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ORIGINAL ARTICLE

Computation of focus values with applications

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Abstract Computation of focus (or focal) values for nonlinear dynamical systems is not only important in theoretical study, but also useful in applications. In this paper, we compare three typical methods for computing focus values, and give a comparison among these methods. Then, we apply these methods to study two practical problems and Hilbert's 16th problem. We show that these different methods have the same computational complexity. Finally, we discuss the "minimal singular point value" problem.

Keywords Focus value · Stability · Limit cycle · Hilbert's 16th problem · Symbolic computation

1 Introduction

Limit cycles commonly exist in nonlinear dynamical systems, in almost all disciplines such as physics, mechanics, electronics, biology, ecology, and economics [1–4]. Limit cycle is closely related to other important dynamical behaviors such as bifurcation and stability. Due to wide occurrence of limit cycles in science and technology, limit cycle theory has been extensively studied by mathematicians, physicists, chemists, engineers, and more recently, by biologists and economists. Limit cycles can be generated from local bifurcations such as Hopf bifurcation [1, 5] or global bifurcations like bifurcation from a center [2]. For local bifurcation to limit cycles, the most powerful tool is the center manifold theory [6] and normal form theory [2, 7–10]. The normal form of Hopf bifurcation can be used to determine bifurcation and stability of limit cycles in the vicinity of an Hopf critical point. To determine the existence of multiple limit cycles in the neighborhood of a degenerate Hopf critical point, one needs to compute the coefficients of the normal form, or more precisely, to compute the focus values of the critical point [9–14]. It should be noted that a "focus value" is also usually called a "focal value" in the literature.

Many methods have been developed for computing focus values and normal forms, including the Poincaré method [15], the Takens method [16, 17], the Lyapunov–Schmidt reduction method [18], the time averaging [2], a perturbation technique based on multiple time scales [8, 9], and the singular point value method [11, 12], etc. The basic ideas of the Poincaré and the Takens method are similar, by using a homological operator to decompose the linear space of vector field at each order and to find the part (normal form) which cannot be removed by nonlinear transformations. Based on the Poincaré method, two pioneering books written by

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Marsden and McCracken [1] and Hassard et al. [5] developed methods for computing the focus values associated with Hopf bifurcation. In particular, in the second book [5], explicit formula was given for the second-order focus value when the first one is zero, even when the dimension of the system is greater than 2. The Lyapunov–Schmidt reduction method [18], on the other hand, is often used by people to find approximation of periodic solutions, emerging from Hopf bifurcation. The basic idea of this method is to project the whole system into the subspace spanned by the eigenvectors associated with a pair of purely imaginary eigenvalues. This method, however, does not yield differential equations but algebraic equations. Strictly speaking, it cannot be employed for stability analysis, but can be used to compute the focus values. As a matter of fact, the idea of this method is similar to that of time averaging and multiple time scales from the view point of projection into subspace. In engineering society, the time averaging and multiple time scales are widely used for computing approximate solutions of oscillators or vibrating systems. Nayfeh [8] was the first one to introduce the multiple time scales to compute the normal form of oscillating systems, described by second-order differential equations. Later, this method was combined with a perturbation technique to form a systematic and unifying procedure [9], which can be directly applied to higher dimensional systems, without application of the center manifold reduction method.

To find the maximal number of multiple limit cycles, one needs to compute the high-order normal forms or high-order focus values, related to degenerate Hopf bifurcations [19, 20]. This requires finding explicit symbolic expressions, giving rise to a crucial problem – computational efficiency, since a non-efficient computer program would quickly run into a loop or terminate. Therefore, it is important to examine the existing methods for computing normal forms and focus values. In this paper, we choose three typical methods for a comparison: the Poincaré method [15] (which was also called the Takens method [16, 17]), a perturbation method using multiple time scales [8, 9], and the singular point value method [11–14]. We shall not discuss the Lyapunov–Schmidt reduction method since the main idea of this method is similar to that of the multiple time scales. We use two practical problems and Hilbert's 16th problem [21] to show that all the three methods have the same computational complexity. We will also discuss the so-called minimal singular point value, which is supposed to be the best in computing focus values, and show that none of the existing methods reaches the best.

The rest of the paper is organized as follows. In the next section, the aforementioned three typical methods for computing focus values (normal forms) are presented. In Section 3, the three methods are applied to two practical problems and compared. In Section 4, the three methods are used to study Hilbert's 16th problem, showing that the three methods have the same computational complexity. In Section 5, a symmetric Liénard system is used to discuss the minimal singular point value. Finally, conclusions are drawn in Section 6.

