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New solitons and periodic wave solutions for nonlinear physical models

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Abstract In this paper, an extended tanh method with computerized symbolic computation is used for constructing the traveling wave solutions of coupled nonlinear equations arising in physics. The obtained solutions include solitons, kinks, and plane periodic solutions. The applied method will be used in further works to establish more entirely new solutions for other kinds of nonlinear evolution equations arising in physics.

Keywords Extended tanh method · Nonlinear physical models · Solitons, kinks, and plane periodic solutions

1 Introduction

Since the world around us is inherently nonlinear, nonlinear evolution equations are widely used to describe complex phenomena in various scientific fields and especially in areas of physics such as plasma physics, fluid mechanics, optical fibers, solid state physics, nonlinear optics, and so on.

One of the most exciting advances of nonlinear science and theoretical physics has been the development of methods to look for exact solutions of nonlinear partial differential equations. The search for exact solutions of nonlinear equations has gained interest in recent years because of the availability of symbolic computational programs such as Mathematica and Maple, which allow us to perform some complicated and tedious algebraic and differential calculations on a computer.

The tanh method is widely used by many researchers such as in [1–17] and by the references therein. The method introduces a unifying method by which one can find exact as well as approximate solutions in a straightforward and systematic way [10]. The tanh method has been subjected to many modifications that mainly depend on the Riccati equation and the solutions of well-known equations. The standard tanh method and the proposed modifications all depend on the balance method, where the linear terms of highest order are balanced with the highest-order nonlinear terms of the reduced equation.

Therefore, the aim of this work is to elucidate further the extended tanh method [10] with an ansatz introduced in this paper and apply it to some nonlinear evolution equations arising from nonlinear physics.

The rest of this paper is arranged as follows. In Sect. 2, we simply provide the mathematical framework of the extended tanh method. In Sect. 3, in order

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to illustrate the method, five nonlinear evolution equations are investigated and abundant exact solutions are obtained which include solitons, kinks and plane periodic solutions. Finally, the discussion and conclusion are given in Sect. 4.

2 Developed extended tanh method

In this section, we give a brief description of the extended tanh method. For the given system of nonlinear evolution equations, say, in two variables

$$N(u,v,u_t,v_t,u_x,v_x,u_{tt},v_{tt},u_{xx},v_{xx},\dots)=0, \quad (1)$$

$$M(u,v,u_t,v_t,u_x,v_x,u_{tt},v_{tt},u_{xx},v_{xx},\dots)=0, \quad (2)$$

we seek the following traveling wave solutions:

$$u(x,t)=u(\xi), \qquad v(x,t)=v(\xi)\xi=x\pm ct$$

which are of important physical significance, with k and c are constants to be determined later. Then system (1) and (2) reduces to a system of nonlinear ordinary differential equations

$$N_0(u,v,u_\xi,v_\xi,u_\xi,v_\xi,u_{\xi\xi},v_{\xi\xi},u_{\xi\xi},v_{\xi\xi},\dots)=0, \quad (3)$$

$$M_0(u,v,u_\xi,v_\xi,u_\xi,v_\xi,u_{\xi\xi},v_{\xi\xi},u_{\xi\xi},v_{\xi\xi},\dots)=0. \quad (4)$$

Introducing a new independent variable in the form

$$Y=\tanh(\mu\xi), \quad \xi=x\pm ct, \quad (5)$$

leads to the change of derivatives

$$\begin{aligned}
\frac{d}{d\xi}&=\mu\left(1-Y^2\right)\frac{d}{dY},\\
\frac{d^2}{d\xi^2}&=-2\mu^2\left(1-Y^2\right)\frac{d}{dY}+\mu^2\left(1-Y^2\right)^2\frac{d^2}{dY^2},\\
\frac{d^3}{d\xi^3}&=2\mu^3\left(1-Y^2\right)\left(3Y^2-1\right)\frac{d}{dY}\\
&\quad-6\mu^3Y\left(1-Y^2\right)^2\frac{d^2}{dY^2}+\mu^3\left(1-Y^2\right)^3\frac{d^3}{dY^3}.
\end{aligned} \quad (6)$$

