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Generalized solitonary and periodic solutions for nonlinear partial differential equations by the Exp-function method

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Abstract In this paper, the Exp-function method with the aid of the symbolic computational system Maple is used to obtain the generalized solitonary solutions and periodic solutions for nonlinear evolution equations arising in mathematical physics, namely, $(2+1)$-dimensional Konopelchenko–Dubrovsky equations, the $(3+1)$-dimensional Jimbo–Miwa equation, the Kadomtsev–Petviashvili (KP) equation, and the $(2+1)$-dimensional sine-Gordon equation. It is shown that the Exp-function method, with the help of symbolic computation, provides a powerful mathematical tool for solving other nonlinear evolution equations arising in mathematical physics.

Keywords Exp-function method · Nonlinear evolution equations · New solitons and periodic solutions

1 Introduction

A large variety of physical, chemical, and biological phenomena are governed by nonlinear evolution equations. The analytical study of nonlinear partial differential equations was of great interest during the last decade. The investigations of the traveling wave solution of nonlinear equations play an important role in the study of nonlinear physical phenomena. The importance of obtaining the exact solutions, if available, of those nonlinear equations facilitates the verification of numerical solutions and aids in the stability analysis of solutions.

Exact traveling wave solutions of nonlinear evolution equations is one of the fundamental areas of study in mathematical physics. These exact solutions when they exist can help one to understand the mechanism of complicated physical phenomena and dynamical processes modeled by these nonlinear evolution equations.

Many effective methods [1–13] have been presented such as the variational iteration method [2, 6, 12], the homotopy perturbation method [3], the F-expansion method [5], as well as others. A complete review of the field is available on [4]. Very recently, He and Wu [14] proposed a straightforward and concise method called the Exp-function method to obtain generalized solitonary solutions and periodic solutions. Applications of the method can be found in [15, 16] for solving nonlinear evolution equations arising in mathematical physics. The solution procedure of this method, with the aid of Maple, is very simple and can easily extended to other kinds of nonlinear evolution equations.

The aim of this paper is to extend the Exp-function method to finding new solitonary solutions, compact-like solutions, and periodic solutions for nonlinear

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evolution equations in mathematical physics. To illustrate the basic idea of the Exp-function method, we consider (2 + 1)-dimensional Konopelchenko–Dubrovsky equation [17–19]

$$u_t - u_{xxx} - 6buu_x + (3/2)a^2u^2u_x - 3v_y + 3avu_x = 0, \quad u_y = v_x, \tag{1}$$

where a and b are constants. Equation (1) is a new nonlinear integrable evolution equation on two spatial dimensions and one temporal. Introducing a complex variation η defined as $\eta = kx + ly + wt$, equation (1) becomes an ordinary differential equation which reads

$$\left[w - \frac{3l^2}{k}\right]u - k^3u'' + \left[\frac{3al}{2} - 3bk\right]u^2 + \frac{a^2}{2}ku^3 = 0, \quad lu = kv, \tag{2}$$

where k, l, and w are constants to be determined later and the prime denotes the differential with respect to η.

In view of the Exp-function method, we assume that the solution of (2) can be expressed in the form

$$u(\eta) = \frac{\sum_{n=-c}^{d} a_n \exp(n\eta)}{\sum_{m=-p}^{q} b_m \exp(m\eta)}, \tag{3}$$

where c, d, p and q are positive integers which are unknown to be determined later, and a_n and b_m are unknown constants. Equation (3) can be rewritten in an alternative form as follows:

$$u(\eta) = \frac{a_c \exp(c\eta) + \cdots + a_{-d} \exp(-d\eta)}{b_p \exp(p\eta) + \cdots + b_{-q} \exp(-q\eta)}. \tag{4}$$

In order to determine values of c and p, we balance the linear term of the highest order in (2) with the highest order nonlinear term u^3 and u'', giving us

$$u^3 = \frac{c_1 \exp[(3c + 3p)\eta] + \cdots}{c_2 \exp[6p\eta] + \cdots}, \tag{5}$$

$$u'' = \frac{c_3 \exp[(c + 5p)\eta] + \cdots}{c_4 \exp[6p\eta] + \cdots}, \tag{6}$$

where c_i are coefficients for simplicity. By balancing the highest order of the Exp-function in (5) and (6), we have

$$c + 5p = 3c + 3p, \tag{7}$$

which leads to the limit

$$p = c. \tag{8}$$

Proceeding in the same manner as illustrated above, we can determine values of d and q. Balancing the linear term of lowest order in (2),

$$u^3 = \frac{\cdots + d_1 \exp[-(q + 5d)\eta]}{\cdots + d_2 \exp[-6q\eta]}, \tag{9}$$

$$u'' = \frac{\cdots + d_3 \exp[-(3q + 3d)\eta]}{\cdots + d_4 \exp[-6q\eta]}, \tag{10}$$

where d_i are determined coefficients only for simplicity, we have

$$-[5d + q] = -[3d + 3q],$$

which leads to the result

$$q = d. \tag{11}$$

Case (1): $p = c = 1$ and $d = q = 1$

We can freely choose the values of c and d, but we will illustrate that the final solution does not strongly depend upon the choice of values of c and d. For simplicity, we set $p = c = 1$ and $d = q = 1$, so that the trial function, (4) becomes

$$u(\eta) = \frac{a_1 \exp(\eta) + a_0 + a_{-1} \exp(-\eta)}{b_1 \exp(\eta) + b_0 + b_{-1} \exp(-\eta)}. \tag{12}$$

