

Species extinction and permanence in a prey–predator model with two-type functional responses and impulsive biological control

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Abstract By introducing impulsive biological control strategy, the dynamic behaviors of the two-prey one-predator model with defensive ability and Holling type-II functional response are investigated. By using Floquet's Theorem and the small amplitude perturbation method, we prove that there exists an asymptotically stable pest-eradication periodic solution when the impulsive period is less than some critical minimum value, and permanence conditions (that is, the impulsive period is greater than some critical maximum value) are established via the method of comparison involving multiple Liapunov functions. It is shown that our impulsive control strategy is more effective than the classical one. Furthermore, the effect of impulsive perturbations on the unforced continuous system is studied. From simulations, we find that the system has more complex dynamic behaviors and is

dominated by periodic, quasi-periodic, and chaotic solutions.

Keywords Impulsive biological control · Functional response · Extinction · Permanence · Quasi-periodic solution

1 Introduction

Pest outbreaks often cause serious ecological and economic problems. For example, in recent years, a large amount of locusts invaded into the region of the northwestern China of Xinjiang Province and Inner Mongolia and caused damages to the crops together with the local locusts. Flea beetle damage to oilseed Brassica crops exceeds \$300 million annually in North America (Courtesy of Lloyd Harris). There are many way to beat agricultural pests. An important method for pest control is chemical control. Pesticides are useful because they quickly kill a significant portion of a pest population and they sometimes provide the only feasible method for preventing economic loss. However, pesticide pollution is not only recognized as a major health hazard to human beings and to natural enemies but also is resisted by pests. This latter reason results in the use of high doses and more toxic materials to combat pests. Beneficial insects are often susceptible to chemical insecticides applied for the target pest. One of the side effects of high rates of pesticide use is that natural enemies and other small animals that

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might otherwise feed on pests are killed and the pest population explodes again after the beneficial insects are killed.

The safety of pesticides has been reviewed whilst advancing activities on the safety of food and environment directly affecting our lives (e.g., establishment of the Food Quality Prevention Act by US Environmental Protection Agency). Pest control methods relying on chemical pesticides have been reviewed with consideration to their resistance and persistence in the environment. In this regard, the Integrated Pest Management (IPM) approach, composed of a combination of various pest control methods (e.g., natural enemy, pheromone of insects), is valued and being promoted. Biological control is another important method to beat the pest [7, 8, 10]. Biological control is the purposeful introduction and establishment of one or more natural enemies from the region of origin of an exotic pest specifically for the purpose of suppressing the abundance of the pest in a new target region to a level at which it no longer causes economic damage. Virtually all insect and mite pests have some natural enemies. One approach to biological control is augmentation, which is the manipulation of existing natural enemies to increase their effectiveness. This can be achieved by the mass production and periodic release of natural enemies of the pest, and by genetic enhancement of the enemies to increase their effectiveness at control. The pioneering project of biological control began in 1888 when the new legendary predator, the vedalia beetle, was imported from Australia and established in California, where it rapidly suppressed populations of cottony cushion scale that had been decimating the developing citrus industry [5].

So natural enemies play a more active role in suppressing insect pests. Many authors have investigated the effect of natural enemies on controlling pests [15, 17, 26]. In these works, the natural enemy is portrayed as only feeding on one kind of pest. However, this is not the actual case. Usually a predator (natural enemy) feeds on several kinds of insect pests. For example, two species of aphids are predominant in small grain crops: the English grain aphid and the oat-bird cherry aphid. Aphids' high reproductive rate enables their populations to quickly build up to levels that can cause an economic loss. However, aphids are usually kept in check by biological control agents, such as lady beetles, parasitic wasps, and syrphid fly maggots, which are often abundant in small grain crops.

In view of these facts, Zhang et al. [25] considered a predator feeding on two kinds of pests. However, the interactions between the predator and prey satisfy the Lotka–Volterra model, which has the unrealistic assumption that the effect of the predation is to reduce the prey's per capita growth rate by a term proportional to the prey and predator populations. As is known, in an ecosystem the interactions between predator and prey are often complicated and diverse. In population dynamics, Holling [12] gave three different kinds of functional response of the predator to the prey density which refer to the change in the density of prey attacked per unit time per predator as the prey density changes, which are monotonic in the first quadrant. If we take into account the time a predator uses in handling the prey it has captured, one finds the predator has a type-II functional response. But some experiments and observations indicate that the non-monotonic response occurs because when the nutrient concentrations reaches a high level an inhibitory effect on the specific growth rate may occur. To model such an inhibitory effect, Andrews [1] suggested a function

$$\phi(x) = \frac{mx}{a + bx + x^2},$$

called the Monod–Haldane function, or Holling type-IV function. Sugie and Howell [22] proposed a simplified Monod–Haldane function of the form:

$$\phi(x) = \frac{mx}{a + x^2},$$

which describes the phenomenon of group defense whereby predation is decreased, or even prevented altogether, due to the increased ability of the prey to better defend or disguise themselves when their numbers are large enough. An example of this phenomenon is introduced by Tener [24].

The four kinds of Holling functional response have been studied [6, 20]. According to Hassel et al. [11], the Holling type-II functional response is the most common type of functional response among arthropod predators. Based on the predator-prey system with Holling II functional response and group defense, we suggest an impulsive differential equation [2, 3, 14] to model the process of periodically releasing natural enemies at different fixed times in Sect. 2. Impulsive equations are found in almost every domain of applied science and have been studied in many investigations: impulsive birth [19, 23], impulsive vaccination [9, 21],

chemotherapeutic treatment of disease [13, 16] and population ecology [4]. They generally describe the phenomena which are subject to steep or instantaneous changes. In Sect. 4, we analyze the dynamic behaviors of such systems. By using Floquet’s theorem and the small amplitude perturbation method, we show that there exists an asymptotically stable pest-eradication periodic solution when the period of impulsive effect is less than some threshold, and we prove the system is permanent under some conditions.