In the context of the tanh function method, many authors [2–7] have used the ansatz

$$u(\xi)=\sum_{i=0}^{M}a_iY^i(\xi), \qquad v(\xi)=\sum_{i=0}^{N}c_iY^i(\xi). \quad (7)$$

In order to generalize this, it is reasonable to introduce the following ansatz [8]:

$$\begin{aligned}
u(\xi)&=\sum_{i=0}^{M}a_iY^i(\xi)+\sum_{i=1}^{M}b_iY^{-i}(\xi),\\
v(\xi)&=\sum_{i=0}^{N}c_iY^i(\xi)+\sum_{i=1}^{N}d_iY^{-i}(\xi),
\end{aligned} \quad (8)$$

in which $a_i, b_i (i=0,1,\dots,M)$ and c_i, $d_i (i=0,1,\dots,N)$ are all real constants to be determined later. The balancing numbers M and N are positive integers which can be determined by balancing the highest-order derivative terms with the highest-power nonlinear terms in (3) and (4). We substitute ansatz (8) and (6) into (3) and (4) using computerized symbolic computation. Equating the coefficients of all power $Y^{\pm i}$ to zero yields a set of algebraic equations for a_i, b_i, c_i, d_i and μ.

3 Applications

In the next section, we will demonstrate the proposed method on five nonlinear evolution equations of special interest in physics. These are the dispersive long-wave equation, the Whitham–Broer–Kaup (WBK) system, the Boussinesq equation, the generalized-Zakharov equations, and the $(2+1)$-dimensional Davey–Stewartson equation.

3.1 The dispersive long-wave equation

The dispersive long-wave equation [18] reads

$$\begin{aligned}
&v_t+vv_x+w_x=0,\\
&w_t+(vw)_x+\frac{1}{3}v_{xxx}=0.
\end{aligned} \quad (9)$$

To look for the traveling wave solutions of (9), we use the transformation $w(x,t)=\sigma(\xi)$, $v(x,t)=\phi(\xi)$, $\xi=k(x+\lambda t)$. Then (9) reduce to

$$\lambda\phi'+\phi\phi'+\sigma'=0, \quad (10)$$

$$\lambda\sigma'+(\sigma\phi)'+\frac{k^3}{2}\phi'''=0, \quad (11)$$

where the primes denote differentiations and λ is a constant to be determined later.

By virtue of the technique of solution, we introduce the ansatz

$$\phi(\xi)=\sum_{i=0}^{N}c_iY^i(\xi)+\sum_{i=1}^{N}d_iY^{-i}(\xi), \quad (12)$$

$$\sigma(\xi)=\sum_{i=0}^{M}a_iY^i(\xi)+\sum_{i=1}^{M}b_iY^{-i}(\xi), \tag{13}$$

where a_i, b_i, c_i, and d_i are constants to be determined later. Balancing ϕ''' and $(\sigma\phi)'$ in (10) and balancing σ' and $\phi\phi'$ in (11), we obtain $N=1$ and $M=2$. Therefore we suppose $u(\xi)$ and $v(\xi)$ in the following form:

$$\phi(\xi)=c_0+c_1Y(\xi)+d_1Y^{-1}(\xi), \tag{14}$$

$$\sigma(\xi)=a_0+a_1Y(\xi)+a_2Y^2(\xi)+b_1Y^{-1}(\xi)+b_2Y^{-2}(\xi). \tag{15}$$

Substituting (14) and (15) with the aid of (6) into (10) and (11) and setting each coefficient of $Y^{\pm i}$ to zero yields a set of equations for a_0, a_1, a_2, c_0, c_1, d_1, b_1, b_2, and μ. Solving the system of algebraic equations with the aid of Maple and the Wu-elimination method [19], we can distinguish three cases, namely:

Case 1:

$$\begin{aligned}&b_1=0,\quad \mu=\mu,\quad a_0=0,\\&b_2=-k^3\mu^2,\quad a_2=-k^3\mu^2,\\&c_0=-\lambda,\quad a_1=0,\quad c_1=d_1=\sqrt{2}k^{3/2}.\end{aligned} \tag{16}$$

Case 2:

$$\begin{aligned}&b_1=0,\quad \mu=\mu,\quad a_0=2\mu^2k^3,\\&b_2=-k^3\mu^2,\quad a_2=-k^3\mu^2,\\&c_0=-\lambda,\quad a_1=0,\quad c_1=d_1=\sqrt{2}k^{3/2}.\end{aligned} \tag{17}$$

Case 3:

$$\begin{aligned}&b_1=0,\quad \mu=\mu,\quad a_0=\mu^2k^3,\\&b_2=-k^3\mu^2,\quad a_2=0,\\&c_0=-\lambda,\quad a_1=0,\quad c_1=0,\quad d_1=\sqrt{2}k^{3/2}.\end{aligned} \tag{18}$$

In view of (16–18), we obtain the following periodic solutions:

$$u_1(x,t)=-\lambda+\sqrt{2}k^{3/2}\tanh\big[\mu k(x+\lambda t)\big]+\sqrt{2}k^{3/2}\coth\big[\mu k(x+\lambda t)\big], \tag{19}$$

$$v_1(x,t)=-k^3k^2\tanh^2\big[\mu k(x+\lambda t)\big]-k^3\mu^2\coth^2\big[\mu k(x+\lambda t)\big], \tag{20}$$

$$u_2(x,t)=-\lambda+\sqrt{2}k^{3/2}\tanh\big[\mu k(x+\lambda t)\big]+\sqrt{2}k^{3/2}\coth\big[\mu k(x+\lambda t)\big], \tag{21}$$

$$v_2(x,t)=2k^3k^2-k^3\mu^2\tanh^2\big[\mu k(x+\lambda t)\big]-k^3\mu^2\coth^2\big[\mu k(x+\lambda t)\big], \tag{22}$$

$$u_3(x,t)=-\lambda+\sqrt{2}k^{3/2}\coth\big[\mu k(x+\lambda t)\big], \tag{23}$$

$$v_3(x,t)=k^3k^2-k^3\mu^2\coth^2\big[\mu k(x+\lambda t)\big], \tag{24}$$

where λ and k are free parameters. It is obvious that the three pairs of solutions obtained by using the extended tanh method were obtained in [20] by using the extended Jacobi elliptic function expansion method.

3.2 The Whitham–Broer–Kaup (WBK) system

A second interactive model is the Whitham–Broer–Kaup system [11],

$$\begin{aligned}&u_t+uu_x+v_x+\mu u_{xx}=0,\\&v_t+(uv)_x+\alpha u_{xxx}-\mu_0v_{xx}=0,\end{aligned} \tag{25}$$

which is a complete integrable model that describes the dispersive long wave in shallow water. In system (25), α and μ_0 are real constants that represent different dispersive powers. If $\alpha=0$ and $\mu_0\neq 0$, (25) become the approximate equations for the long-wave equation. In the case of $\alpha=1$ and $\mu_0=0$, (25) becomes the variant Boussinesq equation. On using $u(x,t)=U(\xi)$, $v(x,t)=V(\xi)$, $\xi=kx+\lambda t$, (25) reduce to

