^2 \eta_i \\
&\quad = \sigma\big[c_{i1}\eta_1 + c_{i2}\eta_2 + P_i^{(2)}(\eta_1,\eta_2) + P_i^{(3)}(\eta_1,\eta_2)\big]; \\
&P_i^{(2)}(\eta_1,\eta_2) = B_1^{(i)}\eta_1^2 + B_2^{(i)}\eta_2^2 + B_3^{(i)}\eta_2\eta_1; \\
&P_i^{(3)}(\eta_1,\eta_2) \\
&\quad = B_4^{(i)}\eta_1^3 + B_5^{(i)}\eta_1^2\eta_2 + B_6^{(i)}\eta_2^2\eta_1 + B_7^{(i)}\eta_2^3; \\
&\quad i = \overline{1,2}.
\end{aligned} \tag{32}$$

Appendix 3 contains the parameters of the system (32). The linear part of the system (32) is rewritten with respect to the normal coordinates, using the equation:

$$\begin{aligned}
&\begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} = \begin{bmatrix} \varsigma_1 & \varsigma_2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix}; \\
&\varsigma_i = \frac{\sigma c_{12}}{p_1^2 - \nu_i^2 - \sigma c_{11}}; \quad i = 1, 2.
\end{aligned} \tag{33}$$

The frequencies $\nu_1$ and $\nu_2$ are determined as roots of biquadrate equation:

$$\nu^4 + \nu^2 R_0 + R_1 = 0, \tag{34}$$

where

$$\begin{aligned}
R_0 &= \sigma(c_{22} + c_{11}) - p_2^2 - p_1^2; \\
R_1 &= p_1^2 p_2^2 - \sigma c_{22} p_1^2 - \sigma c_{11} p_2^2 \\
&\quad + \sigma^2(c_{11}c_{22} - c_{12}c_{21}).
\end{aligned}$$

The system (32) with respect to the variables $(\xi_1, \xi_2)$ takes the following form:

$$\begin{aligned}
&\ddot{\xi}_i + \nu_i^2 \xi_i = \alpha\big[R_i^{(2)}(\xi_1,\xi_2) + R_i^{(3)}(\xi_1,\xi_2)\big]; \\
&R_i^{(2)} = D_1^{(i)}\xi_1^2 + D_2^{(i)}\xi_2^2 + D_3^{(i)}\xi_1\xi_2; \\
&R_i^{(3)} = D_4^{(i)}\xi_1^3 + D_5^{(i)}\xi_1^2\xi_2 + D_6^{(i)}\xi_1\xi_2^2 + D_7^{(i)}\xi_2^3.
\end{aligned} \tag{35}$$

Appendix 4 contains the parameters $\alpha, D_1^{(1)}, \ldots, D_7^{(2)}$.

The nonlinear modes (invariant manifolds) are presented in the form (16) at $k = 2$; $l = 1$. Using the approach from Sect. 2, the coefficients $F_1^{(1)}, F_2^{(1)}, \ldots, F_3^{(2)}$ are obtained in the following form:

$$\begin{aligned}
F_1^{(1)} &= \frac{\alpha D_1^{(2)}(2\nu_1^2 - \nu_2^2)}{\nu_2^2(4\nu_1^2 - \nu_2^2)}; \\
F_2^{(2)} &= -\frac{2\alpha D_1^{(2)}}{4\nu_1^2 - \nu_2^2}; \\
F_3^{(1)} &= \frac{2\alpha D_1^{(2)}}{\nu_2^2(4\nu_1^2 - \nu_2^2)}; \\
F_2^{(1)} &= F_3^{(2)} = F_1^{(2)} = 0.
\end{aligned} \tag{36}$$

Following Sect. 2, the invariant manifold is rewritten in the form (18) with respect to the variables $(\eta_1, \eta_2, s_1, s_2)$:

$$\begin{aligned}
\eta_2 &= \varsigma_1^{-1}\eta_1 + \delta_*\big(F_1^{(1)}\eta_1^2 + F_3^{(1)}s_1^2\big) + \cdots; \\
s_2 &= \varsigma_1^{-1}s_1 + F_2^{(2)}\delta_*\eta_1 s_1 + \cdots,
\end{aligned} \tag{37}$$

where $\delta_* = \frac{(1-\varsigma_2\varsigma_1^{-1})^2}{\varsigma_1(\varsigma_1-\varsigma_2)}$.