2 Model formulation

The model we considered is based on the following system:

$$\begin{cases} x'_1 = x_1(t)[a_1 - x_1(t)] - \frac{x_1(t)x_3(t)}{1+ex_1^2(t)}, \\ x'_2 = x_2[a_2 - x_2(t)] - \frac{x_2(t)x_3(t)}{1+wx_2(t)}, \\ x'_3 = x_3(t)\left[\frac{m_1x_1(t)}{1+ex_1^2(t)} + \frac{m_2x_2(t)}{1+wx_2(t)} - d\right] \end{cases} \quad (2.1)$$

where $x_1(t)$, $x_2(t)$, and $x_3(t)$ are the biomasses of the two prey and the predator at time t , respectively, a_1 and a_2 are the intrinsic growth rates of the two prey, d denotes the death rate of the predator, and m_1 and m_2 are the rates of converting prey into predators. All parameters are positive constants.

For the system (2.1), the following results are obviously true.

Remark 2.1 The equilibrium $(0, 0, 0)$ of system (2.1) is unstable.

Remark 2.2 The equilibrium $(0, 0, x_{3+})$ of system (2.1) doesn’t exist, where $x_{3+} > 0$.

In this paper, with the ideas of group defense and Holling type-II functional response, we consider a two-prey one-predator system with impulsive perturbations:

$$\begin{cases} x'_1 = x_1(t)[a_1 - x_1(t)] - \frac{x_1(t)x_3(t)}{1+ex_1^2(t)}, \\ x'_2 = x_2[a_2 - x_2(t)] - \frac{x_2(t)x_3(t)}{1+wx_2(t)}, \\ x'_3 = x_3(t)\left[\frac{m_1x_1(t)}{1+ex_1^2(t)} + \frac{m_2x_2(t)}{1+wx_2(t)} - d\right], \end{cases} \quad t \neq n\tau, \quad (2.2)$$

$$\begin{cases} x_3(n\tau^+) = x_3(n\tau) + p, \\ x(0^+) = x_0 = (x_1(0), x_2(0), x_3(0))^T \end{cases}$$

where x_1, x_2 and x_3 are defined as above. $x_3(n\tau^+)$ denotes the biomass of predators after the n th release. p is the release amount of predator at time $t = n\tau, n \in N$ and $N = \{1, 2, \dots\}$, and τ is the period of the impulsive effect. $\frac{x_1(t)x_3(t)}{1+ex_1^2(t)}$ describes the relationship between prey x_1 and predator x_3 when the prey exhibits group defense, $\frac{x_2(t)x_3(t)}{1+wx_2(t)}$ is the Holling type-II functional response. All parameters are positive constants. For convenience, we denote the solution of (2.2) by $x(t) = (x_1(t), x_2(t), x_3(t))$.

3 Preliminaries

First, we give some notation, definitions and lemmas which be useful for our main results. Let $R_+ = [0, \infty]$, $R_+^3 = \{x \in R^3 \mid x \geq 0\}$. Denote $f = (f_1, f_2, f_3)$ the map defined by the right hand of system (2.2), and N the set of all non-negative integers. Let $V : R_+ \times R_+^3 \rightarrow R_+$, then V is said to belong to class V_0 if:

- (i) V is continuous in $(n\tau, (n + 1)\tau) \times R_+^3$ and for each $x \in R_+^3, n \in N, \lim_{(t,y) \rightarrow (n\tau^+,x)} V(t, y) = V(n\tau^+, x)$ exists.
- (ii) V is locally Lipschitzian in x .

Definition 3.1 Let $V \in V_0$, then for $(t, x) \in (n\tau, (n + 1)\tau) \times R_+^3$, the upper right derivative of $V(t, x)$ with respect to the impulsive differential system (2.2) is defined as

$$D^+V(t, x) = \lim_{h \rightarrow 0^+} \sup \frac{1}{h} [V(t + h, x + hf(t, x)) - V(t, x)].$$

The solution of the system (2.2) is denoted by $x(t) = (x_1(t), x_2(t), x_3(t)) : R_+ \rightarrow R_+ \times R_+^3$, and is continuously differential on $(n\tau, (n + 1)\tau], n \in N$ and $x(n\tau^+) = \lim_{t \rightarrow n\tau^+} x(t)$ exists. The global existence and uniqueness of the solution of system (2.2) is guaranteed by the smoothness of f (see [2]). The following lemma is obvious.

Lemma 3.1 Suppose $x(t)$ is a solution of (2.2) with $x(0^+) \geq 0$, then $x(t) \geq 0$ for all $t \geq 0$.

Definition 3.2 System (2.2) is said to be permanent if there exist three positive constants m, M and T_0 such that each positive solution $(x_1(t), x_2(t), x_3(t))$ of

system (2.2) satisfied $m \leq x_i(t) \leq M$, for all $t > T_0$, $i = 1, 2, 3$.

We will use the following important comparison theorem on impulsive differential equation [2]:

Lemma 3.2 *Let $V \in V_0$. Assume that*

$$\begin{cases} D^+V(t, x) \leq g(t, V(t, x)), & t \neq n\tau, \\ V(t, x(t^+)) \leq \varphi_n(V(t, x)), & t = n\tau, \end{cases} \quad (3.1)$$

where $g: R_+ \times R_+ \rightarrow R$ is continuous in $(n\tau, (n+1)\tau] \times R_+$ and for $u \in R_+$, $n \in N$, $\lim_{(t,y) \rightarrow (n\tau^+, u)} V(t, y) = V(n\tau^+, u)$ exists, $\varphi_n: R_+ \rightarrow R_+$ is non-decreasing. Let $r(t)$ be the maximal solution of the scalar impulsive differential equation

$$\begin{cases} u'(t) = g(t, u(t)), & t \neq n\tau, \\ u(t^+) = \varphi_n(u(t)), & t = n\tau, \\ u(0^+) = u_0, \end{cases} \quad (3.2)$$

existing on $[0, \infty)$. Then $V(0^+, x_0) \leq u_0$ implies that $V(t, x(t)) \leq r(t)$, $t \geq 0$, where $x(t)$ is any solution of (2.2).

Similar results can be obtained when all the directions of the inequalities in (3.1) are reversed. Note that if we have some smoothness conditions of g to guarantee the existence and uniqueness of solutions for (3.2), then $r(t)$ is exactly the unique solution of (3.2).