$$\lambda U'+kUU'+kV'+k^2\mu U''=0, \tag{26}$$

$$\lambda V'+k(UV)'+\alpha k^3U'''-\mu k^2V''=0. \tag{27}$$

Integrating (26) and (27) once, we get

$$\lambda U+\frac{k}{2}U^2+kV+k^2\mu U'=C_1, \tag{28}$$

$$\lambda V+kUV+\alpha k^3U''-\mu k^2V'=C_2, \tag{29}$$

where C_1 and C_2 are integration constants. From (28), we deduce

$$V(\xi)=\frac{C_1}{k}-\frac{U^2}{2}-\mu kU'-\frac{\lambda}{k}U. \tag{30}$$

Substituting (30) into (29) yields

$$k_1U''+k_2U^3+k_3U^2+k_4U+k_5=0, \tag{31}$$

$$\begin{aligned} &k_1=(\alpha+\mu^2)k^3, \qquad k_2=-\frac{k}{2}, \qquad k_3=-\frac{3}{2}\lambda,\\ &k_4=\left(\frac{C_1}{k}-\frac{\lambda^2}{k}\right), \qquad k_5=\frac{C_1\lambda}{k}-C_2. \end{aligned} \tag{32}$$

Balancing the linear term of the highest order with the nonlinear term leads to the following ansatz:

$$U(\xi)=a_0+a_1Y(\xi)+b_1Y^{-1}(\xi). \tag{33}$$

Substituting (33) with the aid of (6) into (31) and collecting the coefficients of $Y^{\pm i}$, we obtain a set of algebraic equations for a_0, a_1, b_1, and μ. Solving this system with the aid of Maple we obtain the following two sets of solutions:

The first set:

$$\begin{aligned} &\lambda=-ka_0, \qquad \mu=\mu, \qquad a_1=-b_1, \qquad b_1=b_1,\\ &a_0=a_0, \qquad \mu_0=\mu_0, \qquad k=k,\\ &\alpha=-\frac{1}{4}\frac{4k^2\mu_0\mu^2-b_1^2}{k^2\mu^2},\\ &C_1=-k^2b_1^2-\frac{1}{2}k^2a_0^2,\\ &C_2=-ka_0b_1^2-\frac{1}{2}ka_0k^3+a_0k^2b_1^2+\frac{1}{2}k^2a_0^3. \end{aligned} \tag{34}$$

The second set:

$$\begin{aligned} &\lambda=-ka_0, \qquad \mu=\mu, \qquad a_1=0, \qquad b_1=b_1,\\ &a_0=a_0, \qquad \mu_0=\mu_0, \qquad k=k,\\ &\alpha=-\frac{1}{4}\frac{4k^2\mu_0\mu^2-b_1^2}{k^2\mu^2}\\ &C_1=\frac{1}{2}k^2b_1^2-\frac{1}{2}k^2a_0^2,\\ &C_2=\frac{1}{2}ka_0b_1^2-\frac{1}{2}ka_0^3k-\frac{1}{2}k^2a_0b_1^2+\frac{1}{2}k^2a_0. \end{aligned} \tag{35}$$

The first set gives the periodic solutions

$$\begin{aligned} u_1(x,t)={}&a_0-b_1\tanh\big[k\mu(x-a_0t)\big]\\ &+b_1\coth\big[k\mu(x-a_0t)\big], \end{aligned} \tag{36}$$

$$\begin{aligned} v_1(x,t)={}&\frac{-k^2b_1^2-\frac{1}{2}a_0^2}{k}\\ &-\frac{1}{2}\big[a_0-b_1\tanh\big[k\mu(x-a_0t)\big]\\ &+b_1\coth\big[k\mu(x-a_0t)\big]\big]^2\\ &-\mu_0k\big[-b_1\big(1-\tanh\big[k\mu(x-a_0t)\big]^2\big)k\mu\\ &-b_1\big(1-\tanh\big[k\mu(x-a_0t)\big]^2\big)k\mu\coth^2\\ &\times\big[\mu(kx-ka_0t)\big]\\ &+a_0\big[a_0-b_1\tanh\big[k\mu(x-a_0t)\big]\\ &+b_1\coth\big[k\mu(x-a_0t)\big]\big]. \end{aligned} \tag{37}$$