The motions on the normal mode (37) are described by one nonlinear ordinary differential equation:

$$\begin{aligned}
&\ddot{x}_1 + p_1^2 x_1 \\
&\quad = \sigma\big[f_1\cos\Omega_1 t + G_0^{(2)} + G_1^{(2)}x_1 + G_2^{(2)}x_1^2 \\
&\qquad + G_3^{(2)}\dot{x}_1^2 + G_4^{(2)}x_1^3 + G_5^{(2)}x_1\dot{x}_1^2\big].
\end{aligned} \tag{38}$$

The parameters of this equation $G_0^{(2)}; G_1^{(2)}; \ldots$ are presented in Appendix 5.

The periodic motions in the system (38) are presented in the form (4). Then parameters $(\Omega, A_0, A_2, A_3)$ are determined from the system of nonlinear algebraic equations. This system has the following form:

$$\tilde{\Phi}_\nu(\Omega, A_0, A_2, A_3) = 0; \quad \nu = \overline{1,4}. \tag{39}$$

The functions $\tilde{\Phi}_\nu$; $\nu = \overline{1,4}$ are presented in Appendix 6.

Now the stability of periodic motions is considered. The equations, which describe small perturbations $(\Delta q_1, \Delta q_2)$ close to the periodic motions of the dynamical system (26), have the following form:

$$\begin{aligned} &\Delta\ddot{q}_1+\pi^4\Delta q_1\\ &\quad+\frac{\pi^4}{4}\big[a_{11}(t)\Delta q_1+a_{12}(t)\Delta q_2\big]=0;\\ &\Delta\ddot{q}_2+16\pi^4\Delta q_2\\ &\quad+\pi^4\big[a_{21}(t)\Delta q_1+a_{22}(t)\Delta q_2\big]=0, \end{aligned} \tag{40}$$

where

$$\begin{aligned} a_{11}(t)&=3q_1^2+6\lambda_1 q_1+2\lambda_1^2+4q_2(q_2+2\lambda_2);\\ a_{12}(t)&=8(q_1+\lambda_1)(q_2+\lambda_2);\\ a_{21}(t)&=2(q_1+\lambda_1)(q_2+\lambda_2);\\ a_{22}(t)&=q_1(q_1+2\lambda_1)+4q_2(q_2+2\lambda_2)\\ &\quad+8(q_2+\lambda_2)^2. \end{aligned} \tag{41}$$

Using the formulas (27, 31, 33, 37), the equations with respect to the variables $q_1$, $q_2$ and the solutions of (38) are derived in the following form:

$$\begin{aligned} q_1&=r_1x_1+r_2\big[x_{0,2}+\varsigma_1^{-1}(x_1-x_{0,1})\\ &\quad+\delta_*\big(F_1^{(1)}(x_1-x_{0,1})^2+F_3^{(1)}\dot{x}_1^2\big)\big];\\ q_2&=x_1+x_{0,2}+\varsigma_1^{-1}(x_1-x_{0,1})\\ &\quad+\delta_*\big(F_1^{(1)}(x_1-x_{0,1})^2+F_3^{(1)}\dot{x}_1^2\big). \end{aligned}$$

The approximate solution of (2) has the form (4). In stability analysis, this solution is limited by two summands: $x_1=A_0+A_1\cos\Omega t$. Then the functions (42) have the following form:

$$a_{\nu\mu}=a_{\nu\mu}^{(0)}+a_{\nu\mu}^{(1)}\cos(\Omega t);\qquad \nu=1,2;\ \mu=1,2. \tag{42}$$

The parameters $a_{\nu\mu}^{(0)}$, $a_{\nu\mu}^{(1)}$ $(\nu=1,2;\ \mu=1,2)$ are give in Appendix 7. The results of numerical calculations of stability are given in the next section.

## 4 Numerical analysis of vibrations

Now the numerical analysis of shallow arch forced vibrations (21) is carried out. The following numerical values of the parameters are taken: $\lambda_1=5$; $\lambda_2=0.8$; $f=0.05$. The vibrations with moderate amplitudes close to $q_1=q_2=0$ are considered. The snap-through motions of arch are not studied in this paper. The eigenfrequencies of linear vibrations are the following:

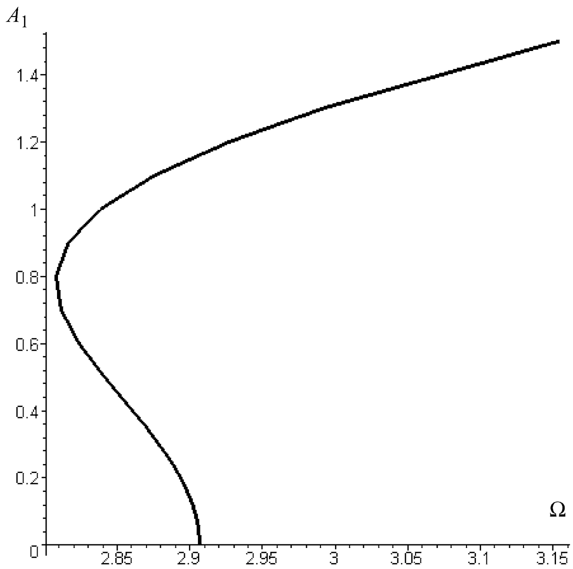


**Fig. 2** The backbone curve

$$p_1=2.9067;\qquad p_2=5.1158. \tag{43}$$

The free nonlinear vibrations are studied now. The nonlinear modes [13] are used for these calculations. Figure 2 shows the backbone curve of the free vibrations. At small amplitudes of free vibrations, the backbone curve is soft and at big amplitudes, the backbone curve is hard.

The coordinates $q_1$ and $q_2$ of (26) are not normal modes of linear part of the problem. Therefore, the numbers $\pi^2$ and $4\pi^2$ from (26) are not the eigenfrequency. The first eigenfrequency is determined by (43) and the backbone curve (Fig. 2) is close to this frequency.

The procedure from Sect. 3 is applied to calculate forced vibrations. The first harmonic of the Fourier series $A_1$ of forced vibrations versus the frequency of excitation $\Omega_1$ is shown on Fig. 4 by solid lines and the backbone curve is shown by dotted line on the same figure. Note, the right branch of the frequency response with respect to the backbone curve has $A_1>0$ and the left branch of the frequency response has $A_1<0$. Figure 3 shows the modulus of the amplitude $A_1$ versus $\Omega$. The solid line shows the stable solutions and the dotted line—unstable one. As an example, Fig. 4 shows the surface of nonlinear mode at $A_1=-1.2$.

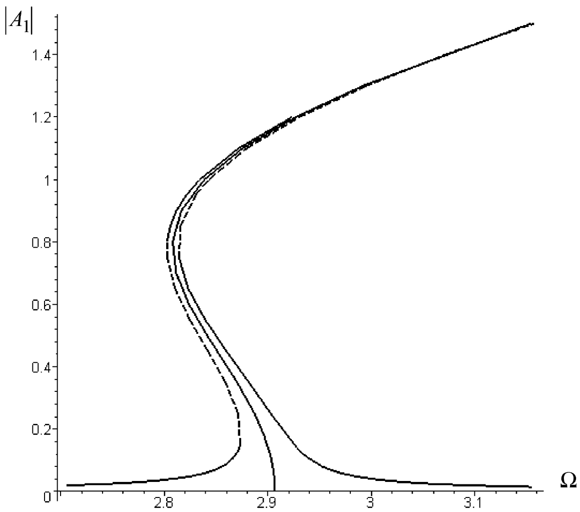


**Fig. 3** The frequency response and backbone curves

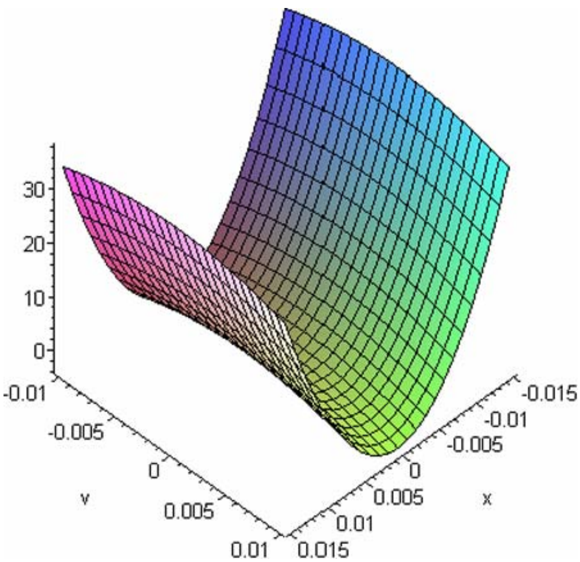


**Fig. 4** Nonlinear mode of forced vibrations

## 5 Conclusions

The iterative method for forced vibrations analysis in nonlinear system is suggested in this paper. The basis of this procedure is Rausher method and nonlinear modes. Due to the use of the Rausher method, the nonautonomous dynamical systems are replaced by iterative process. On each step of this process, the pseudo-autonomous dynamical system is solved. Nonlinear modes are used to analyze the dynamics of the pseudo-autonomous system.

Using of Rausher method suggesting a closeness of forced vibration to nonlinear normal mode is justified if one deals with strongly nonlinear vibrations.

The suggested method can be used for the systems without internal resonances. The generalization of this method for the systems with internal resonances will be treated in the future papers of the author. Moreover, this method can be used for the system with multi-harmonic force. In this case, the coefficients $\alpha_0, \alpha_1, \alpha_2, \ldots$ of the expansion (3) are determined by more complex formulas. This method can be generalized for the discrete systems with dissipation. This will be done in future publications of the author.