The function we will use is in the form $-du$. For convenience, we give some basic properties of the following system.

$$\begin{cases} u'(t) = -du(t), & t \neq n\tau, \\ u(t^+) = u(t) + p, & t = n\tau, \\ u(0^+) = u_0. \end{cases} \quad (3.3)$$

Clearly, system (3.3) has a positive periodic solution

$$u^*(t) = \frac{p \exp(-d(t - n\tau))}{1 - \exp(-d\tau)},$$

$$t \in (n\tau, (n+1)\tau], n \in N,$$

$$u^*(0^+) = \frac{p}{1 - \exp(-d\tau)}.$$

Since the solution of the system (3.3) is

$$u(t) = \left(u(0^+) - \frac{p}{1 - \exp(-d\tau)} \right) \exp(-dt) + u^*(t),$$

$$t \in (n\tau, (n+1)\tau], u(0^+) \geq 0,$$

we have

Lemma 3.3 *System (3.3) has a positive periodic solution $u^*(t)$ and for every solution $u(t)$ of (3.3), it follows that $u(t) \rightarrow u^*(t)$ as $t \rightarrow \infty$.*

Therefore, the system (2.2) has a pest-eradication periodic solution

$$(0, 0, x_3^*(t)) = \left(0, 0, \frac{p \exp(-d(t - n\tau))}{1 - \exp(-d\tau)} \right)$$

for $t \in (n\tau, (n+1)\tau]$.

4 Extinction and permanence

First, we study the stability of the prey-eradication periodic solution.

Theorem 4.1 *Let $(x_1(t), x_2(t), x_3(t))$ be any solution of (2.2), then $(0, 0, x_3^*(t))$ is asymptotically stable provided that*

$$\tau \leq \min \left\{ \frac{P}{a_1 d}, \frac{P}{a_2 d} \right\} \triangleq \tau_{\min}.$$

Proof The local stability of periodic solution $(0, 0, x_3^*(t))$ may be determined by considering the behavior of small amplitude perturbations of the solution. Define

$$x_1(t) = u_1(t), \quad x_2(t) = u_2(t),$$

$$x_3(t) = x_3^*(t) + u_3(t)$$

where $u_i(t)$ ($i = 1, 2, 3$) are small perturbations. Equation (2.2) can be expanded in a Taylor series after neglecting higher order terms, the linearized equations read:

$$\left. \begin{cases} u'_1(t) = (a_1 - x_3^*(t))u_1(t), \\ u'_2(t) = (a_2 - x_3^*(t))u_2(t), \\ u'_3(t) = m_1 x_3^*(t)u_1(t) \\ \quad + m_2 x_3^*(t)u_2(t) \\ \quad - du_3(t), \\ u_3(n\tau^+) = u_3(n\tau). \end{cases} \right\} 0 < t < \tau, \quad (4.1)$$

Let $\Phi(t)$ be fundamental matrix of (4.1), then $\Phi(t)$ must satisfy

$$\frac{d\Phi(t)}{dt} = \begin{pmatrix} a_1 - x_3^*(t) & 0 & 0 \\ 0 & a_2 - x_3^*(t) & 0 \\ m_1 x_3^*(t) & m_2 x_3^*(t) & -d \end{pmatrix} \Phi(t)$$

$$\triangleq A\Phi(t), \quad (4.2)$$

and $\Phi(0) = I$, the identity matrix. The linearization of the resetting impulsive conditions of (2.2) becomes

$$\begin{pmatrix} u_1(n\tau^+) \\ u_2(n\tau^+) \\ u_3(n\tau^+) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} u_1(n\tau) \\ u_2(n\tau) \\ u_3(n\tau) \end{pmatrix}.$$

Thus, the monodromy matrix of (4.1) is

$$M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \Phi(\tau).$$

From (4.2), we have $\Phi(\tau) = \Phi(0) \exp(\int_0^\tau A dt) \triangleq \Phi(0) \exp(\bar{A})$. Let μ_1, μ_2 and μ_3 be eigenvalues of matrix \bar{A} , then

$$\begin{aligned} \mu_1 &= \frac{a_1 d \tau - p}{d}, & \mu_2 &= \frac{a_2 d \tau - p}{d}, \\ \mu_3 &= -d\tau < 0. \end{aligned}$$

Therefore, all eigenvalues of M , namely, $\exp(\mu_1)$, $\exp(\mu_2)$, and $\exp(\mu_3)$ have absolute values less than one if and only if $\tau \leq \min\{\frac{p}{a_1 d}, \frac{p}{a_2 d}\}$. According to Floquet’s theory of impulsive differential equations, the two-pest eradication solution $(0, 0, x_3^*(t))$ is locally stable. This completes the proof. \square

In the following, we show that all solutions of (2.2) are uniformly upper bounded.

Theorem 4.2 *There exists a constant $M > 0$, such that $x_i(t) \leq M$ ($i = 1, 2, 3$) for each solution $(x_1(t), x_2(t), x_3(t))$ of system (2.2) with all t large enough.*

Proof Define a function $V(t, x)$ as

$$V(t, x) = m_1 x_1(t) + m_2 x_2(t) + x_3(t).$$

It is clear that $V \in V_0$. We calculate the upper right derivative of $V(t, x)$ along a solution of system (2.2) and get the following impulsive differential equation:

$$\begin{cases} D^+ V(t) + \lambda V(t) \\ = \sum_{i=1}^2 [m_i(a_i + \lambda)x_i - b_i m_i x_i^2] \\ + (\lambda - d)x_3, \\ t \neq n\tau, \\ V(t^+) = V(t) + p, \quad t = n\tau. \end{cases} \quad (4.3)$$