The second set gives the singular solitary wave solutions

$$u_2(x,t)=a_0+b_1\coth\big[k\mu(x-a_0t)\big], \tag{38}$$

$$\begin{aligned} v_2(x,t)={}&\frac{(1/2)k^2b_1^2-\frac{1}{2}a_0^2k^2}{k}\\ &-\frac{1}{2}\big[a_0+b_1\coth\big[k\mu(x-a_0t)\big]\big]^2\\ &+\mu_0k^2b_1\big(1-\tanh^2\big[k\mu(x-a_0t)\big]\big)\mu\\ &\times\coth^2\big[k\mu(x-a_0t)\big]\\ &+a_0\big[a_0+b_1\coth\big[k\mu(x-a_0t)\big]\big], \end{aligned} \tag{39}$$

where a_0, b_1, μ_0, and k are free parameters. These solutions obtained by using the extended tanh method were obtained in [11] using the tanh method.

3.3 The Boussinesq equation

In this case, we consider the Boussinesq equation [21]:

$$u_t+v_x+uu_x+\alpha u_{xxt}=0, \tag{40}$$

$$v_t+(uv)_x+\beta u_{xxx}=0, \tag{41}$$

where α and β are arbitrary constants. Substituting $u(x,t)=u(\xi)$, $v(x,t)=v(\xi)$, $\xi=x+\lambda t$ into (40) and (41) and integrating once yields

$$\begin{aligned} &\lambda u+\frac{1}{2}u^2+v+\alpha\lambda u''=c_1,\\ &(\lambda+u)v+\beta u''=c_2, \end{aligned} \tag{42}$$

where c_1 and c_2 are constants of integration. Balancing u'' with u^2, we find $N = M = 2$. We may expand the solutions of (42) in the form

$$u(\xi) = c_0 + c_1 Y(\xi) + c_2 Y^2(\xi) + d_1 Y^{-1}(\xi) + d_2 Y^{-2}(\xi), \tag{43}$$

$$v(\xi) = a_0 + a_1 Y(\xi) + a_2 Y^2(\xi) + b_1 Y^{-1}(\xi) + b_2 Y^{-2}(\xi). \tag{44}$$

Inserting (43) and (44) into (42) and collecting the coefficients of $Y^{\pm i}$, we obtain a set of algebraic equations for a_0, a_1, a_2, c_0, c_1, c_2, d_1, d_2, b_1, b_2, and μ. Solving this system with the aid of the Maple software package, we have

$$\begin{aligned}
&a_0 = -\frac{3}{8}\frac{\beta}{\alpha}, \qquad a_2 = \frac{3\beta}{16\alpha}, \qquad b_2 = \frac{3}{16}\frac{\beta}{\alpha},\\
&c_1 = d_1 = a_1 = b_1 = 0, \qquad d_2 = c_2 = \frac{3\sqrt{-\frac{\beta}{\alpha}}}{8},\\
&c_0 = \frac{3\beta}{4\alpha\sqrt{\frac{-\beta}{\alpha}}}, \qquad \lambda = \sqrt{-\frac{\beta}{\alpha}}, \qquad \mu = \sqrt{\frac{-1}{32\alpha}}.
\end{aligned} \tag{45}$$

In view of this we obtain the following bell-shaped solitary wave solutions

$$u_1(x,t) = \frac{3\beta}{4\alpha\sqrt{\frac{-\beta}{\alpha}}} + \frac{3\sqrt{-\frac{\beta}{\alpha}}}{8}\operatorname{sech}^2\left[\sqrt{\frac{-1}{32\alpha}}\left(x + \sqrt{-\frac{\beta}{\alpha}}t\right)\right], \tag{46}$$

$$v_1(x,t) = -\frac{3}{8}\frac{\beta}{\alpha} + \frac{3\beta}{16\alpha}\operatorname{sech}^2\left[\sqrt{\frac{-1}{32\alpha}}\left(x + \sqrt{-\frac{\beta}{\alpha}}t\right)\right], \tag{47}$$

where α, β, and λ are constants. It is to be noted that the solutions obtained using the proposed method agree well with those obtained in [21] using the homogeneous balance method.