As an example, this method is applied to calculate the forced vibrations of the shallow arch.

The nonlinear modes are very effective to analyze the dynamical system with many degrees-of-freedom [14, 17]. It is possible to apply nonlinear modes to the high dimension models, which are obtained as a result of finite element discretization of elastic medium. Using the method suggested in this paper, it is possible to study forced vibrations of high dimension nonlinear systems, too. Moreover, this method is effective to analyze the models with finite degree-of-freedom, which are obtained as a result of finite element discretization of nonlinear medium.

## Appendix 1: Parameters of discrete model of shallow arch

$$A_1^{(1)} = a_1 r_1^2 + a_2 + a_3 r_1;$$

$$A_2^{(1)} = a_1 r_2^2 + a_2 + a_3 r_2;$$

$$A_3^{(1)} = 2a_1 r_1 r_2 + 2a_2 + a_3(r_1 + r_2);$$

$$A_4^{(1)} = \left(4 + r_1^2\right)(r_2 - 0.25 r_1);$$

$$A_5^{(1)} = 0.25 r_1^2 r_2 - 2r_1 + 2r_1 r_2^2 + 11 r_2;$$

$$A_6^{(1)} = \frac{5}{4} r_1 r_2^2 - r_1 + r_2^3 + 10 r_2;$$

$$A_7^{(1)} = 0.75 r_2^3 + 3 r_2;$$

$$A_1^{(2)} = b_1 r_1^2 + b_2 + b_3 r_1; \; A_2^{(2)} = b_1 r_2^2 + b_2 + b_3 r_2;$$

$$A_3^{(2)} = 2b_1 r_1 r_2 + 2b_2 + b_3(r_1 + r_2);$$

$$A_4^{(2)} = -0.75 r_1^3 - 3 r_1;$$

$$A_5^{(2)} = -\frac{5}{4} r_1^2 r_2 + r_2 - r_1^3 - 10 r_1;$$

$$A_6^{(2)} = -\frac{1}{4}r_1 r_2^2 + 2r_2 - 2r_1^2 r_2 - 11r_1;$$

$$A_7^{(2)} = \left(0.25r_2^2 + 1\right)(r_2 - 4r_1),$$

where

$$a_1 = r_2\lambda_2 - 0.75\lambda_1; \qquad a_2 = 12r_2\lambda_2 - \lambda_1;$$

$$a_3 = 2r_2\lambda_1 - 2\lambda_2;$$

$$b_1 = -r_1\lambda_2 + 0.75\lambda_1; \qquad b_2 = -12r_1\lambda_2 + \lambda_1;$$

$$b_3 = -2r_1\lambda_1 + 2\lambda_2.$$

## Appendix 2: Nonlinear algebraic equation of harmonic balance method

$$\begin{aligned}
\Phi_1 = {} & p_1^2 A_0 - \sigma A_1^{(1)}\left(A_0^2 + 0.5\left[A_1^2 + A_2^2 + A_3^2\right]\right) \\
& - \sigma A_4^{(1)}\left[0.75A_1^2 A_2 \right. \\
& \left. + 1.5\left(A_0 A_3^2 + A_0 A_1^2 + A_0 A_2^2 + A_1 A_2 A_3\right)\right];
\end{aligned}$$

$$\begin{aligned}
\Phi_2 = {} & A_1\left(p_1^2 - \Omega_1^2\right) \\
& - \sigma A_1^{(1)}(2A_0 A_1 + A_1 A_2 + A_2 A_3) \\
& - \sigma A_4^{(1)}\left[3\left(A_0^2 A_1 + A_0 A_1 A_2 + A_0 A_2 A_3\right) \right. \\
& + 1.5A_1\left(A_2^2 + A_3^2\right) \\
& \left. + 0.75\left(A_1^2 A_3 + A_2^2 A_3 + A_1^3\right)\right];
\end{aligned}$$

$$\begin{aligned}
\Phi_3 = {} & A_2\left(p_1^2 - 4\Omega_1^2\right) \\
& - \sigma A_1^{(1)}\left(2A_0 A_2 + 0.5A_1^2 + A_1 A_3\right) \\
& - \sigma A_4^{(1)}\left[1.5\left(A_2 A_3^2 + A_0 A_1^2 \right.\right. \\
& \left. + A_1^2 A_2 + A_1 A_2 A_3\right) \\
& \left. + 3\left(A_0^2 A_2 + A_0 A_1 A_3\right) + 0.75A_2^3\right];
\end{aligned}$$

$$\begin{aligned}
\Phi_4 = {} & \left(p_1^2 - 9\Omega_1^2\right)A_3 - \sigma A_1^{(1)}(2A_0 A_3 + A_1 A_2) \\
& - \sigma A_4^{(1)}\left[0.75\left(A_1 A_2^2 + A_3^3\right) \right. \\
& + 3\left(A_0^2 A_3 + A_0 A_1 A_2\right) \\
& \left. + 1.5\left(A_2^2 + A_1^2\right)A_3 + 0.25A_1^3\right].
\end{aligned}$$