Let $0 < \lambda < d$, then $\sum_{i=1}^2 [m_i(a_i + \lambda)x_i - b_i m_i x_i^2] + (\lambda - d)x_3$ is bounded. Select λ_0 and λ_1 such that

$$\begin{cases} D^+ V(t) \leq -\lambda_0 V(t) + \lambda_1, & t \neq n\tau, \\ V(t^+) = V(t) + p, & t = n\tau, \end{cases}$$

where λ_0, λ_1 are two positive constants. According to Lemma 3.2, we have

$$\begin{aligned} V(t) &\leq \left(V(0^+) - \frac{\lambda_1}{\lambda_0} \right) \exp(-\lambda_0 t) \\ &\quad + \frac{p(1 - \exp(-n\lambda_0 \tau))}{\exp(\lambda_0 \tau) - 1} \exp(\lambda_0 \tau) \\ &\quad \times \exp(-\lambda_0(t - n\tau)) + \frac{\lambda_1}{\lambda_0}, \end{aligned}$$

where $t \in (n\tau, (n + 1)\tau]$. Hence,

$$\lim_{t \rightarrow \infty} V(t) \leq \frac{\lambda_1}{\lambda_0} + \frac{p \exp(\lambda_0 \tau)}{\exp(\lambda_0 \tau) - 1}.$$

So $V(t)$ is uniformly ultimately bounded. Hence, by the definition of $V(t)$, it is known that there exists a constant $M > 0$ such that $x_i(t) \leq M$ ($i = 1, 2, 3$) for t large enough. The proof is completed. \square

In the following, let us investigate the permanence of system (2.2).

Theorem 4.3 *System (2.2) is permanent provided that*

$$d > \max \left\{ \frac{m_1 \sqrt{e}}{1 + \sqrt{e^3}}, \frac{m_2 a_2}{1 + w a_2} \right\}$$

and

$$\tau > \max \left\{ \frac{p}{a_1(d - \frac{m_2 a_2}{1 + w a_2})}, \frac{p}{a_2(d - \frac{m_1 \sqrt{e}}{1 + \sqrt{e^3}})} \right\} \triangleq \tau_{\max}.$$

Proof Suppose $x(t)$ is a solution of (2.2) with $x_0 > 0$. From Theorem 4.2, we may assume $x_3(t) \leq M$ and $M > \max\{a_1, a_2\}$ for $t > 0$. Note that

$$x_1'(t) \leq x_1(t)(a_1 - x_1(t)),$$

considering the comparison equation

$$\begin{cases} f'(t) = f(t)(a_1 - f(t)), \\ f(0) = x_1(0), \end{cases}$$

we have $x_1(t) \leq f(t)$ and $f(t) \rightarrow a_1$ as $t \rightarrow \infty$, thus $x_1(t) \leq a_1 + \varepsilon_1$ ($\varepsilon_1 > 0$) for t large enough. Without

loss of generality, we may assume $x_1(t) \leq a_1 + \varepsilon_1$ for $t > 0$. Similarly, one can assume $x_2(t) \leq a_2 + \varepsilon_1$ ($\varepsilon_1 > 0$) for $t > 0$. Let $\bar{k}_3 = \frac{p \exp(-d\tau)}{1 - \exp(-d\tau)} - \varepsilon_3 > 0$, $\varepsilon_3 > 0$. According to Lemmas 3.2 and 3.3, we have $x_3(t) > \bar{k}_3$ for all t large enough. In the following, we want to find $\bar{k}_1 > 0$ and $\bar{k}_2 > 0$ such that $x_i(t) \geq \bar{k}_i$ ($i = 1, 2$) for t large enough. This induction involves two steps.

Step 1. Because of the conditions of the theorem

$$d > \max \left\{ \frac{m_1 \sqrt{e}}{1 + \sqrt{e^3}}, \frac{m_2 a_2}{1 + w a_2} \right\},$$

one can select $k_i > 0$ ($i = 1, 2$) and $\varepsilon > 0$ small enough such that

$$0 < k_1 < \min \left\{ a_1, \frac{1}{\sqrt{e}} \right\}, \quad 0 < k_2 < a_2,$$

$$\delta_1 = \frac{m_1 k_1}{1 + e k_1^2} < d - \Delta_1 - \varepsilon$$

$$\left(\text{where } \Delta_1 \triangleq \frac{m_2 a_2}{1 + w a_2} \right), \quad d - \Delta_1 > 0,$$

$$\delta_2 = \frac{m_2 k_2}{1 + w k_2} < d - \Delta_2 \quad \left(\text{where } \Delta_2 \triangleq \frac{m_1 \sqrt{e}}{1 + \sqrt{e^3}} \right),$$

and

$$\delta_3 = \delta_1 + \delta_2 < d.$$

It will be proved that there exist $t_1, t_2 \in (0, \infty)$ such that $x_i(t_i) \geq k_i$ ($i = 1, 2$), otherwise there will be three cases.

- (i) There exists a $t_2 > 0$, such that $x_2(t_2) \geq k_2$, but $x_1(t) < k_1$ for all $t > 0$.
- (ii) There exists a $t_1 > 0$, such that $x_1(t_1) \geq k_1$, but $x_2(t) < k_2$ for all $t > 0$.
- (iii) $x_1(t) < k_1, x_2(t) < k_2$ for all $t > 0$.

Let us consider the three cases.

Case (i). In this case, by the condition of theorem

$$\tau > \max \left\{ \frac{P}{a_1(d - \frac{m_2 a_2}{1 + w a_2})}, \frac{P}{a_2(d - \frac{m_1 \sqrt{e}}{1 + \sqrt{e^3}})} \right\} \triangleq \tau_{\max},$$

choose $\varepsilon'_1 > 0$ small enough such that

$$\sigma_1 = a_1 \tau - k_1 \tau - \frac{P}{d - \delta_1 - \Delta_1 - \varepsilon} - \varepsilon'_1 \tau > 0.$$

According to the assumption above, it is easy to see

$$x'_3(t) \leq x_3(t)(-d + \delta_1 + \Delta_1 + \varepsilon).$$

By Lemmas 3.2 and 3.3, it follows that $x_3(t) \leq y(t)$ and $y(t) \rightarrow \bar{y}(t)$, where

$$\bar{y}(t) = \frac{p \exp((-d + \delta_1 + \Delta_1 + \varepsilon)(t - n\tau))}{1 - \exp((-d + \delta_1 + \Delta_1 + \varepsilon)\tau)},$$

$$t \in (n\tau, (n + 1)\tau],$$

and $y(t)$ is the solution of the following equation

$$\begin{cases} y'(t) = (-d + \delta_1 + \Delta_1 + \varepsilon)y(t), & t \neq n\tau, \\ y(t^+) = y(t) + p, & t = n\tau, \\ y(0^+) = x_{03}. \end{cases} \quad (4.4)$$