3.4 Example: the generalized Zakharov equations

The generalized Zakharov equations for the complex envelope $\psi(x,t)$ of the high-frequency wave and the real low-frequency field $v(x,t)$ in the form [22] are:

$$i\psi_t + \psi_{xx} - 2\lambda|\psi|^2\psi + 2\psi v = 0, \tag{48}$$

$$v_{tt} - v_{xx} + \left(|\psi|^2\right)_{xx} = 0, \tag{49}$$

where the cubic term in (48) describes the nonlinear self-interaction in the high-frequency subsystem. This cubic term corresponds to a self-focusing effect in plasma physics. The coefficient λ is a real constant that can be a positive or negative number. Let us assume the traveling wave solution of (48) and (49) has the form

$$\begin{aligned}
&\psi(x,t) = e^{i\eta}u(\xi), \qquad v = v(\xi),\\
&\eta = \alpha x + \beta t, \qquad \xi = k(x - 2\alpha t),
\end{aligned} \tag{50}$$

where $u(\xi)$ and $v(\xi)$ are real functions and the constants α, β, and k are to be determined. Substituting (50) into (48) and (49), we have

$$k^2 u'' + 2uv - \left(\alpha^2 + \beta\right)u - 2\lambda u^3 = 0, \tag{51}$$

$$k^2\left(4\alpha^2 - 1\right)v'' + k^2\left(u^2\right)'' = 0. \tag{52}$$

In order to simplify ODEs (51) and (52), integrating (51) once, setting the constant of integration to zero, and integrating again yields

$$v(\xi) = \frac{u^2}{(1-4\alpha^2)} + C, \quad \text{if } \alpha^2 \neq \frac{1}{4}, \tag{53}$$

where C is the integration constant. Inserting (53) into (52), we have

$$k^2 u'' + \left[2C - \alpha^2 - \beta\right]u + 2\left[\frac{1}{1-4\alpha^2} - \lambda\right]u^3 = 0. \tag{54}$$

Balancing u'' with u^3 gives $M = 1$. Consequently, the extended tanh method becomes

$$u(\xi) = a_0 + a_1 Y(\xi) + b_1 Y^{-1}(\xi). \tag{55}$$

Substituting (55) and making use of (6) into (54) yields a set of algebraic equations for a_0, a_1, b_1 and μ.

Solving this system, we have two sets of solutions, namely,

The first set:

$$a_0=0,\qquad \mu=\mu,\qquad k=\frac{\sqrt{C-\frac{\beta}{2}-\frac{\alpha^2}{2}}}{2\mu},$$
$$b_1=\sqrt{\frac{-2C+4\alpha^4-\alpha^2-\beta+4\beta\alpha^2-8C\alpha^2}{8-8\lambda+32\lambda\alpha^2}},\tag{56}$$
$$a_1=\frac{2C+4\alpha^4-\alpha^2-\beta+4\beta\alpha^2-8C\alpha^2}{8\sqrt{\frac{-2C+4\alpha^4-\alpha^2-\beta+4\beta\alpha^2-8C\alpha^2}{8-8\lambda+32\lambda\alpha^2}}(1-\lambda+4\lambda\alpha^2)}.$$

The second set:

$$a_0=a_1=0,\qquad \mu=\mu,$$
$$k=\frac{\sqrt{C-\frac{\beta}{2}-\frac{\alpha^2}{2}}}{2\mu},\tag{57}$$
$$b_1=\sqrt{\frac{-2C-8C\alpha^2-\alpha^2-\beta+4\alpha^2+4\beta\alpha^2}{2-2\lambda+8\lambda\alpha^2}}.$$