Substituting (12) into (2), equating to zero the coefficients of all powers of $\exp(n\eta)$ yields a set of algebraic equations for a_0, b_0, a_1, a_{-1}, b_{-1}, k, l and w. Solving the system of algebraic equations with the aid of Maple, we obtain

$$\begin{aligned} a_1 &= a_1, \qquad a_0 = \frac{a_0b_0(a^2a_1^2 - 4k^2)}{a^2a_1^2 + 2k^2}, \\ l &= \frac{2(a^2a_1^2k - 3bka_1 - k^3)}{3aa_1}, \\ w &= \frac{-a^2a_1^2k^2 + 6bk^2a_1 - 3aa_1kl + 6l^2}{2k}, \\ a_{-1} &= \frac{a_1b_0^2(a^4a_1^4 - 5a^2a_1^2k^2 + 4k^4)}{4(a^2a_1^2 + 2k^2)^2}, \\ b_{-1} &= \frac{b_0^2(a^4a_1^4 - 5a^2a_1^2k^2 + 4k^4)}{4(a^2a_1^2 + 2k^2)^2}, \end{aligned} \tag{13}$$

where $b_1 = 1$ and b_0 are constants. Inserting (13) into (12) admits the following generalized solitonary solution of (1):

$$u(x,t) = \left[a_1 \exp(\eta) + \frac{a_0 b_0 (a^2 a_1^2 - 4k^2)}{a^2 a_1^2 + 2k^2} + \frac{a_1 b_0^2 (a^4 a_1^4 - 5a^2 a_1^2 k^2 + 4k^4)}{4(a^2 a_1^2 + 2k^2)^2} \exp(-\eta)\right] \times \left[\exp(\eta) + b_0 + \frac{b_0^2 (a^4 a_1^4 - 5a^2 a_1^2 k^2 + 4k^4)}{4(a^2 a_1^2 + 2k^2)^2} \exp(-\eta)\right]^{-1},$$
$$\eta = kx + wt + ly. \tag{14}$$

Case (2): $p = c = 2$ and $d = q = 2$

As mentioned above, the values of c and d can be freely chosen. We set $p = c = 2$ and $d = q = 2$, then trial function (4) becomes

$$u(\eta) = \left[a_1 \exp(\eta) + a_2 \exp(2\eta) + a_0 + a_{-1} \exp(-\eta) + a_{-2} \exp(-2\eta)\right] \times \left[\exp(2\eta) + b_1 \exp(\eta) + b_0 + b_{-1} \exp(-\eta) + b_{-2} \exp(-2\eta)\right]^{-1}. \tag{15}$$

In (15) there are some parameters, so we set $b_{-1} = b_1 = 0$ for simplicity, and the trial function is simplified as follows:

$$u(\eta) = \left[a_1 \exp(\eta) + a_2 \exp(2\eta) + a_0 + a_{-1} \exp(-\eta) + a_{-2} \exp(-2\eta)\right] \times \left[\exp(2\eta) + b_0 + b_{-2} \exp(-2\eta)\right]^{-1}. \tag{16}$$

Substituting (16) into (2) and equating to zero the coefficients of all powers of $\exp(n\eta)$ yields a set of algebraic equations for $a_1, a_{-1}, a_0, a_2, a_{-2}, b_0, b_{-2}, k, l$, and w. Solving the system of algebraic equations with the aid of Maple, we obtain

$$a_2 = \frac{k}{a}, \qquad a_1 = a_1, \qquad a_0 = \frac{a^2 a_1 + 2b_0 k^2}{2ak},$$
$$a_{-1} = \frac{a_1 (a_1^2 a^2 + 4b_0 k^2)}{4k^2}, \qquad l = \frac{2bk}{a},$$
$$a_{-2} = \frac{a a_1^2 (a_1^2 a^2 + 4b_0 k^2)}{16k^2}, \tag{17}$$
$$b_{-2} = -\frac{a^2 a_1^2 (a_1^2 a^2 + 4b_0 k^2)}{16k^4},$$
$$w = -\frac{k(-24b^2 + a^2 k^2)}{2a^2},$$

where b_0 and a_1 are free parameters.