Therefore there exists a $T_1 > 0$ such that

$$\begin{aligned} x_3(t) &\leq y(t) \leq \bar{y}(t) + \varepsilon'_1, \\ x'_1(t) &\geq x_1(t)(a_1 - k_1 - (\bar{y}(t) + \varepsilon'_1)). \end{aligned} \quad (4.5)$$

Let $N_1 \in \mathbb{N}$ and $N_1 \tau > T_1$. Integrating (4.5) on $(n\tau, (n + 1)\tau]$ ($n \geq N_1$), we have

$$\begin{aligned} x_1((n + 1)\tau) &\geq x_1(n\tau) \exp \left(\int_{n\tau}^{(n+1)\tau} [a_1 - k_1 - (\bar{y}(t) + \varepsilon'_1)] dt \right) \\ &= x_1(n\tau) \exp(\sigma_1). \end{aligned}$$

Then

$$\begin{aligned} x_1((N_1 + k)\tau) &\geq x_1(N_1 \tau) \exp(k\sigma_1) \rightarrow \infty, \\ &\text{as } k \rightarrow \infty, \end{aligned}$$

which is a contradiction to the boundedness of $x_1(t)$.

Case (ii). Let ε'_2 small enough such that

$$\sigma_2 = a_2 \tau - k_2 \tau - \frac{P}{d - \delta_2 - \Delta_2} - \varepsilon'_2 \tau > 0.$$

Similar to Case (i), one can find a contradiction to the boundedness of $x_2(t)$.

Case (iii). Using the assumption in Case (iii), it is easy to see

$$\begin{aligned} x'_3(t) &\leq x_3(t) \left(-d + \frac{m_1 k_1}{1 + e k_1^2} + \frac{m_2 k_2}{1 + w k_2} \right) \\ &= x_3(t)(-d + \delta_1 + \delta_2). \end{aligned}$$

In the same manner as Case (i), we can find a contradiction.

From the three cases above, it is concluded that there exist $t_1 > 0, t_2 > 0$ such that $x_1(t_1) \geq k_1$ and $x_2(t_2) \geq k_2$.

Step 2. If $x_1(t) \geq k_1$ for all $t \geq t_1$, then our aim is obtained. Otherwise, if $x_1(t) < k_1$ for some $t \geq t_1$. Setting $t^* = \inf_{t \geq t_1} \{x_1(t) < k_1\}$, then $x_1(t) \geq k_1$ for $t \in [t_1, t^*)$. It is easy to see $x_1(t^*) = k_1$, since $x_1(t)$ is continuous. Suppose $t^* \in [n_1\tau, (n_1 + 1)\tau], n_1 \in N$. Select $n_2, n_3 \in N$ such that

$$n_2\tau > \ln\left(\frac{\varepsilon'_1}{M+p}\right) / (-d + \delta_1 + \Delta_1 + \varepsilon),$$

$$\exp((n_2 + 1)\sigma_3\tau) \exp(n_3\sigma_1) > 1,$$

where $\sigma_3 = a_1 - k_1 - M < 0$. Set $\bar{t} = n_2\tau + n_3\tau$. We claim that there must exist a $t' \in ((n_1 + 1)\tau, (n_1 + 1)\tau + \bar{t})$ such that $x_1(t') \geq k_1$. Otherwise, $x_1(t) < k_1$, for all $t \in ((n_1 + 1)\tau, (n_1 + 1)\tau + \bar{t}]$. Considering (4.4) with $y((n_1 + 1)\tau^+) = x_3((n_1 + 1)\tau^+)$, we have

$$y(t) = \left((y(n_1 + 1)\tau^+) - \frac{p}{1 - \exp((-d + \delta_1 + \Delta_1 + \varepsilon)\tau)} \right) \times \exp((-d + \delta_1 + \Delta_1 + \varepsilon)(t - (n_1 + 1)\tau)) + \bar{y}(t),$$

for $t \in (n\tau, (n + 1)\tau], n_1 + 1 \leq n < n_1 + 1 + n_2 + n_3$. Then

$$|y(t) - \bar{y}(t)| < (M + p) \exp(-(d - \delta_1 - \Delta_1 - \varepsilon)n_2\tau) < \varepsilon'_1,$$

$$x_3(t) \leq y(t) \leq \bar{y}(t) + \varepsilon'_1,$$

for $(n_1 + n_2 + 1)\tau \leq t \leq (n_1 + 1)\tau + \bar{t}$, which implies (4.4) holds on $[(n_1 + n_2 + 1)\tau, (n_1 + 1)\tau + \bar{t}]$. Similar as in Step 1, we have

$$x_1((n_1 + n_2 + 1 + n_3)\tau) \geq x_1((n_1 + n_2 + 1)\tau) \exp(n_3\sigma_1).$$

System (2.2) gives

$$x'_1(t) \geq x_1(a_1 - k_1 - M) = \sigma_3 x_1(t). \tag{4.6}$$

Integrating (4.6) on $(t^*, (n_1 + n_2 + 1)\tau]$, yields

$$x_1((n_1 + n_2 + 1)\tau) \geq k_1 \exp(\sigma_3(n_1 + 1)\tau).$$

Then

$$x_1((n_1 + 1 + n_2 + n_3)\tau) \geq k_1 \exp(\sigma_3(n_1 + 1)\tau) \exp(n_3\sigma_1) > k_1,$$

which is a contradiction. Let $\bar{t} = \inf_{t \geq t^*} \{x_1(t) \geq k_1\}$, then $x_1(\bar{t}) = k_1$ and (4.6) holds for $t \in [t^*, \bar{t}]$. Integrating (4.6) on $[t^*, \bar{t}]$ yields

$$x_1(t) \geq x_1(t^*) \exp(\sigma_3(t - t^*)) \geq \exp k_1((1 + n_2 + n_3)\tau).$$

Let $\bar{k}_1 \triangleq k_1 \exp(\sigma_3(1 + n_2 + n_3)\tau)$. For $t > \bar{t}$ the same argument can be continued since $x_1(\bar{t}) \geq k_1$, hence $x_1(t) \geq \bar{k}_1$ for all $t > \bar{t}$.