## Appendix 3: Parameters of the system (32)

The dynamical system (32) has the following parameters:

$$B_1^{(1)} = f_1(\alpha_2 + 3\alpha_3 x_{0,1}) + A_1^{(1)} + \bar{\lambda}_{11};$$

$$B_2^{(1)} = A_2^{(1)} + \bar{\lambda}_{13}; \qquad B_3^{(1)} = A_3^{(1)} + \bar{\lambda}_{12};$$

$$B_4^{(1)} = A_4^{(1)} + f_1\alpha_3; \qquad B_5^{(1)} = A_5^{(1)};$$

$$B_6^{(1)} = A_6^{(1)}; \qquad B_7^{(1)} = A_7^{(1)};$$

$$B_1^{(2)} = -f_1\alpha_2 - 3f_1\alpha_3 x_{0,1} + A_1^{(2)} + \lambda_{21};$$

$$B_2^{(2)} = A_2^{(2)} + \bar{\lambda}_{23}; \qquad B_3^{(2)} = A_3^{(2)} + \bar{\lambda}_{22};$$

$$B_4^{(2)} = A_4^{(2)} - f_1\alpha_3; \qquad B_5^{(2)} = A_5^{(2)};$$

$$B_6^{(2)} = A_6^{(2)}; \qquad B_7^{(2)} = A_7^{(2)};$$

$$\bar{\lambda}_{i1} = 3A_4^{(i)} x_{0,1} + A_5^{(i)} x_{0,2};$$

$$\bar{\lambda}_{i2} = 2A_5^{(i)} x_{0,1} + 2A_6^{(i)} x_{0,2};$$

$$\bar{\lambda}_{i3} = A_6^{(i)} x_{0,1} + 3A_7^{(i)} x_{0,2};$$

$$c_{11} = f_1\alpha_1 + 2f_1\alpha_2 x_{0,1} + 3f_1\alpha_3 x_{0,1}^2 + a_{11} + b_{11};$$

$$c_{12} = a_{12} + b_{12};$$

$$c_{21} = -f_1\left(\alpha_1 + 2\alpha_2 x_{0,1} + 3\alpha_3 x_{0,1}^2\right) + a_{21} + b_{21};$$

$$c_{22} = a_{22} + b_{22};$$

$$b_{i1} = 3A_4^{(i)} x_{0,1}^2 + 2A_5^{(i)} x_{0,1} x_{0,2} + A_6^{(i)} x_{0,2}^2;$$

$$b_{i2} = A_5^{(i)} x_{0,1}^2 + 2A_6^{(i)} x_{0,1} x_{0,2} + 3A_7^{(i)} x_{0,2}^2;$$

$$a_{i1} = 2A_1^{(i)} x_{0,1} + A_3^{(i)} x_{0,2};$$

$$a_{i2} = 2A_2^{(i)} x_{0,2} + A_3^{(i)} x_{0,1}; \quad i = 1, 2.$$

## Appendix 4: Parameters of the system (35)

The parameters of the system (35) are determined as

$$D_\mu^{(1)} = c_\mu^{(1)} - \varsigma_2 c_\mu^{(2)};$$

$$D_\mu^{(2)} = -c_1^{(1)} + \varsigma_1 c_1^{(2)}; \quad \mu = \overrightarrow{1,7}.$$

$$c_4^{(i)} = B_4^{(i)}\varsigma_1^3 + B_5^{(i)}\varsigma_1^2 + B_6^{(i)}\varsigma_1 + B_7^{(i)}; \quad i = 1, 2;$$

$$\begin{aligned}
c_5^{(i)} = {} & 3\varsigma_1^2\varsigma_2 B_4^{(i)} + B_5^{(i)}\left(2\varsigma_1\varsigma_2 + \varsigma_1^2\right) \\
& + B_6^{(i)}(2\varsigma_1 + \varsigma_2) + 3B_7^{(i)};
\end{aligned}$$

$$\begin{aligned}
c_6^{(i)} = {} & 3B_4^{(i)}\varsigma_1\varsigma_2^2 + B_5^{(i)}\left(\varsigma_2^2 + 2\varsigma_1\varsigma_2\right) \\
& + B_6^{(i)}(\varsigma_1 + 2\varsigma_2) + 3B_7^{(i)};
\end{aligned}$$