In case k, w, and l are imaginary numbers, the obtained solitonary solution (17) reduces to the periodic solution or compact-like solution. We write $k = iK$, $w = i\beta$, and $l = i\alpha$ and using the following transformation

$$\exp\left(\pm i[2Kx + 2\alpha y + 2\beta t]\right) = \cos[2Kx + 2\alpha y + 2\beta t] \pm i \sin[2Kx + 2\alpha y + 2\beta t], \tag{18}$$

the new periodic solution of (1) follows:

$$u(x,t) = \left[[a_2 + a_{-2}] \cos[2Kx + 2\alpha y + 2\beta t] + [a_1 + a_{-1}] \cos[Kx + \alpha y + \beta t] + a_0\right] \times \left[[1 + b_{-2}] \cos[2Kx + 2\alpha y + 2\beta t] + [b_1 + b_{-1}] \cos[Kx + \alpha y + \beta t] + b_0\right]^{-1}, \tag{19}$$

$$\alpha = \frac{-2bK}{a}, \tag{20}$$

$$\beta = -\frac{-K(-24b^2 - a^2 K^2)}{2a^2}, \tag{21}$$

with a_2, a_0, a_{-1}, a_{-2} and b_{-2} given by (17).

So by means of the Exp-function method, we obtain the generalized solitonary solution and a periodic solution for nonlinear evolution equations arising in mathematical physics. To illustrate its effectiveness and convenience, we consider in the next section three nonlinear equations, namely, the $(3 + 1)$-dimensional Jimbo–Miwa equation, the K-P equation, and the $(2 + 1)$-dimensional sine-Gordon equation.

2 New applications

2.1 The $(2 + 1)$-dimensional Jumbo–Miwa equation

Consider the $(2 + 1)$-dimensional Jimbo–Miwa equation [20] in the form

$$u_{xxxy} + 3(uu_y)_x + 3u_{xx}\partial_x^{-1} u_y + 3u_x u_y + 3u_{yt} - 3u_{zz} = 0. \tag{22}$$

Using the transformation $\eta = kx + dy + ez + wt$, then (22) becomes

$$k^3\, du'''' + 6\, dk\, uu'' + 6k\, du'^2 + 3lu'' = 0, \tag{23}$$

where $l = dw - e^2$, k, d, e, and w are constants to be determined later.

According to the Exp-function method, we assume that the solution of (23) can be expressed in the following form

$$u(\eta) = \frac{\sum_{n=-c}^{d} a_n \exp(n\eta)}{\sum_{m=-p}^{q} b_m \exp(m\eta)}, \tag{24}$$

where c, d, p, and q are unknown positive integers to be determined later, and a_n and b_m are unknown constants. Equation (4) can be rewritten in an alternative form as follows:

$$u(\eta) = \frac{a_c \exp(c\eta) + \cdots + a_{-d} \exp(-d\eta)}{b_p \exp(p\eta) + \cdots + b_{-q} \exp(-q\eta)}. \tag{25}$$

To determine the values of c and p, we balance the linear term of the highest order in (23) with the highest-order nonlinear term u'''' and u'^2 to get

$$u'^2 = \frac{c_3 \exp[2(c+7p)\eta] + \cdots}{c_4 \exp[16p\eta] + \cdots}, \tag{26}$$

$$u'''' = \frac{c_1 \exp[(c+15p)\eta] + \cdots}{c_2 \exp[16p\eta] + \cdots}. \tag{27}$$

Balancing the highest order Exp-functions in (26) and (27), we have

$$15p + c = 14p + 2c, \tag{28}$$

which leads to the result

$$p = c. \tag{29}$$

Similarly, we determine the values of d and q. Balancing the linear term of lowest order in (23),

$$u'^2 = \frac{d_3 \exp[-2(7q+d)\eta] + \cdots}{d_4 \exp[-16q\eta] + \cdots}, \tag{30}$$

$$u'''' = \frac{d_1 \exp[-(15q+d)\eta] + \cdots}{d_2 \exp[-16q\eta] + \cdots}, \tag{31}$$

where d_i are determined coefficients only for simplicity, we have

$$-2[7q + d] = -[d + 15q],$$

which leads to the result

$$q = d. \tag{32}$$

Case (1): $p = c = 1$ and $d = q = 1$

We can freely choose the values of c and d, but we will illustrate that the final solution does not strongly depend upon the choice of values of c and d. For simplicity, we set $p = c = 1$ and $d = q = 1$. The trial function (25) becomes

$$u(\eta) = \frac{a_1 \exp(\eta) + a_0 + a_{-1} \exp(-\eta)}{\exp(\eta) + b_0 + b_{-1} \exp(-\eta)}. \tag{33}$$