Since $x_1(\bar{t}) \geq k_1$, then the same result holds for all $t > t_1$. Thus, we conclude $x_1(t) \geq \bar{k}_1$ for $t > t_1$.

Similarly, we can prove $x_2(t) \geq \bar{k}_2$ for all $t > t_2$. This completes the proof. \square

Remark 4.1 The conditions of Theorem 4.3 is sufficient.

Remark 4.2 From [26] and [15], one knows that the corresponding systems are permanent when the stability of pest-eradication periodic solutions disappears. But for system (2.2), actually this is not the case. For example, let $a_1 = 1.6, a_2 = 3, e = 1, w = 1, d = 0.5, m_1 = 0.9, m_2 = 0.5, b_1 = 1, b_2 = 1, p = 1$, then $\frac{p}{a_1 d} \approx 1.25, \frac{p}{a_2 d} \approx 0.6667$. Choose $\tau = 3$, then $\tau > \max\{\frac{p}{a_1 d}, \frac{p}{a_2 d}\}$. But x_1 is extinct, and x_2 and x_3 are permanent.

Remark 4.3 From [18], it is shown that the extinct critical value τ_{\min} and the permanent value τ_{\max} are the same for the single functional response system. But for the combined system (2.2), both of them are different, and the behaviors of the systems are difficult to tell apart.

Remark 4.4 From the numerical simulations in Fig. 2, one infers that there exists a positive solution to system (2.2).

5 Numerical analysis

Our focus so far has been on the asymptotic stability of the pest-eradication periodic solution and the permanence of the system (2.2). Now we will study the effects of the impulsive period τ on the complexity of the system (2.2).

Let $a_1 = 3.1, a_2 = 3, \omega = e = 1, d = 0.7, m_1 = 0.9$ and $m_2 = 0.8$. Then system (2.1) has four bounded equilibria which are unstable and the system does not have positive equilibrium. By computer simulations, system (2.1) exists a positive periodic solution in R_3^+ , see Fig. 1a and b. Since system (2.1) cannot be solved explicitly and system (2.2) cannot be rewritten as equivalent difference equations, it is difficult to study them analytically.

First of all, we introduce additional predators into the predator-prey system (2.1) and consider the impulsive effect of τ (the control parameter) on the system (2.1) where $p = 2$. The effect τ may be documented by stroboscopically sampling one of the variables over a range of τ values. In Fig. 2a, we have displayed bifurcation diagrams for the population x_2 as τ increases from 1.5 to 8.5 with initial values $(x_{10}, x_{20}, x_{30}) = (2, 3, 2)$. When the impulsive period $\tau < 0.952$, the pest-eradication periodic solution is stable. From Theorem 4.3, d satisfies the condition, and the system is permanent when the impulsive period $\tau > 6.452$. The two conclusions are confirmed by Fig. 2a. When $0.952 < \tau < 6.452$, the dynamic behavior of species x_1, x_2 and x_3 are complicated.

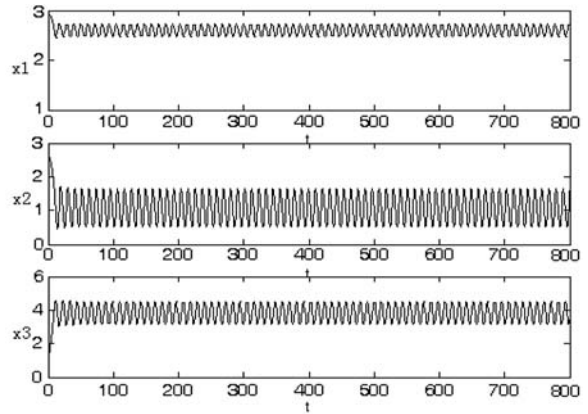
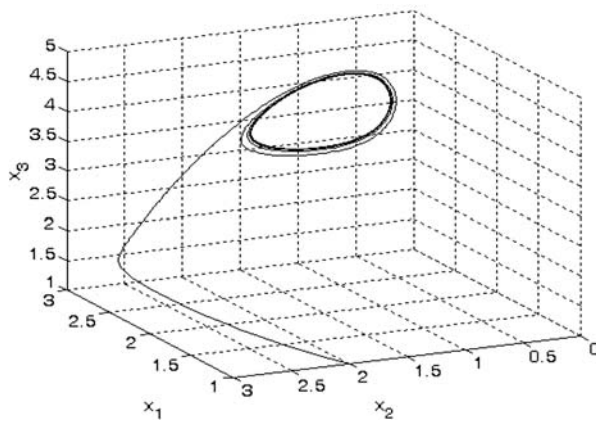


Fig. 1 For system (2.1) exists a periodic solution. **a** Phase portrait of periodic solution. **b** Time series of periodic solution

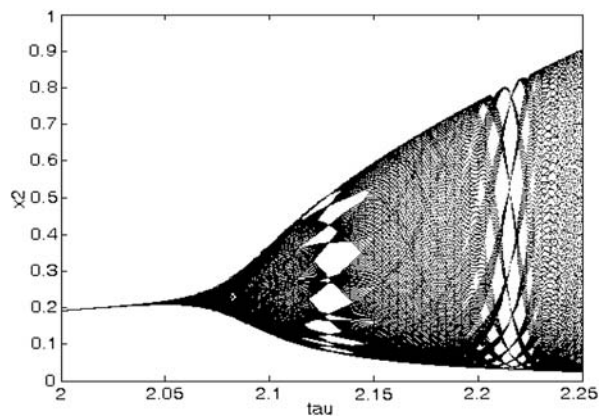
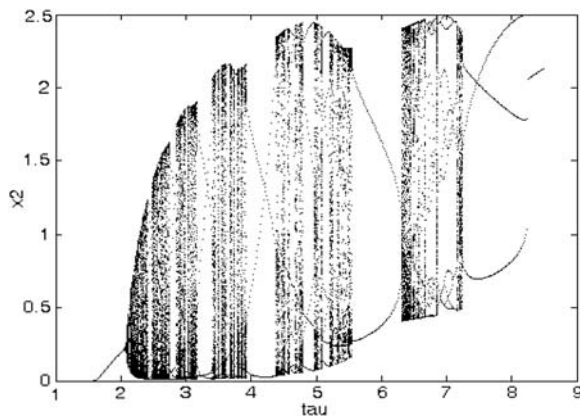