2 Typical methods for computing focus values

In this section, we compare three typical methods for computing focus values, which are widely used in applications associated with Hopf bifurcation.

We first introduce the Poincaré method, and then discuss a perturbation method and the singular point value method. Note that the Poincaré method and the singular point value method are only applicable to two-dimensional systems, while the perturbation method can be directly applied to n-dimensional systems. In other words, if one wants to apply the Poincaré method or the singular point value method to compute focus value, one must first apply the center manifold theory, while the perturbation method combines the center manifold theory and normal form theory in one unified approach.

2.1 The Poincaré method

Consider the following general two-dimensional system:

$$\frac{d\boldsymbol{x}}{dt} = L\boldsymbol{x} + \boldsymbol{f}(\boldsymbol{x}) \equiv \boldsymbol{v}_1 + \boldsymbol{f}_2(\boldsymbol{x}) + \boldsymbol{f}_3(\boldsymbol{x}) + \cdots + \boldsymbol{f}_k(\boldsymbol{x}) + \cdots, \tag{1}$$

where $\boldsymbol{x} = (x_1, x_2)^{\mathrm{T}} \in \boldsymbol{R}^2$, $\boldsymbol{f}_k(\boldsymbol{x})$ denotes the kth-order vector homogeneous polynomials of $\boldsymbol{x}$, and

$$L = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \tag{2}$$

with $\boldsymbol{v}_1 = L\boldsymbol{x} \equiv J\boldsymbol{x}$. (Usually, J is used to denote the Jacobian matrix. Here, L is used in consistence with the Lie bracket notation.) It is assumed that all eigenvalues of L have zero real parts, implying that the dynamics of system (1) is described on a two-dimensional center manifold.

The basic idea of the Poincaré normal form theory is to find a near-identity nonlinear transformation,

$$\boldsymbol{x} = \boldsymbol{y} + \boldsymbol{h}(\boldsymbol{y}) \equiv \boldsymbol{y} + \boldsymbol{h}_2(\boldsymbol{y}) + \boldsymbol{h}_3(\boldsymbol{y}) + \cdots + \boldsymbol{h}_k(\boldsymbol{y}) + \cdots, \tag{3}$$

such that the resulting system

$$\frac{d\boldsymbol{y}}{dt} = L\boldsymbol{y} + \boldsymbol{g}(\boldsymbol{y}) \equiv L\boldsymbol{y} + \boldsymbol{g}_2(\boldsymbol{y}) + \boldsymbol{g}_3(\boldsymbol{y}) + \cdots + \boldsymbol{g}_k(\boldsymbol{y}) + \cdots \tag{4}$$

becomes as simple as possible. Here, $\boldsymbol{h}_k(\boldsymbol{y})$ and $\boldsymbol{g}_k(\boldsymbol{y})$ denote some kth-order vector homogeneous polynomials of $\boldsymbol{y}$.

To apply the normal form theory, first define an operator as follows:

$$L_k: \ \mathcal{H}_k \mapsto \mathcal{H}_k,$$
$$U_k \in \mathcal{H}_k \mapsto L_k(U_k) = [U_k, \boldsymbol{v}_1] \in \mathcal{H}_k, \tag{5}$$

where $\mathcal{H}_k$ denotes a linear vector space consisting of the kth-order vector homogeneous polynomials. The operator $[U_k, \boldsymbol{v}_1]$ is called the Lie bracket, defined as

$$[U_k, \boldsymbol{v}_1] = D\boldsymbol{v}_1 \cdot U_k - DU_k \cdot \boldsymbol{v}_1. \tag{6}$$

Next, define the space $\mathcal{R}_k$ as the range of L_k, and the complementary space of $\mathcal{R}_k$ as $\mathcal{K}_k = \mathrm{Ker}(L_k)$. Thus,

$$\mathcal{H}_k = \mathcal{R}_k \oplus \mathcal{K}_k, \tag{7}$$

and we can then choose bases for $\mathcal{R}_k$ and $\mathcal{K}_k$. Consequently, a vector homogeneous polynomial $\boldsymbol{f}_k \in \mathcal{H}_k$ can be split into two parts: one is spanned on the basis of $\mathcal{R}_k$ and the other on that of $\mathcal{K}_k$. Normal form theory shows that the part of $\boldsymbol{f}_k$ belonging to $\mathcal{R}_k$ can be eliminated while the part belonging to $\mathcal{K}_k$ must be retained, which is called a normal form.