Substituting (33) into (23), equating to zero the coefficients of all powers of $\exp(n\eta)$ yields a set of algebraic equations for $a_0, b_0, a_1, a_{-1}, b_{-1}, k, d, e$, and w. Solving the system of algebraic equations, we obtain

$$\begin{aligned}
&b_{-1} = (1/4)b_0^2, \qquad a_{-1} = a_{-1},\\
&a_1 = \frac{4a_{-1}}{b_0^2}, \qquad a_0 = \frac{k^2 b_0^2 + 4a_{-1}}{b_0},\\
&k = k, \qquad d = d, \qquad e = e, \qquad b_0 = b_0,\\
&w = \frac{1}{3}\frac{24kda_{-1} + k^3 db_0^2 - 3b_0^2 e^2}{db_0^2},
\end{aligned} \tag{34}$$

where b_0 and a_{-1} are free parameters. Inserting (34) into (33) yields the following generalized solitonary solution of (22):

$$\begin{aligned}
&u(x,t) = \frac{\frac{4a_{-1}}{b_0^2} e^{[\eta]} + \frac{k^2 b_0^2 + 4a_{-1}}{b_0} + a_{-1} e^{[-\eta]}}{e^{[\eta]} + b_0 + \frac{1}{4b_0^2} e^{-[\eta]}},\\
&\eta = kx + dy + ez + wt.
\end{aligned} \tag{35}$$

When k, d, e, and w are imaginary numbers, the obtained solitonary solution (35) can be converted into a periodic solution or compact-like solution. We write $k = iK$, $d = iD$, $e = iE$, and $w = i\alpha$, and using the transformation

$$\begin{aligned}
&\exp\bigl(\pm i[Kx + Dy + Ez + \alpha t]\bigr)\\
&\quad = \cos[Kx + Dy + Ez + \alpha t]\\
&\qquad \pm i \sin[Kx + Dy + Ez + \alpha t],
\end{aligned}$$

then (35) yields

$$u(x,t)=\Big[a_{-1}\big[1+(1/4)b_0^2\big] \times \cos[Kx+Dy+Ez+\alpha t] + \frac{-K^2b_0^2+4a_{-1}}{b_0}+a_{-1}\big[\big(4/b_0^2\big) - 1\big]i\sin[Kx+Dy+Ez+\alpha t]\Big] \times \big[\big[1+(1/4)b_0^2\big] \times \cos[Kx+Dy+Ez+\alpha t] + b_0+\big[1-(1/4)b_0^2\big] \times i\sin[Kx+Dy+Ez+\alpha t]\big]^{-1}. \quad (36)$$

To look for a periodic solution, the imaginary part in (36) must be zero, requiring that

$$b_0=2. \quad (37)$$

Inserting (37) into (36), we obtain a periodic solution which reads

$$u(x,t)=\big[2a_{-1}\cos[Kx+Dy+Ez+\alpha t] + 2\big(a_{-1}-K^2\big)\big] \times \big[2\cos[Kx+Dy+Ez+\alpha t]+b_0\big]^{-1},$$
$$\alpha=\frac{1}{3}\frac{-24KDa_{-1}-K^3Db_0^2-3b_0^2E^2}{Db_0^2}. \quad (38)$$

Case (2): $p=c=2$ and $d=q=2$

As mentioned above, the values of c and d can be freely chosen. If we set $p=c=2$ and $d=q=2$, then the trial function (25) becomes

$$u(\eta)=\big[a_1\exp(\eta)+a_2\exp(2\eta)+a_0 + a_{-1}\exp(-\eta)+a_{-2}\exp(-2\eta)\big] \times \big[\exp(2\eta)+b_1\exp(\eta)+b_0 + b_{-1}\exp(-\eta)+b_{-2}\exp(-2\eta)\big]^{-1}. \quad (39)$$

In (39) there are some parameters, and so we set $b_{-1}=b_1=0$ for simplicity. The trial function is simplified as follows:

$$u(\eta)=\big[a_1\exp(\eta)+a_2\exp(2\eta)+a_0 + a_{-1}\exp(-\eta)+a_{-2}\exp(-2\eta)\big] \times \big[\exp(2\eta)+b_0+b_{-2}\exp(-2\eta)\big]^{-1}. \quad (40)$$

By the same manipulation illustrated above, we determine the coefficients

$$a_{-2}=\frac{1}{16}\frac{a_2a_1^4}{k^8},\qquad b_{-2}=\frac{1}{16}\frac{a_1^4}{k^8},$$
$$a_{-1}=\frac{1}{4}\frac{a_1^3}{k^4},\qquad a_0=\frac{a_1^2(a_2+2k^2)}{2k^4},$$
$$b_0=\frac{1}{2}\frac{a_1^2}{k^4},\qquad k=k,\qquad d=d,\qquad e=e, \quad (41)$$
$$a_2=a_2,\qquad a_1=a_1,$$
$$w=\frac{-6kda_2-k^3d+3e^2}{3d},$$

where a_2 and a_1 are free parameters. Substituting (41) into (40) yields the following generalized solitonary solution of (22):

$$u(x,t)=\Big[a_2e^{[2\eta]}+a_1e^{[\eta]}-\frac{a_1^2(a_2+2k^2)}{2k^4} + \frac{1}{4}\frac{a_1^3}{k^4}e^{[-\eta]}+\frac{1}{16}\frac{a_2a_1^4}{k^8}e^{[-2\eta]}\Big] \times \Big[\frac{1}{16}\frac{a_1^4}{k^8}e^{[-2\eta]}+e^{[2\eta]}-\frac{1}{2}\frac{a_1^2}{k^4}\Big]^{-1}. \quad (42)$$