Fig. 2 **a** The bifurcation diagram of population x_2 with respect to τ on [1.5, 8.5] for system (2.2) with initial values (2, 3, 2). **b** An expanded view of **a** with respect to τ on [2, 2.25]. Note that below $\tau = 2.04$ the behavior is periodic. Above $\tau = 2.04$ there are quasi-periodic behavior, period-doublings, intermittency and crises, and in the vicinity of $\tau = 8.2$, the solution from period-3 turns to period-1

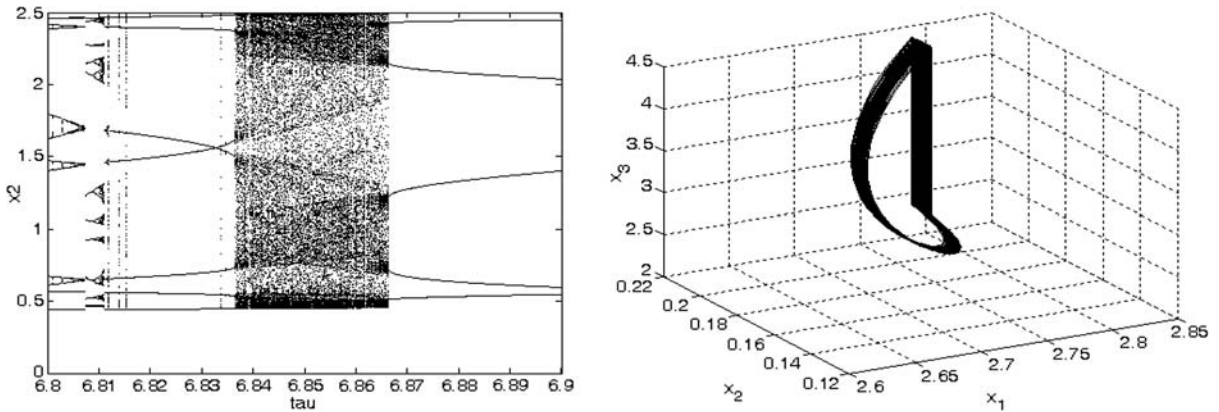


Fig. 3 **a** Expanded view of Fig. 2a with respect to τ on $[6.8, 6.9]$ for system (2.2). **b** First-order quasi-periodic when $\tau = 2.05$, with initial values $(2, 3, 2)$

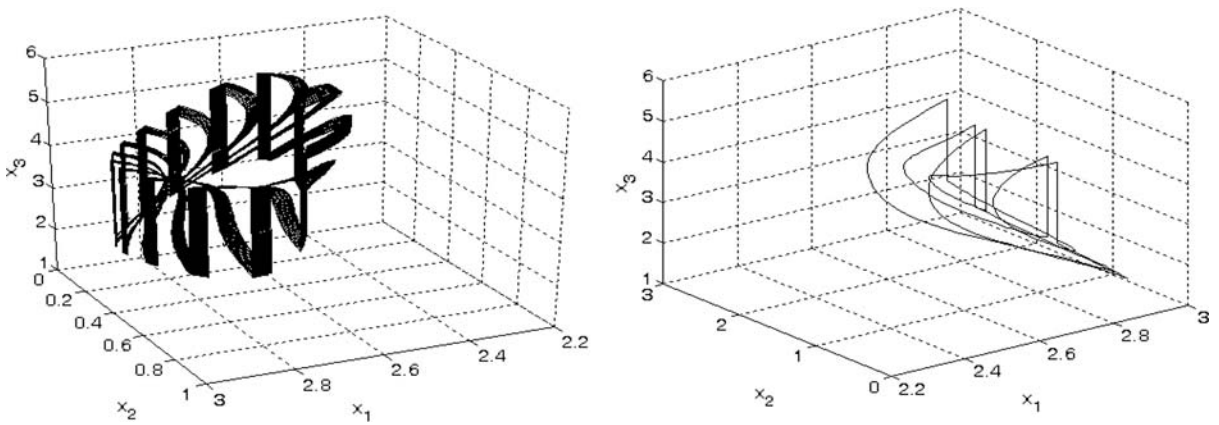


Fig. 4 **a** A high-order quasi-periodic solution for system (2.2) when $\tau = 2.05$, with initial values $(2, 3, 2)$. **b** A period-5 solution when $\tau = 6.88$ with initial values $(2, 3, 2)$

With τ increasing from 1.5 to 8.5, there are some interesting phenomena. Zooming in, we get Figs. 2b and 3a. In both figures, there are four important respects to be illustrated as follows.

First, in Fig. 3a, there is period-5 solution (Fig. 4b) for system (2.2) when $\tau = 6.88$. However, in Fig. 2b, there is a “window” which is composed of “ribbons” between $\tau = 2.2$ and $\tau = 2.25$. What dynamic behavior do those “ribbons” represent? Choosing $\tau = 2.22$, the phase portrait plot shows that the orbit is a high-order quasi-periodic solution (Fig. 4a) which is compared with a first-order quasi-periodic one (Fig. 3b).

Second, we must be careful to distinguish the smear of points that occurs on Fig. 2b due to quasi-periodicity from the smear of points on Fig. 3a in the chaotic regions. One should notice the banded struc-

ture of the quasi-periodic behavior and compare that to the “random scatter” of the chaotic. For example, Fig. 5a shows a high order quasi-periodicity when $\tau = 2.14$, and Fig. 5b shows a strange attractor when $\tau = 6.84$. That is, on bifurcation diagrams, the dynamic behavior of the “black” region is either chaotic or quasi-periodic which can be distinguished by two methods.