$$c_7^{(i)} = B_4^{(i)}\varsigma_2^3 + B_5^{(i)}\varsigma_2^2 + B_6^{(i)}\varsigma_2 + B_7^{(i)};$$

$$c_1^{(i)} = B_1^{(i)}\varsigma_1^2 + B_2^{(i)} + B_3^{(i)}\varsigma_1;$$
$$c_2^{(i)} = B_1^{(i)}\varsigma_2^2 + B_2^{(i)} + B_3^{(i)}\varsigma_2;$$
$$c_3^{(i)} = 2B_1^{(i)}\varsigma_1\varsigma_2 + 2B_2^{(i)} + B_3^{(i)}(\varsigma_1 + \varsigma_2);$$
$$\alpha = \frac{\sigma}{\varsigma_1 - \varsigma_2}.$$

## Appendix 5: The parameters of the system (38)

The parameters of the system (38) are the following:

$$G_0^{(2)} = A_7^{(1)}G_0^{(1)3} + A_2^{(1)}G_0^{(1)2};$$
$$G_1^{(2)} = A_6^{(1)}G_0^{(1)2} + 3A_7^{(1)}G_0^{(1)2}G_1^{(1)} + 2A_2^{(1)}G_0^{(1)}G_1^{(1)} + A_3^{(1)}G_0^{(1)};$$
$$G_2^{(2)} = A_5^{(1)}G_0^{(1)} + 2A_6^{(1)}G_0^{(1)}G_1^{(1)} + A_7^{(1)}\big[G_0^{(1)}\big(2G_0^{(1)}G_2^{(1)} + G_1^{(1)2}\big) + G_2^{(1)}G_0^{(1)2} + 2G_1^{(1)2}G_0^{(1)}\big] + A_1^{(1)} + A_2^{(1)}G_1^{(1)2} + 2A_2^{(1)}G_0^{(1)}G_2^{(1)} + A_3^{(1)}G_1^{(1)};$$
$$G_3^{(2)} = 3A_7^{(1)}G_0^{(1)2}G_3^{(1)} + 2A_2^{(1)}G_0^{(1)}G_3^{(1)};$$
$$G_4^{(2)} = A_4^{(1)} + A_5^{(1)}G_1^{(1)} + A_6^{(1)}G_1^{(1)2} + 2A_6^{(1)}G_0^{(1)}G_2^{(1)} + A_7^{(1)}\big[4G_0^{(1)}G_1^{(1)}G_2^{(1)} + G_1^{(1)}\big(2G_0^{(1)}G_2^{(1)} + G_1^{(1)2}\big)\big] + 2A_2^{(1)}G_1^{(1)}G_2^{(1)} + A_3^{(1)}G_2^{(1)};$$
$$G_5^{(2)} = 2A_6^{(1)}G_0^{(1)}G_3^{(1)} + 6A_7^{(1)}G_0^{(1)}G_3^{(1)}G_1^{(1)} + 2A_2^{(1)}G_1^{(1)}G_3^{(1)} + A_3^{(1)}G_3^{(1)};$$
$$G_0^{(1)} = x_{0,2} - \varsigma_1^{-1}x_{0,1} + \delta_* F_1^{(1)}x_{0,1}^2;$$
$$G_1^{(1)} = \varsigma_1^{-1} - 2x_{0,1}F_1^{(1)}\delta_*;$$
$$G_2^{(1)} = \delta_* F_1^{(1)};\ G_3^{(1)} = \delta_* F_3^{(1)}.$$

## Appendix 6: The functions $\tilde{\Phi}_\nu$

$$\begin{aligned}
&\tilde{\Phi}_1(\Omega, A_0, A_1, A_2, A_3)\\
&= G_0^{(2)} + G_1^{(2)}A_0 + G_2^{(2)}\big[A_0^2 + 0.5\big(A_1^2 + A_2^2 + A_3^2\big)\big]\\
&\quad + G_3^{(2)}\left(\frac{\Omega^2}{2}A_1^2 + 2\Omega^2 A_2^2\right)\\
&\quad + G_4^{(2)}\left(\frac{3}{4}A_1^2A_2 + \frac{3}{2}A_0A_3^2 + \frac{3}{2}A_0A_1^2 + \frac{3}{2}A_0A_2^2 + A_0^3 + \frac{3}{2}A_1A_2A_3\right)\\
&\quad + G_5^{(2)}\left(\frac{7}{2}\Omega^2A_1A_2A_3 + \frac{3}{4}\Omega^2A_1^2A_2 + \frac{\Omega^2}{2}A_0A_1^2 + 2\Omega^2A_0A_2^2 + \frac{9}{2}\Omega^2A_0A_3^2\right)\\
&\quad - \frac{p_1^2}{\sigma}A_0;
\end{aligned}$$