It should be noted that if we set $k=iK$, $d=iD$, $e=iE$, and $w=i\alpha$, eliminating the imaginary part, we obtain the periodic solution of (22) in the form

$$u(x,t)=\Big[a_2\Big(1+\frac{a_1^4}{16K^8}\Big)\cos[2Kx+2Dy + 2Ez+2\alpha t]+a_1\Big(1+\frac{a_1^2}{4K^4}\Big) \times \cos[Kx+Dy+Ez+\alpha t] - \frac{a_1^2(a_2-2K^2)}{2K^4}\Big]\Big[\Big(1+\frac{a_1^4}{16K^8}\Big) \times \cos[2Kx+2Dy+2Ez+2\alpha t] - \frac{a_1^2}{2K^4}\Big]^{-1},$$
$$\alpha=\frac{6KDa_2-K^3D-3E^2}{3D}. \quad (43)$$

Case (3): $p=c=2$ and $d=q=1$

We consider the case $p=c=2$ and $d=q=1$. (25) can then be expressed as

$$u(\eta)=\frac{a_2\exp(2\eta)+a_1\exp(\eta)+a_0+a_{-1}\exp(-\eta)}{\exp(2\eta)+b_0+b_1\exp(\eta)+b_{-1}\exp(-\eta)}. \tag{44}$$

By a simple calculation using Maple, we obtain the coefficients a_0, a_{-1}, b_0, w, k, a_2, d, e, and b_{-1} (see Appendix A). For simplicity's sake, we omit this calculation here. In this case, we obtain the periodic wave solution of (22) as follows:

$$\begin{aligned}u(x,t)=\big[&a_2\cos[2Kx+2Dy+2Ez+2\alpha t]\\&+[a_1+a_{-1}]\cos[2Kx+2Dy+2Ez\\&+2\alpha t]+a_0\big]\\&\times\big[\cos[2Kx+2Dy+2Ez+2\alpha t]\\&+b_0+[b_1+b_{-1}]\cos[Kx+Dy\\&+Ez+\alpha t]\big]^{-1},\end{aligned} \tag{45}$$

where a_0, a_{-1}, b_0, and b_{-1} are given by (A.1) and a_2, a_1, and b_1 are constants.

2.2 (2 + 1)-dimensional sine-Gordon equation

The (2 + 1)-dimensional sine-Gordon equation [21] reads

$$\begin{aligned}&u_t+u_{xxx}+u_{yyy}+3\big(u\partial_y^{-1}u_x\big)_x\\&\quad+3\big(u\partial_x^{-1}u_y\big)_y=0.\end{aligned} \tag{46}$$

By introducing a complex variation $\eta=kx+cy+wt$, then (46) reduces to

$$kcwu'+\big[k^4c+kc^4\big]u'''+\big[6k^3+6c^3\big]uu'=0. \tag{47}$$

According to the Exp-function method, we assume that the solution of (47) is as follows:

$$u(\eta)=\frac{\sum_{n=-c}^{d}a_n\exp(n\eta)}{\sum_{m=-p}^{q}b_m\exp(m\eta)}, \tag{48}$$

where c, d, p, and q are unknown positive integers to be determined later, and a_n and b_m are unknown constants. Equation (48) can be rewritten in an alternative form as follows:

$$u(\eta)=\frac{a_c\exp(c\eta)+\cdots+a_{-d}\exp(-d\eta)}{b_p\exp(p\eta)+\cdots+b_{-q}\exp(-q\eta)}. \tag{49}$$

In order to determine values of c and p, we balance the linear term of the highest order in (47) with the highest order nonlinear term u''' and uu', yielding

$$uu'=\frac{c_3\exp[(2c+3p)\eta]+\cdots}{c_4\exp[5p\eta]+\cdots}, \tag{50}$$

$$u'''=\frac{c_1\exp[(c+4p)\eta]+\cdots}{c_2\exp[5\eta]+\cdots}. \tag{51}$$

By the balancing highest order of the Exp-function in (50) and (51), we have

$$c+4p=3p+2c, \tag{52}$$

which leads to the limit

$$p=c. \tag{53}$$

As mentioned above, we can determine values of d and q. Balancing the linear term of lowest order in (47),