On the one hand, one can distinguish them by a skiagraph of the phase portrait. For example, Figs. 6a and b are skiagraphs of Figs. 5a and b on the plane x_1-x_2 , respectively. On the other hand, to verify that the regions on Fig. 3a with a scattering of trajectory points are actually chaotic, we should compute the (average) Lyapunov exponent for the trajectory. The Lyapunov exponent is 0 for quasi-periodic behavior and

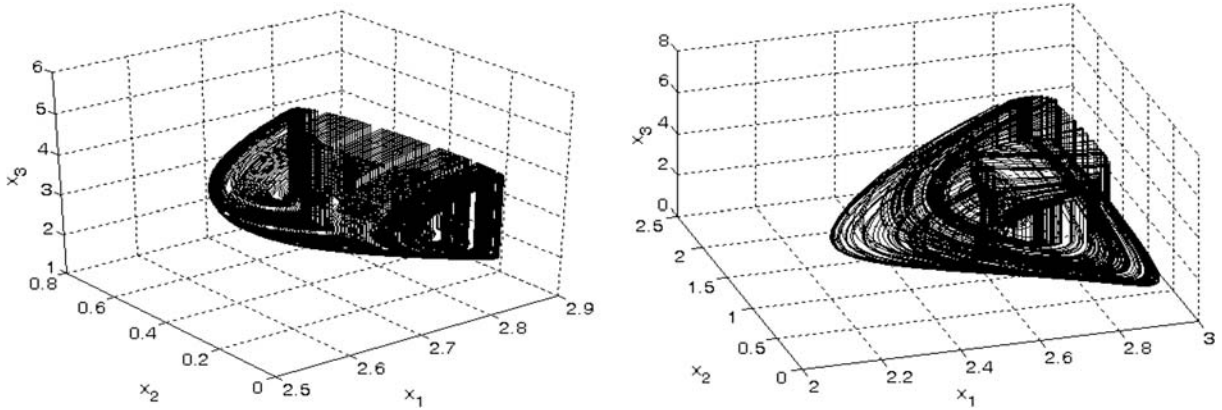


Fig. 5 **a** A high order of quasi-periodicity for system (2.2), where $\tau = 2.14$. **b** A strange attractor, where $\tau = 6.84$, with initial values (2, 3, 2)

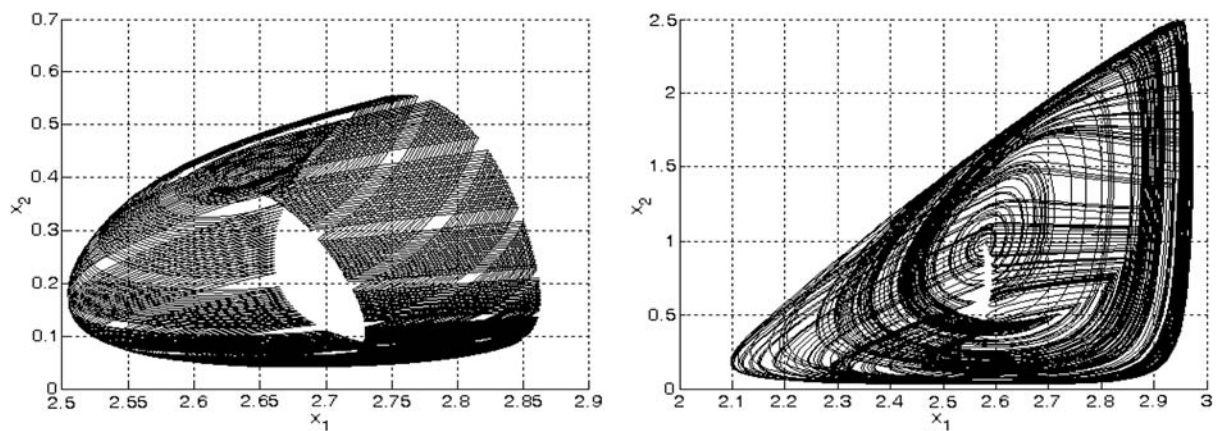


Fig. 6 **a** A skiagraph of Fig. 5a on the plane x_1-x_2 for system (2.2). **b** A skiagraph of Fig. 5b on the plane x_1-x_2

positive for chaotic behavior. In the quasi-periodic scenario, the system begins again with a limit cycle trajectory. As the control parameter changes, a second periodicity appears in the behavior of the system. If the ratio of the period of the second type of motion to the period of the first is not rational, then the motion is quasi-periodic. Under some circumstances, if the control parameter is changed further, the motion becomes chaotic. However, the quasi-periodic route to chaos is likely to lead to chaotic behavior, but not necessarily.

Finally, we remark on intermittency and crises. The intermittency route to chaos is characterized by dynamics with irregularly occurring bursts of chaotic behavior interspersed with intervals of apparently periodic behavior. A crisis is a bifurcation event in which a chaotic attractor and its basin of attraction suddenly

disappear or suddenly change in size as some control parameter is adjusted.

6 Discussion

In this work, by introducing impulsive effect concerning biological control strategy, we investigate the dynamic behaviors of a two-prey one-predator model in which one prey has defensive ability and another prey has a Holling type-II functional response to the predator. By using Floquet's theorem and the small amplitude perturbation method, we prove that there exists an asymptotically stable pest-eradication periodic solution when the impulsive period is less than some critical value τ_{\min} , and permanence conditions are established via the method of comparison involving mul-

tiple Liapunov functions when the impulsive period is greater than some critical value τ_{\max} . With the increase of impulsive period τ , the system has a unique positive solution. In the following, let us compare the results of system (2.1) with systems (2.2).

System (2.1) is a model with a classical biological control technique. From Remarks 2.1 and 2.2, it can be shown that the pest eradication equilibrium $(0, 0, x_{3+})$ doesn't exist, and the equilibrium $(0, 0, 0)$ is unstable, which means that one cannot eradicate pests steadily. However, from the impulsive control system (2.2), the pest can be eradicated provided that the impulsive period τ is less than a threshold $\min\{\frac{p}{a_1d}, \frac{p}{a_2d}\}$, where p is the release amount of the natural enemy, which can be controlled in advance. So it is shown that the impulsive control strategy is more effective than the classical one.

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