In view of (57), we obtain the following solutions

$$v_1(x,t)=\Bigg[\frac{2C+4\alpha^4-\alpha^2-\beta+4\beta\alpha^2-8C\alpha^2}{8\sqrt{\frac{-2C+4\alpha^4-\alpha^2-\beta+4\beta\alpha^2-8C\alpha^2}{8-8\lambda+32\lambda\alpha^2}}(1-\lambda+4\lambda\alpha^2)}\tanh\Bigg[\frac{\sqrt{C-\frac{\beta}{2}-\frac{\alpha^2}{2}}}{2}(x-2\alpha t)\Bigg]$$
$$\times\sqrt{\frac{-2C+4\alpha^4-\alpha^2-\beta+4\beta\alpha^2-8C\alpha^2}{8-8\lambda+32\lambda\alpha^2}}\coth\Bigg[\frac{\sqrt{C-\frac{\beta}{2}-\frac{\alpha^2}{2}}}{2}(x-2\alpha t)\Bigg]\Bigg]^2\Big/\big(1-4\alpha^2\big)+C,\tag{58}$$

$$\psi_1(x,t)=e^{i(\alpha x+\beta t)}\Bigg[\frac{2C+4\alpha^4-\alpha^2-\beta+4\beta\alpha^2-8C\alpha^2}{8\sqrt{\frac{-2C+4\alpha^4-\alpha^2-\beta+4\beta\alpha^2-8C\alpha^2}{8-8\lambda+32\lambda\alpha^2}}(1-\lambda+4\lambda\alpha^2)}\tanh\Bigg[\frac{\sqrt{C-\frac{\beta}{2}-\frac{\alpha^2}{2}}}{2}(x-2\alpha t)\Bigg]$$
$$\times\sqrt{\frac{-2C+4\alpha^4-\alpha^2-\beta+4\beta\alpha^2-8C\alpha^2}{8-8\lambda+32\lambda\alpha^2}}\coth\Bigg[\frac{\sqrt{C-\frac{\beta}{2}-\frac{\alpha^2}{2}}}{2}(x-2\alpha t)\Bigg]\Bigg],\tag{59}$$

$$v_2(x,t)=\frac{-2C-8C\alpha^2-\alpha^2-\beta+4\alpha^2+4\beta\alpha^2}{2-2\lambda+8\lambda\alpha^2(1-4\alpha^2)}\coth^2\Bigg[\frac{\sqrt{C-\frac{\beta}{2}-\frac{\alpha^2}{2}}}{2}(x-2\alpha t)\Bigg]+C,$$
$$\psi_2(x,t)=e^{i(\alpha x+\beta t)}\sqrt{\frac{-2C-8C\alpha^2-\alpha^2-\beta+4\alpha^2+4\beta\alpha^2}{2-2\lambda+8\lambda\alpha^2}}\coth\Bigg[\frac{\sqrt{C-\frac{\beta}{2}-\frac{\alpha^2}{2}}}{2}(x-2\alpha t)\Bigg],\tag{60}$$

where C, α, and β are constants. It is worth noting that the obtained (58–60) compare well with those obtained in [22] using the F-expansion method and Jacobi elliptic function method.

3.5 Example: $(2+1)$-dimensional Davey–Stewartson equation

The two-dimensional Davey–Stewartson equation [22] reads

$$iu_t+u_{xx}-u_{yy}-2|u|^2u-2uv=0,$$
$$v_{xx}+v_{yy}+2\big(|u|^2\big)_{xx}=0.\tag{61}$$

This equation is completely integrable and often used to describe the long-time evolution of a two-dimensional wave packet. Using the following wave variables

$$u=e^{i\theta}U(\xi),\qquad v=V(\xi),$$
$$\theta=px+qy+rt,\qquad \xi=kx+cy+dt,\tag{62}$$

where p, q, r, k, c, and d are real constants, converts (61) into the ODE

$$\begin{aligned}&(q^2-p^2-r)U+(k^2-c^2)U''-2U^3-2UV=0,\\&(k^2+c^2)V''+(U^2)''=0.\end{aligned} \tag{63}$$

Accordingly, the extended tanh method is of the form

$$U(\xi)=c_0+c_1Y(\xi)+d_1Y^{-1}(\xi), \tag{64}$$

$$\begin{aligned}V(\xi)&=a_0+a_1Y(\xi)+a_2Y^2(\xi)+b_1Y^{-1}(\xi)\\&\quad+b_2Y^{-2}(\xi).\end{aligned} \tag{65}$$