$$uu'=\frac{d_3\exp[-(q+4d)\eta]+\cdots}{d_4\exp[-5q\eta]+\cdots}, \tag{54}$$

$$u'''=\frac{d_1\exp[-(2q+3d)\eta]+\cdots}{d_2\exp[-5\eta]+\cdots}, \tag{55}$$

where d_i are coefficients determined only for simplicity. We have

$$-[q+4d]=-[2q+3d],$$

which leads to the limit

$$q=d. \tag{56}$$

Case (1): $p=c=1$ and $d=q=1$

For simplicity, we set $p=c=1$ and $d=q=1$. The trial function (49) becomes

$$u(\eta)=\frac{a_1\exp(\eta)+a_0+a_{-1}\exp(-\eta)}{\exp(\eta)+b_0+b_{-1}\exp(-\eta)}. \tag{57}$$

Substituting (57) into (47) and equating to zero the coefficients of all powers of $\exp(n\eta)$ yields a set of algebraic equations for a_0, b_0, a_1, a_{-1}, b_{-1}, k, c, and w. Solving the system of algebraic equations, we obtain

$$\begin{aligned}&c=c,\qquad k=k,\qquad a_1=a_1,\qquad a_{-1}=\frac{1}{4}a_1b_0^2,\\&a_0=b_0kc+b_0a_1,\\&w=-\frac{6c^3a_1+c^4k+k^4c+6k^3a_1}{kc},\end{aligned} \tag{58}$$

where a_1 and b_0 are free parameters. Inserting (58) into (57) admits the following generalized solitonary solution of (46):

$$u(x,t)=\frac{a_1e^{[\eta]}+b_0kc+b_0a_1+(1/4)a_1b_0^2e^{[-\eta]}}{e^{[\eta]}+b_0+(1/4)b_0^2e^{[-\eta]}},\quad \eta=kx+cy+wt. \tag{59}$$

In case k, w, and c are imaginary numbers, the obtained solitonary solution (59) reduces to the periodic solution. We write $k=iK$, $c=iC$, and $w=i\alpha$ and using the transformation

$$\exp(\pm i[Kx+\alpha t+Cy]) = \cos[Kx+\alpha t+Cy]\pm i\sin[Kx+\alpha t+Cy],$$

obtain the periodic solution of (46) in the following form:

$$\begin{aligned} u(x,t)&=\big[a_1[1+(1/4)b_0^2]\cos[Kx+\alpha t+Cy] - b_0KC+a_1b_0\big] \\ &\quad\times\big[[1+(1/4)b_0^2]\cos[Kx+\alpha t+Cy] + b_0\big]^{-1}, \\ \alpha&=-\frac{6C^3a_1-C^4K+K^4c+6K^3a_1}{KC}. \end{aligned} \tag{60}$$

Case (2): $p=c=2$ and $d=q=2$

As mentioned above, the values of c and d can be freely chosen. We set $p=c=2$ and $d=q=2$, so that the trial function (49) becomes

$$\begin{aligned} u(\eta)&=\big[a_1\exp(\eta)+a_2\exp(2\eta)+a_0+a_{-1}\exp(-\eta) + a_{-2}\exp(-2\eta)\big] \\ &\quad\times\big[\exp(2\eta)+b_1\exp(\eta)+b_0 + b_{-1}\exp(-\eta)+b_{-2}\exp(-2\eta)\big]^{-1}. \end{aligned} \tag{61}$$

In (61) there are some parameters, so we set $b_{-1}=b_1=0$ for simplicity. The trial function is simplified as follows:

$$\begin{aligned} u(\eta)&=\big[a_1\exp(\eta)+a_2\exp(2\eta)+a_0 + a_{-1}\exp(-\eta)+a_{-2}\exp(-2\eta)\big] \\ &\quad\times\big[\exp(2\eta)+b_0+b_{-2}\exp(-2\eta)\big]^{-1}. \end{aligned} \tag{62}$$

By the same manipulation as illustrated above, we obtain

$$\begin{aligned} &c=c,\qquad k=k, \\ &a_1=\sqrt{-2b_0}kc,\qquad a_{-1}=\frac{1}{2}\sqrt{-2b_0}kcb_0, \\ &b_{-2}=\frac{1}{4}b_0^2,\qquad a_{-2}=\frac{1}{4}a_2b_0^2, \\ &a_0=b_0(a_2+2kc), \\ &w=-\frac{(kc^4+6k^3a_2+6c^3a_2+k^4c)}{kc}, \end{aligned} \tag{63}$$

where a_2 and b_0 are free parameters. Substituting (63) into (62) yields

$$\begin{aligned} u(x,t)&=\Big[a_2e^{[2\eta]}+\sqrt{-2b_0}kce^{[2\eta]}+b_0(a_2+2kc) \\ &\quad-\frac{1}{2}\sqrt{-2b_0}kcb_0e^{[-\eta]}+\frac{1}{4}a_2b_0^2e^{[-2\eta]}\Big] \\ &\quad\times\big[e^{[2\eta]}+b_0+(1/4)b_0^2e^{[-2\eta]}\big]^{-1}. \end{aligned} \tag{64}$$