It should be pointed out here that in general, $\mathrm{Ker}(L_k)$ does not always complement the range $\mathcal{R}(L_k)$; however, it does for the case of simple (non-degenerate) Hopf bifurcation. Certainly, it does not imply that it is always the best choice of complement. As a matter of fact, how to choose the $\mathcal{R}_k$ which best complements the range $\mathcal{R}_k$ is still an open problem, as discussed in Section 5.

By applying the Poincaré normal form theory [15], one can find the kth-order normal form $\boldsymbol{g}_k(\boldsymbol{y})$, while the part belonging to $\mathcal{R}_k$ can be removed by appropriately choosing the coefficients of the nonlinear transformation $\boldsymbol{h}_k(\boldsymbol{y})$. The "form" of the normal form $\boldsymbol{g}_k(\boldsymbol{y})$ is mainly determined by the linear vector $\boldsymbol{v}_1$. However, it depends not only on the basis of the complementary space $\mathcal{K}_k$, but also on the choice of the complementary space itself. Once $\mathcal{K}_k$ is chosen, one may apply the matrix method [2] to find a basis of the space $\mathcal{R}_k$ and then determine a basis of the complementary space $\mathcal{K}_k$. Having chosen the basis of $\mathcal{K}_k$, the form of $\boldsymbol{g}_k(\boldsymbol{y})$ can be determined, which actually represents the normal form.

In general, when one applies the normal form theory to a system, one can find the "form" of the normal form (i.e., the basis of the complementary space $\mathcal{K}_k$), but not explicit expressions. However, in practical applications, solutions for the normal form and the nonlinear transformation need to be found explicitly. To achieve this, one may assume a general form of the nonlinear transformation and then substitute it back into the differential equation, with the aid of normal form theory, to obtain the kth-order algebraic equations by balancing the coefficients of the homogeneous polynomial terms. These algebraic equations are then used to determine the coefficients of the normal form and the nonlinear transformation. Thus, the key step in the computation of the kth-order normal form is to find the kth-order algebraic equations.

When the eigenvalues of a Jacobian involve one or more pure imaginary pairs, a complex analysis may simplify the solution procedure. It has actually been noted that the real analysis given in [22] yields coupled algebraic equations, while it will be seen below that the complex analysis can decouple the algebraic equations.

Thus, introduce the linear transformation

$$\begin{cases} x_1 = \dfrac{1}{2}(z + \overline{z}), \\ x_2 = \dfrac{i}{2}(z - \overline{z}), \end{cases} \quad \text{i.e.} \quad \begin{cases} z = x_1 - ix_2, \\ \overline{z} = x_1 + ix_2, \end{cases} \tag{8}$$

where $i = \sqrt{-1}$, and $\overline{z}$ is the complex conjugate of z. Then, the linear part of system (1) becomes $\boldsymbol{v}_1 = x_2 \frac{\partial}{\partial x_1} - x_1 \frac{\partial}{\partial x_2} = (iz - i\overline{z})^{\mathrm{T}}$. Indeed, applying the transformation (8) into system (1) yields

$$\frac{dz}{dt} = iz + f(z, \overline{z}), \quad \frac{d\overline{z}}{dt} = -i\overline{z} + \overline{f}(z, \overline{z}), \tag{9}$$

where f is a polynomial in z and $\overline{z}$ starting from the second-order terms, and $\overline{f}$ is the complex conjugate of f. Here, for convenience, we use the same notation $\boldsymbol{f} = (f, \overline{f})^{\mathrm{T}}$ for the complex analysis. To find the normal form of Hopf singularity, one may use a nonlinear transformation given by

$$z = y + \sum h_k(y, \overline{y}), \quad \overline{z} = \overline{y} + \sum \overline{h}_k(y, \overline{y}), \tag{10}$$

and determine the basis $\boldsymbol{g}_k$ for the complementary space of $\mathcal{K}_k$, or use the Poincaré normal form theory to determine the so-called resonant terms. It is well known that the "resonant" terms are given in the form of $z^j \overline{z}^{j-1}$ (e.g., see [2]), and the kth-order normal form is given by

$$\boldsymbol{g}_k(y, \overline{y}) = \begin{pmatrix} (b_{1k} + ib_{2k})\, y^{(k+1)/2} \overline{y}^{(k-1)/2} \\ (b_{1k} - ib_{2k})\, \overline{y}^{(k+1)/2} y^{(k-1)/2} \end{pmatrix}, \tag{11}$$

where the b_{1k} and b_{2k} are real coefficients to be determined. Therefore, the normal form can be written as

$$\begin{aligned} \frac{dy}{dt} &= iy + \sum_{m=1}^{\infty} g_{2m+1}(y, \overline{y}), \\ \frac{d\overline{y}}{dt} &= -i\overline{y} + \sum_{m=1}^{\infty} \overline{g}_{2m+1}(y, \overline{y}). \end{aligned} \tag{12}$$

It is easy to see from Equation (12) that the normal form contains odd-order terms only, as expected. In normal form computation, the two kth-order coefficients b_{1k} and b_{2k} should, in general, be retained in the normal form.