$$\begin{aligned}
&\tilde{\Phi}_2(\Omega, A_0, A_1, A_2, A_3)\\
&= f_1 + G_1^{(2)}A_1 + G_2^{(2)}(2A_0A_1 + A_1A_2 + A_2A_3)\\
&\quad + G_3^{(2)}\big(2\Omega^2A_1A_2 + 6\Omega^2A_2A_3\big)\\
&\quad + G_4^{(2)}\left(3A_0^2A_1 + \frac{3}{4}A_1^2A_3 + \frac{3}{2}A_1A_2^2 + \frac{3}{2}A_1A_3^2 + \frac{3}{4}A_2^2A_3 + \frac{3}{4}A_1^3 + 3A_0A_2A_3 + 3A_0A_1A_2\right)\\
&\quad - \frac{A_1}{\sigma}\big(p_1^2 - \Omega^2\big)\\
&\quad + G_5^{(2)}\left(\frac{9}{2}\Omega^2A_1A_3^2 + \frac{5}{4}\Omega^2A_1^2A_3 + \frac{1}{4}\Omega^2A_1^3 + 2\Omega^2A_0A_1A_2 + 6\Omega^2A_0A_2A_3 + 2\Omega^2A_2^2A_3 + 2\Omega^2A_1A_2^2\right);
\end{aligned}$$

$$\begin{aligned}
&\tilde{\Phi}_3(\Omega, A_0, A_1, A_2, A_3)\\
&= G_1^{(2)}A_2 + G_2^{(2)}\big(2A_0A_2 + 0.5A_1^2 + A_1A_3\big)\\
&\quad + G_3^{(2)}\big(3\Omega^2A_1A_3 - 0.5\Omega^2A_1^2\big)\\
&\quad + G_4^{(2)}\big(1.5A_2A_3^2 + 1.5A_0A_1^2 + 1.5A_1^2A_2 + 3A_0^2A_2 + 1.5A_1A_2A_3 + 3A_0A_1A_3 + 0.75A_2^3\big)\\
&\quad + G_5^{(2)}\big(0.5\Omega^2A_2A_1^2 + 2.5\Omega^2A_1A_2A_3 + 4.5\Omega^2A_2A_3^2 - 0.5\Omega^2A_0A_1^2 + \Omega^2A_2^3 + 3\Omega^2A_0A_1A_3\big) - \frac{A_2}{\sigma}\left(p_1^2 - 4\Omega^2\right);
\end{aligned}$$

$$\begin{aligned}
&\tilde{\Phi}_4(\Omega, A_0, A_1, A_2, A_3)\\
&= G_1^{(2)}A_3 + G_2^{(2)}\big(2A_0A_3 + A_1A_2\big)\\
&\quad - G_3^{(2)}2\Omega^2A_1A_2 + G_4^{(2)}\big(0.75A_1A_2^2 + 3A_0^2A_3
\end{aligned}$$

$$+1.5A_2^2A_3+1.5A_1^2A_3+0.25A_1^3$$
$$+3A_0A_1A_2+0.75A_3^3\big)$$
$$+G_5^{(2)}\big(0.5\Omega^2A_3A_1^2+2\Omega^2A_3A_2^2+2.25\Omega^2A_3^3$$
$$-0.25\Omega^2A_1^3-2\Omega^2A_0A_1A_2\big)$$
$$-\frac{A_3}{\sigma}\big(p_1^2-9\Omega^2\big).$$