It is to be noted that if we set $k=iK$, $c=iC$, and $w=i\alpha$ in (64), we can obtain a new periodic solution of (46):

$$\begin{aligned} u(x,t)&=\big[a_2\big[1+(1/4)b_0^2\big]\cos[2Kx+2Cy+2\alpha t] \\ &\quad-(1/2)\sqrt{-2b_0}KC\cos[Kx+Cy+\alpha t] \\ &\quad+b_0[a_2-2KC]\big] \\ &\quad\times\big[\big[1+(1/4)b_0^2\big]\cos[2Kx+2Cy \\ &\quad+2\alpha t]+b_0\big]^{-1}, \\ \alpha&=\frac{(KC^4-6K^3a_2-6C^3a_2+K^4C)}{KC}. \end{aligned} \tag{65}$$

Case (3): $p=c=2$ and $d=q=1$

We consider the case $p=c=2$ and $d=q=1$. Equation (49) can then be expressed as

$$u(\eta)=\frac{a_2\exp(2\eta)+a_1\exp(\eta)+a_0+a_{-1}\exp(-\eta)}{\exp(2\eta)+b_0+b_1\exp(\eta)+b_{-2}\exp(-2\eta)}. \tag{66}$$

By simple calculation using Maple, we have

$$a_{-1}=0,\qquad w=-\frac{(kc^4+6k^3a_2+6c^3a_2+k^4c)}{kc},$$

$$k=k,\qquad c=c,\qquad a_0=a_2b_0,$$
$$b_{-1}=0,\qquad a_2=a_2,\qquad b_1=2\sqrt{b_0},\tag{67}$$
$$a_1=\left[2kc\sqrt{b_0}+2a_2\sqrt{b_0}\right],$$

where a_0, a_2, and b_0 are parameters. From (67) and (66), we obtain

$$u(x,t)=\frac{a_2e^{[2\eta]}+[2kc\sqrt{b_0}+2a_2\sqrt{b_0}]e^{[2\eta]}+a_2b_0}{e^{[2\eta]}+b_0+2\sqrt{b_0}e^{[\eta]}}.\tag{68}$$

In this case, the periodic solution of (68) yields

$$\begin{aligned}u(x,t)=\big[&a_2\cos[2Kx+2\alpha t+2Cy]\\&+2\sqrt{b_0}(a_2-KC)\cos[Kx+\alpha t+Cy]\\&+a_2b_0\big]\\&\times\big[\cos[2Kx+2\alpha t+2Cy]+b_0\\&+2\sqrt{b_0}\cos[Kx+\alpha t+Cy]\big]^{-1},\\ \alpha=-&\frac{(KC^4-6K^3a_2-6C^3a_2-K^4C)}{KC}.\end{aligned}\tag{69}$$

2.3 The Kadomtsev–Petviashvili (KP) equation

In this case we consider the Kadomtsev–Petviashvili equation [22] in the form

$$\partial_x[u_t+uu_x+\beta u_{xxx}]+\frac{\alpha}{2}u_{yy}=0,\tag{70}$$

where α and β are constants. Using the transformation $\eta=kx+ly-wt$, then (70) becomes

$$-kwu''+k^2u'^2+k^2uu''+\beta k^4u''''+\frac{\alpha l^2}{2}u''=0.\tag{71}$$

Case (1): $p=c=1$ and $d=q=1$

Substituting (57) into (71) yields a set of algebraic equations for a_0, b_0, a_1, a_{-1}, b_{-1}, k, and w. Solving the system of algebraic equations, we get

$$\begin{aligned}&a_0=a_1b_0+6\beta k^2b_0,\qquad k=k,\qquad l=l,\\&w=\frac{1}{2}\frac{2k^2a_1+\alpha l^2+2\beta k^4}{k},\\&a_{-1}=\frac{1}{4}a_1b_0^2,\qquad b_{-1}=\frac{1}{4}b_0^2.\end{aligned}\tag{72}$$

In case $k=iK$, $l=iL$, and $w=i\alpha$ are imaginary numbers, we get the new periodic wave solution of (70) as follows:

$$\begin{aligned}u(x,t)=\big[&a_1\big[1+(1/4)b_0^2\big]\cos[Kx+Ly+\alpha t]\\&+a_1b_0-6\beta K^2b_0\big]\\&\times\big[\big[1+(1/4)b_0^2\big]\cos[Kx+Ly+\alpha t]\\&+b_0\big]^{-1},\end{aligned}\tag{73}$$