Finally, based on Equations (4) and (9)–(11), one can determine the algebraic equations order by order starting from $k = 2$, and then apply the normal form theory to solve for the coefficients b_{1k} (k is odd) explicitly in terms of the original system coefficients.

Summarizing the above results gives the following theorem.

Theorem 1. *For system (1) with L given by Equation (2), the normal form is given by Equation (12), where b_{1k} is the kth-order focus value.*

2.2 A perturbation method

Now, we briefly describe a perturbation technique based on multiple time scales for computing normal forms (e.g., see [8, 9]). This technique is not only applicable to Hopf singularity [8, 9], but also to other singularities such as Hopf-zero [23], double-Hopf [24, 25], etc. The basic idea of the normal form theory is to apply successive coordinate transformations to obtain a simplified form that is qualitatively equivalent to the original system in the vicinity of an equilibrium point. Normal form theory is usually employed after an application of the center manifold theory, so as to reduce the original equation to a lower-dimensional system. The perturbation technique has been used to develop a unified approach to directly compute the normal forms of Hopf and degenerate Hopf bifurcations for general n-dimensional systems without the application of the center manifold theory [9].

Consider a general n-dimensional differential equation:

$$\frac{d\boldsymbol{x}}{dt} = J\boldsymbol{x} + \boldsymbol{f}(\boldsymbol{x}), \quad \boldsymbol{x} \in R^n, \quad \boldsymbol{f}: R^n \to R^n, \tag{13}$$

where $J\boldsymbol{x}$ represents the linear terms of the system, and the nonlinear function $\boldsymbol{f}$ is assumed to be analytic with $\boldsymbol{x} = \boldsymbol{0}$ being an equilibrium point of the system, i.e., $\boldsymbol{f}(\boldsymbol{0}) = \boldsymbol{0}$. Further, assume that the Jacobian of system (13), evaluated at the equilibrium point $\boldsymbol{0}$, contains one pair of purely imaginary eigenvalues $\pm i$, and thus, the Jacobian of system (13) may be assumed in the Jordan canonical form:

$$J = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & A \end{bmatrix}, \quad A \in R^{(n-2)\times(n-2)}, \tag{14}$$

where A is *stable* (i.e., all of its eigenvalues have negative real parts).

The basic idea of the perturbation technique based on multiple scales is as follows: Instead of a single time variable, multiple independent variables or scales are used in the expansion of the system response. To achieve this, introducing the new independent

time variables $T_k = \epsilon^k t,\ k = 0, 1, 2, \ldots$ yields partial derivatives with respect to T_k as follows:

$$\frac{d}{dt} = \frac{\partial T_0}{\partial t}\frac{\partial}{\partial T_0} + \frac{\partial T_1}{\partial t}\frac{\partial}{\partial T_1} + \frac{\partial T_2}{\partial t}\frac{\partial}{\partial T_2} + \cdots = D_0 + \epsilon D_1 + \epsilon^2 D_2 + \cdots, \tag{15}$$

where $D_k = \frac{\partial}{\partial T_k}$. Then, assume that the solutions of system (13) in the neighborhood of $\boldsymbol{x} = \mathbf{0}$ are expanded in series as

$$\boldsymbol{x}(t;\epsilon) = \epsilon \boldsymbol{x_1}(T_0, T_1, \ldots) + \epsilon^2 \boldsymbol{x_2}(T_0, T_1, \ldots) + \cdots. \tag{16}$$

Note that the same perturbation parameter, ϵ, is used in both the time and space scalings in (15) and (16). In other words, this perturbation approach treats time and space in the same scaling.