Substituting (65) and (64) into (63) and proceeding as before, we obtain two sets of solutions

The first set:

$$\begin{aligned}&d_1=d_1,\qquad c_1=-d_1,\qquad a_1=b_1=0,\\&c_0=0,\qquad a_2=b_2=-\frac{d_1^2}{k^2+c^2},\\&a_0=-\frac{r}{2}+2d_1^2+\frac{(q^2-p^2)}{2}.\end{aligned} \tag{66}$$

The second set:

$$\begin{aligned}&d_1=d_1,\qquad c_1=d_1,\qquad a_1=b_1=0,\\&c_0=0,\qquad a_2=b_2=-\frac{d_1^2}{k^2+c^2},\\&a_0=-\frac{1}{2}\big[c^2(-q^2+p^2+r)+8c^2d_1^2-4d_1^2\\&\qquad-k^2(q^2-p^2-r)+8k^2d_1^2\big]\big[k^2+c^2\big]^{-1},\\&\mu=\sqrt{\frac{-k^2+c^2-1}{-k^4+c^4}}d_1.\end{aligned} \tag{67}$$

These yield the following solutions:

$$\begin{aligned}u_1(x,t)&=-d_1\tanh\big[\mu(kx+cy+dt)\big]\\&\quad+d_1\coth\big[\mu(kx+cy+dt)\big],\end{aligned} \tag{68}$$

$$\begin{aligned}v_1(x,t)&=-\frac{r}{2}+2d_1^2+\frac{(q^2-p^2)}{2}\\&\quad-\frac{d_1}{(k^2+c^2)}\operatorname{sech}^2\big[\mu(kx+cy+dt)\big],\end{aligned} \tag{69}$$

$$\begin{aligned}u_2(x,t)&=d_1\tanh\left[\sqrt{\frac{-k^2+c^2-1}{-k^4+c^4}}d_1(kx+cy+dt)\right]\\&\quad+d_1\coth\left[\sqrt{\frac{-k^2+c^2-1}{-k^4+c^4}}d_1\right.\\&\quad\left.\times(kx+cy+dt)\right],\end{aligned} \tag{70}$$

$$\begin{aligned}v_2(x,t)&=-\frac{1}{2}\big[c^2(-q^2+p^2+r)+8c^2d_1^2-4d_1^2\\&\quad-k^2(q^2-p^2-r)+8k^2d_1^2\big]\big[k^2+c^2\big]^{-1}\\&\quad-\frac{d_1^2}{k^2+c^2}\operatorname{sech}^2\left[\sqrt{\frac{-k^2+c^2-1}{-k^4+c^4}}d_1\right.\\&\quad\left.\times(kx+cy+dt)\right],\end{aligned} \tag{71}$$

where p, q, r, and c are constants. It is clear that the two pairs of solutions are in excellent agreement with those obtained [22] by using the extended Jacobi elliptic function expansion method.

4 Discussion and conclusion

In this paper, the extended tanh method with symbolic computation on the computer is used for constructing broad classes of periodic traveling wave solutions of five nonlinear equations arising in nonlinear physics. These are the variant Boussinesq equation, the dispersive long-wave equation, the Whitham–Broer–Kaup (WBK) system, the generalized Zakharov equations, and the $(2+1)$-dimensional Davey–Stewartson equation.

In this work, we presented a generalized extended tanh method based on the general ansatz (8) in which the exponent of tanh function may take both positive and negative values in contrast to the solution based on ansatz (7) where its exponent can only assume only positive values. This, of course, leads to the conclusion that the extended tanh method with the ansatz (8) can be used to improve the tanh function method [1–9] with the solution ansatz (7).

Finally, it is worthwhile to mention that the proposed method is reliable and effective and gives more solutions. This method will be applied in further research to establish more new solutions for other kinds of nonlinear equations.

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