$$\alpha=-\frac{1}{2}\frac{-2K^2a_1-\alpha L^2+2\beta K^4}{K}.\tag{74}$$

Case (2): $p=c=2$ and $d=q=1$

By a simple calculation in Maple, we have

$$\begin{aligned}&k=k,\qquad l=l,\qquad a_2=\frac{a_0-10k^2\beta b_0}{b_0},\\&w=\frac{b_0l^2\alpha-18b_0\beta k^4+2k^2a_0}{2b_0k},\\&a_1=\frac{3(6k^2\beta b_0+a_0)\sqrt{(-1/6)b_0}}{b_0},\\&b_1=\frac{3}{2}\sqrt{(-1/6)b_0},\qquad b_{-1}=\frac{5}{3}\sqrt{(-1/6)b_0}b_0,\\&a_{-1}=\frac{5}{3}\sqrt{(-1/6)b_0}\big(a_0-10k^2\beta b_0\big),\end{aligned}\tag{75}$$

where a_0 and b_0 are constants.

In this case we obtain the following periodic solution of (70) as

$$\begin{aligned}u(x,t)=\big[&a_2\cos[Kx+Ly+\alpha t]\\&+[a_1+a_{-1}]\cos[Kx+Ly+\alpha t]+a_0\big]\\&\times\big[\cos[2Kx+2Ly+2\alpha t]+b_0\\&+[b_1+b_{-1}]\cos[Kx+Ly+\alpha t]\big]^{-1},\\ \alpha=-&\frac{-b_0L^2\alpha-18b_0\beta K^4-2K^2a_0}{2b_0K},\end{aligned}\tag{76}$$

where a_2, a_{-1}, b_1, b_{-1}, and a_{-1} are given by (75).

3 Conclusion

In this paper, the Exp-function method with the computerized symbolic computation system Maple is used for finding the generalized solitonary solutions and periodic solutions to nonlinear evolution equations arising in mathematical physics. The validity of this method has been tested by applying it successfully to the $(2+1)$-dimensional Konopelchenko–Dubrovsky equations, the $(3+1)$-dimensional Jimbo–Miwa equation, the Kadomtsev–Petviashvili (KP) equation, and the $(2+1)$-dimensional sine-Gordon equation.

Finally, it is worthwhile to mention that the Exp-function method is straightforward, concise, and is a promising and powerful new method for other nonlinear evolution equations in mathematical physics. Its applications are worth further study.

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References

1. He, J.H.: Int. J. Nonlinear Sci. Numer. Simul. **6**(2), 207–208 (2005)
2. He, J.H.: Int. J. Non-Linear Mech. **34**(4), 699–708 (1999)
3. He, J.H.: Periodic solutions and bifurcations of delay differential equations. Phys. Lett. A **347**(4–6), 228–230 (2005)
4. He, J.H.: Int. J. Mod. Phys. B **20**(18), 2561–2568 (2006)
5. Abdou, M.A.: Chaos Solitons Fractals **31**, 95–104 (2007)
6. Abdou, M.A., Soliman, A.A.: Phys. D **211**, 1–8 (2005)
7. El-Wakil, S.A., Abdou, M.A.: Phys. Lett. A **353**, 40 (2006)
8. He, J.H., Wu, X.H.: Chaos Solitons Fractals **29**, 108 (2006)
9. El-Wakil, S.A., Abdou, M.A.: Chaos Solitons Fractals **31**, 1256–1264 (2007)
10. El-Wakil, S.A., Abdou, M.A.: Chaos Solitons Fractals **31**, 840–852 (2007)
11. El-Wakil, S.A., Abdou, M.A.: Phys. Lett. A **358**, 275–282 (2006)
12. Odibat, Z.M., Momani, S.: Int. J. Nonlinear Sci. Numer. Simul. **7**, 27–34 (2006)
13. El-Wakil, S.A., Abdou, M.A.: New exact traveling wave solutions for two nonlinear physical models. Nonlinear Anal. (2007, in press)
14. He, J.H., Wu, X.H.: Exp-function method for nonlinear wave equations. Chaos Solitons Fractals **30**, 700–708 (2006)
15. He, J.H., Abdou, M.A.: New periodic solutions for nonlinear evolution equations using Exp-function method. Chaos Solitons Fractals (2007, in press)
16. Zhang, S.: Application of Exp-function method to high-dimensional nonlinear evolution equation. Chaos Solitons Fractals (2007, in press)
17. Sheng, Z.: Chaos Solitons Fractals **31**(4), 951–959 (2007)
18. Sheng, Z.: Chaos Solitons Fractals **30**(5), 1213–1220 (2006)
19. Wazwaz, A.M.: Math. Comput. Model. **45**(3–4), 473–479 (2007)
20. Xu, G.Q.: Chaos Solitons Fractals **30**(1), 71–76 (2006)
21. Hu, H.C., Lou, S.Y., Chow, K.W.: Chaos Solitons Fractals **31**(5), 1213–1222 (2007)
22. Liu, G.T., Fan, T.Y., Phys. Lett. A **345**, 161–166 (2005)