Substituting the formulas (15) and (16) into system (13), and solving the resulting ordered linear differential equations finally yields the normal form, given in polar coordinates (a detailed procedure can be found in [9]):

$$\frac{dr}{dt} = \frac{\partial r}{\partial T_0}\frac{\partial T_0}{\partial t} + \frac{\partial r}{\partial T_1}\frac{\partial T_1}{\partial t} + \frac{\partial r}{\partial T_2}\frac{\partial T_2}{\partial t} + \cdots = D_0 r + D_1 r + D_2 r + \cdots, \tag{17}$$

$$\frac{d\phi}{dt} = \frac{\partial \phi}{\partial T_0}\frac{\partial T_0}{\partial t} + \frac{\partial \phi}{\partial T_1}\frac{\partial T_1}{\partial t} + \frac{\partial \phi}{\partial T_2}\frac{\partial T_2}{\partial t} + \cdots = D_0\phi + D_1\phi + D_2\phi + \cdots, \tag{18}$$

where $D_i r$ and $D_i \phi$ are uniquely determined, implying that the normal form given by Equations (17) and (18) is unique. Further, it has been shown [9] that the derivatives $D_i r$ and $D_i \phi$ are functions of r only, and only $D_{2k} r$ and $D_{2k}\phi$ are non-zero, which can be expressed as $D_{2k} r = v_{2k+1} r^{2k+1}$ and $D_{2k}\phi = t_{2k+1} r^{2k}$, where both v_{2k+1} and t_{2k+1} are expressed in terms of the original system's coefficients. The coefficient v_{2k+1} is called the kth-order focus value of the Hopf-type critical point (the origin).

Theorem 2. *Suppose the general n-dimensional system (13) has an Hopf-type singular point at the origin, i.e., the linearized system of (13) has one pair of purely imaginary eigenvalues and the remaining eigenvalues have negative real parts. Then the normal form of system (13) for the Hopf or generalized Hopf bifurcations up to the* $(2k+1)$*st order term is given by*

$$\frac{dr}{dt} = r(v_1 + v_3 r^2 + v_5 r^4 + \cdots + v_{2k+1} r^{2k}), \tag{19}$$

$$\frac{d\theta}{dt} = 1 + \frac{d\phi}{dt} = 1 + t_3 r^2 + t_5 r^4 + \cdots + t_{2k+1} r^{2k}, \tag{20}$$

where the constants $v_{2k+1} = D_{2k} r / r^{2k+1}$ *and* $t_{2k+1} = D_{2k}\phi / r^{2k+1}$ *are explicitly expressed in terms of the original system parameters, and* $D_{2k} r$ *and* $D_{2k}\phi$ *are obtained recursively using multiple time scales.*

2.3 The singular point value method

This iterative method computes focus values by computing the singular point quantities (see [12–14] for details).

To introduce this method, consider the following planar polynomial differential system:

$$\frac{dx}{dt'} = \delta x - y + \sum_{k=2}^{\infty} X_k(x, y),$$

$$\frac{dy}{dt'} = x + \delta y + \sum_{k=2}^{\infty} Y_k(x, y), \tag{21}$$

where $X_k(x, y)$ and $Y_k(x, y)$ are homogeneous polynomials of x, y of degree k. The origin $(x, y) = (0, 0)$ is a singular point of system (21), which is either a focus or a linear center (when $\delta = 0$). Since we are interested in the computation of focus values, we assume $\delta = 0$ in the following analysis.

Introducing the transformations given by

$$z = x + iy, \quad w = x - iy, \quad T = it', \quad i = \sqrt{-1}, \tag{22}$$

into system (21) results in

$$\frac{dz}{dT} = z + \sum_{k=2}^{\infty} Z_k(z, w) = Z(z, w),$$

$$\frac{dw}{dT} = -w - \sum_{k=2}^{\infty} W_k(z, w) = -W(z, w), \tag{23}$$

where z, w, and T are complex variables, and

$$\begin{aligned} Z_k(z,w) &= \sum_{\alpha+\beta=k} a_{\alpha\beta} z^\alpha w^\beta, \\ W_k &= \sum_{\alpha+\beta=k} b_{\alpha\beta} w^\alpha z^\beta. \end{aligned} \tag{24}$$

Systems (21) and (23) are said to be concomitant.

If system (21) is a real planar differential system, then the coefficients of system (23) must satisfy the following conjugate conditions:

$$\overline{a_{\alpha\beta}} = b_{\alpha\beta}, \quad \alpha \geq 0, \quad \beta \geq 0, \quad \alpha+\beta \geq 2. \tag{25}$$

By the following transformations:

$$z = re^{i\theta}, \quad w = re^{-i\theta}, \quad T = it, \tag{26}$$

system (23) can be transformed into

$$\begin{aligned} \frac{dr}{dt} &= \frac{ir}{2} \sum_{m=1}^{\infty} \sum_{\alpha+\beta=m+2} \left[a_{\alpha(\beta-1)} - b_{\beta(\alpha-1)}\right] e^{i(\alpha-\beta)\theta} r^m, \\ \frac{d\theta}{dt} &= 1 + \frac{1}{2} \sum_{m=1}^{\infty} \sum_{\alpha+\beta=m+2} \left[a_{\alpha(\beta-1)} \right. \\ &\quad \left. + b_{\beta(\alpha-1)}\right] e^{i(\alpha-\beta)\theta} r^m. \end{aligned} \tag{27}$$