## Appendix 7: The parameters of the system (40)

$$a_{11}^{(0)}=3B_0^2+6\lambda_1B_0+2\lambda_1^2+4(D_0^2+0.5D_1^2)$$
$$+8\lambda_2D_0+\frac{3}{2}B_1^2;$$
$$a_{11}^{(1)}=6B_0B_1+6\lambda_1B_1+8D_0D_1+8\lambda_2D_1;$$
$$a_{12}^{(0)}=8(D_0+\lambda_2)(B_0+\lambda_1)+4D_1B_1;$$
$$a_{12}^{(1)}=8B_1(D_0+\lambda_2)+4D_1(B_0+\lambda_1);$$
$$a_{21}^{(0)}=0.25a_{12}^{(0)};\qquad a_{21}^{(1)}=0.25a_{12}^{(1)};$$
$$a_{22}^{(0)}=B_0(B_0+2\lambda_1)+0.5B_1^2+4D_0(D_0+2\lambda_2)$$
$$+6D_1^2+8(D_0+\lambda_2)^2;$$
$$a_{22}^{(1)}=2B_0B_1+2\lambda_1B_1+4(2D_1D_0+2\lambda_2D_1)$$
$$+16D_1(D_0+\lambda_2);$$
$$B_0=\alpha_1+\alpha_2A_0+\alpha_3A_0^2+0.5\alpha_3A_1^2+0.5\alpha_4A_1^2\Omega^2;$$
$$B_1=\alpha_2A_1+2\alpha_3A_0A_1;$$
$$D_0=\beta_1+\beta_2A_0+\beta_3A_0^2+0.5\beta_3A_1^2+0.5\beta_4A_1^2\Omega^2;$$
$$D_1=\beta_2A_1+2\beta_3A_0A_1;$$
$$\alpha_1=r_2x_{0,2}-r_2\varsigma_1^{-1}x_{0,1}+r_2\delta_*F_1^{(1)}x_{0,1}^2;$$
$$\alpha_2=r_1+r_2\varsigma_1^{-1}-2x_{0,1}r_2\delta_*F_1^{(1)};$$
$$\alpha_3=r_2\delta_*F_1^{(1)};$$
$$\alpha_4=r_2\delta_*F_3^{(1)};$$
$$\beta_1=x_{0,2}-\varsigma_1^{-1}x_{0,1}+\delta_*F_1^{(1)}x_{0,1}^2;$$
$$\beta_2=1+\varsigma_1^{-1}-2x_{0,1}\delta_*F_1^{(1)};$$
$$\beta_3=\delta_*F_1^{(1)};$$
$$\beta_4=\delta_*F_3^{(1)}.$$

## References


1. Nayfeh, A.H., Mook, D.T.: Nonlinear Oscillations. Wiley, New York (1979)
2. Nayfeh, A.H.: Introduction to Perturbation Techniques. Wiley, New York (1981)
3. Wiggins, S.: Introduction to Applied Nonlinear Dynamical Systems and Chaos. Springer, Berlin (1990)
4. Sanders, J.A., Verhulst, F.: Averaging Methods in Nonlinear Dynamical Systems. Springer, Berlin (1985)
5. Parker, T.S., Chua, L.O.: Practical Numerical Algorithms for Chaotic Systems. Springer, New York (1989)
6. Vakakis, A.F., Manevitch, L.I., Mikhlin, Yu.V., Pilipchuk, V.N., Zevin, A.A.: Normal Modes and Localization in Nonlinear Systems. Wiley Interscience, New York (1996)
7. Manevitch, L.I., Mikhlin, Yu.V., Pilipchuk, V.N.: Method of Normal Modes for Essential Nonlinear Systems. Nauka, Moscow (1989) (in Russian)
8. Rauscher, M.: Steady oscillations of system with nonlinear and unsymmetrical elasticity. J. Appl. Mech. A **5**, 169 (1938)
9. Stoker, J.J.: Nonlinear Vibrations. Interscience, New York (1950)
10. Kauderer, H.: Nonlinear Mechanics. Springer, Berlin (1958) (in German)
11. Rosenberg, R.M.: Steady-state forced vibrations. Int. J. Non-Linear Mech. **1**, 95–108 (1966)
12. Avramov, K.V., Mikhlin, Y.V.: Snap-through truss as an absorber of forced oscillations. J. Sound Vib. **29**, 705–722 (2006)
13. Shaw, S.W., Pierre, C.: Normal modes for nonlinear vibratory systems. J. Sound Vib. **164**, 58–124 (1993)
14. Shaw, S.W., Pierre, C., Pesheck, E.: Modal analysis-based reduced-order models for nonlinear structures- an invariant manifolds approach. Shock Vib. Dig. **31**, 3–16 (1999)
15. Avramov, K.V., Mikhlin, Y.V.: Snap-through truss as a vibration absorber. J. Vib. Control **10**, 291–308 (2004)
16. Mikhlin, Y.V., Reshetnikova, S.N.: Dynamic interaction of an elastic system and essential nonlinear absorber. J. Sound Vib. **283**, 91–120 (2005)
17. Avramov, K.V., Pierre, C., Shiraeva, N.: Flexural–flexural–torsional nonlinear vibrations of pre-twisted rotating beams with asymmetrical cross section. J. Vib. Control (2007, in press)
18. Hsu, C.S.: Stability of shallow arches against snap-through under timewise step loads. ASME J. Appl. Mech. **33**, 31–39 (1968)
19. Breslavsky, I., Avramov, K.V., Mikhlin, Yu., Kochurov, R.: Nonlinear modes of snap-through motions of shallow arch. J. Sound Vib. (2007, to be submitted)