For a complex constant h, $|h| \ll 1$, we may write the solution of (27) satisfying the initial condition $r|_{\theta=0} = h$ as

$$r = \widetilde{r}(\theta, h) = h + \sum_{k=2}^{\infty} v_k(\theta) h^k. \tag{28}$$

Evidently, if system (21) is a real system, then $v_{2k+1}(2\pi)$ $(k = 1, 2, \ldots)$ is the kth-order focal (or focus) value of the origin.

For system (23), we can uniquely derive the following formal series:

$$\begin{aligned} \varphi(z,w) &= z + \sum_{k+j=2}^{\infty} c_{kj} z^k w^j, \\ \psi(z,w) &= w + \sum_{k+j=2}^{\infty} d_{k,j} w^k z^j, \end{aligned} \tag{29}$$

such that

$$\begin{aligned} \frac{d\varphi}{dT} &= \varphi + \sum_{j=1}^{\infty} p_j \varphi^{j+1} \psi^j, \\ \frac{d\psi}{dT} &= -\psi - \sum_{j=1}^{\infty} q_j \psi^{j+1} \varphi^j. \end{aligned} \tag{30}$$

Let $\mu_0 = 0$, and $\mu_k = p_k - q_k$, $k = 1, 2, \ldots$, where μ_k is called the kth-order singular point quantity of the origin of system (23) [12]. Based on the singular quantities, we can define the following concepts:

- If $\mu_0 = \mu_1 = \cdots = \mu_{k-1} = 0$ and $\mu_k \neq 0$, then the origin of system (23) is called the *kth-order weak critical singular point*. In other words, k is the multiplicity of the origin of system (23).
- If $\mu_k = 0$ for $k = 1, 2, \ldots,$ then the origin of system (23) is called an *extended center (complex center)*.
- If system (21) is a real autonomous differential system with the concomitant system (23), then for the origin, the kth-order focus quantity v_{2k+1} of system (21) and the kth-order quantity of the singular point of system (23) have the relation given in the following theorem [11, 13].

Theorem 3. *Given system (21) ($\delta = 0$) or (23), for any positive integer m, the following assertion holds:*

$$\begin{aligned} v_{2k+1}(2\pi) &= i\pi \left(\mu_k + \sum_{j=1}^{k-1} \xi_k^{(j)} \mu_j \right), \\ & \quad k = 1, 2, \ldots, \end{aligned} \tag{31}$$

where $\xi_m^{(j)}$ $(j = 1, 2, \ldots, k-1)$ are polynomial functions of coefficients of system (23).

The following recursive formulas are used for computing the singular point quantities of system (23) [12]: $c_{11} = 1$, $c_{20} = c_{02} = c_{kk} = 0$, $k = 2, 3, \ldots,$ and $\forall(\alpha, \beta)$, $\alpha \neq \beta$, and $m \geq 1$:

$$\begin{aligned} C_{\alpha\beta} &= \frac{1}{\beta-\alpha} \sum_{k+j=3}^{\alpha+\beta+2} [(\alpha-k+1) a_{k,j-1} \\ &\quad - (\beta-j+1) b_{j,k-1}] C_{\alpha-k+1,\beta-j+1}, \end{aligned} \tag{32}$$

and

$$\mu_m = \sum_{k+j=3}^{2m+4} [(m-k+2)a_{k,j-1} - (m-j+2)b_{j,k-1}]C_{m-k+2,m-j+2}, \tag{33}$$

where $a_{kj} = b_{kj} = C_{kj} = 0$ for $k < 0$ or $j < 0$.

It is clearly seen from Equation (31) that

$$\mu_0 = \mu_1 = \mu_2 = \cdots = \mu_{k-1} = 0 \iff v_1 = v_3 = v_5 = \cdots = v_{2k-1} = 0.$$

(Note that here $\mu_0 = v_1 = \delta$ is the linear focus value.) Therefore, when determining the conditions such that $v_1 = v_3 = v_5 = \cdots = v_{2k-1} = 0$, one can instead use the equations $\mu_0 = \mu_1 = \mu_2 = \cdots = \mu_{k-1} = 0$. If the μ_k's are simpler than the v_{2k+1}'s, then this method is better than the method of directly computing v_{2k+1}. However, in general, these μ_k are not necessarily simpler than v_{2k+1}, as will be seen in the next section.

It should be pointed out that since the normal form is not unique, the focus values obtained by using different methods are not necessarily the same. However, the first non-zero focus value must be identical (neglecting a constant multiplier). This implies that for different focus values obtained by using different approaches, solutions to the equations $v_1 = v_3 = v_5 = \cdots = v_{2k-1} = 0$ (or $\mu_0 = \mu_1 = \mu_3 = \cdots = \mu_{k-1} = 0$) must be identical.

For the three methods described above, symbolic programs have been developed using Maple, which will be used in the following two sections.

3 Application to practical problems

3.1 Brusselator model

The well-known Brusselator model is used here to verify the results obtained in the previous section. The model is described by [26]

$$\begin{aligned} \frac{dw_1}{dt} &= A - (1+B)w_1 + w_1^2 w_2, \\ \frac{dw_2}{dt} &= Bw_1 - w_1^2 w_2, \end{aligned} \tag{34}$$

where $A, B > 0$ are parameters. The system has a unique equilibrium point:

$$w_{1e} = A, \quad w_{2e} = \frac{B}{A}. \tag{35}$$

Evaluating the Jacobian of the system at the equilibrium point shows that an Hopf bifurcation occurs at the critical point $B = 1 + A^2$. Let

$$B = 1 + A^2 + \mu, \tag{36}$$

where μ is a perturbation parameter. Then, the Jacobian has eigenvalues $\lambda = \pm Ai$. Suppose $A = 1$, and then introduce the transformation $w_1 = w_{1e} + x_1$ and $w_2 = w_{2e} - x_1 + x_2$ into (34) to obtain the following new system:

$$\begin{aligned} \frac{dx_1}{dt} &= x_2 + \mu x_1 + \mu x_1^2 + 2x_1x_2 - x_1^3 + x_1^2 x_2, \\ \frac{dx_2}{dt} &= -x_1. \end{aligned} \tag{37}$$

Now, at the critical point defined by $\mu = 0$, apply the three methods described in the previous section to compute the first-order focus value. Maple programs are employed to obtain the following results:

The Poincaré method: $b_{13} = -\frac{3}{8}$.
The perturbation method: $v_3 = -\frac{3}{8}$.
The singular point value method: $\mu_1 = \frac{3}{4}i$.

It is seen that $b_{13} = v_3 = \frac{i}{2}\mu_1$. Ignoring the constant factor $\frac{i}{2}$, the three methods give the identical result for the first-order focus value: $-\frac{3}{8}$. This shows that the limit cycles bifurcating from the critical point $\mu = 0$ in the vicinity of the equilibrium point (w_{1c}, w_{2c}) is supercritical, i.e., the bifurcating limit cycles are stable since the first-order focus value is negative.

Further, computing the second-order focus values gives

The Poincaré method: $b_{15} = -\frac{1}{96}$.
The perturbation method: $v_5 = -\frac{1}{96}$.
The singular point value method: $\mu_2 = -\frac{67}{48}i$.

This indicates that the Poincaré method and the perturbation method still give the same second-order focus value, but the singular point method yields a different μ_2 (ignoring the difference factor $\frac{i}{2}$). This is not

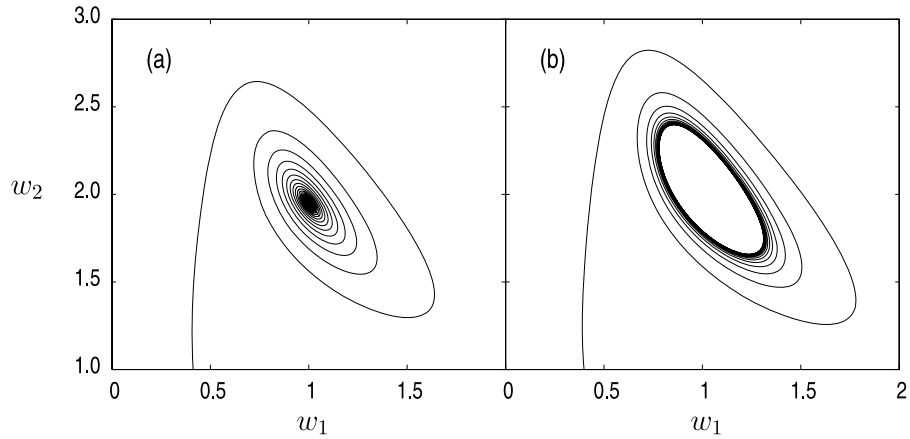

Fig. 1 Simulated trajectories of the Brusselator model (34) with $A = 1.0$ and initial point $(w_1, w_2) = (2, -1)$. (a) Convergent to the stable equilibrium point $\boldsymbol{w}^+$ when $B = 1.95$. (b) Convergent to a stable limit cycle when $B = 2.05$

surprising since here $\mu_1 \neq 0$ and the second-order focus value is a combination of μ_1 and μ_2.

Further computation shows that

The Poincaré method: $b_{17} = -\frac{2695}{36864}$.
The perturbation method: $v_7 = -\frac{4543}{36864}$.
The singular point value method: $\mu_3 = \frac{6239}{2304}i$.

For the third-order focus value, even the Poincaré method and the perturbation method give different results, as expected.

Numerical simulation results based on the original system (34) are shown in Fig. 1, which clearly indicates that when $A = 1.0$ and $B = 1.95$, the trajectory converges to the stable equilibrium point $w_{1c} = 1$, $w_{2c} = 1.95$ (see Fig. 1(a)); while when $A = 1.0$, $B = 2.05$, the equilibrium point becomes unstable and a stable limit cycle bifurcates from the equilibrium point (see Fig. 1(b)).

3.2 Induction machine model

A model of induction machine is used to demonstrate the application of the theorems obtained in the previous section. The model is based on the one discussed in [27] and the same notations are adopted here. Since in this paper we are mainly interested in the application of focus value computation, we will not give a detailed derivation of the model.

Induction machine (or asynchronous machine) is widely used in industrial applications. The behavior of induction machine has been studied for years, but the main attention has been focused on steady-state solutions due to the complexity of the model (even with simplifying assumptions). In order to study the dynamical behavior of the model such as instability and bifurcations, it needs to determine the conditions of the bifurcation (critical) points.

The model is described by a system of seven ordinary differential equations, given as follows:

$$
\begin{aligned}
\frac{d\phi_{qs}}{dt} &= \omega_b\left\{u_q - \phi_{ds} + \frac{r_s}{X_{1s}}\left[X_{aq}\left(\frac{\phi_{qs}}{X_{1s}} + \frac{\phi'_{qr}}{X'_{1r}}\right) - \phi_{qs}\right]\right\},\\
\frac{d\phi_{ds}}{dt} &= \omega_b\left\{u_d + \phi_{qs} + \frac{r_s}{X_{1s}}\left[X_{aq}\left(\frac{\phi_{ds}}{X_{1s}} + \frac{\phi'_{dr}}{X'_{1r}}\right) - \phi_{ds}\right]\right\},\\
\frac{d\phi_{0s}}{dt} &= \omega_b\left\{\frac{r_s}{X_{1s}}(-\phi_{0s})\right\},\\
\frac{d\phi'_{qr}}{dt} &= \omega_b\left\{-(1-\omega_r)\phi'_{dr} + \frac{r'_r}{X'_{1r}}\left[X_{aq}\left(\frac{\phi_{qs}}{X_{1s}} + \frac{\phi'_{qr}}{X'_{1r}}\right) - \phi'_{qr}\right]\right\},\\
\frac{d\phi'_{dr}}{dt} &= \omega_b\left\{(1-\omega_r)\phi'_{qr} + \frac{r'_r}{X'_{1r}}\left[X_{aq}\left(\frac{\phi_{ds}}{X_{1s}} + \frac{\phi'_{dr}}{X'_{1r}}\right) - \phi'_{dr}\right]\right\},\\
\frac{d\phi_{0r}}{dt} &= \omega_b\left\{\frac{r'_r}{X'_{1r}}(-\phi'_{0r})\right\},\\
\frac{d\omega'_r}{dt} &= \frac{1}{2H}\left\{\frac{X_{ad}}{X_{1s}X'_{1r}}(\phi_{qs}\phi'_{dr} - \phi_{ds}\phi'_{qr}) - T_L\right\},
\end{aligned}
\tag{38}
$$

where, except the state variables, all the variables are system parameters. Letting

$$
w_1 = \phi_{qs}, \quad w_2 = \phi_{ds}, \quad w_3 = \phi_{0s}, \quad w_4 = \phi'_{qr},
$$
$$
w_5 = \phi'_{dr}, \quad w_6 = \phi'_{0r}, \quad w_7 = \omega_r,
$$

and substituting proper parameter values to Equation (38) yields a model of a 3hp induction machine as $\frac{d\boldsymbol{w}}{dt} = \boldsymbol{f}(\boldsymbol{w}, V)$; namely,

$$\begin{aligned}
\frac{dw_1}{dt} &= -\frac{3}{10}w_1 - w_2 + \frac{3}{10}w_4 + V,\\
\frac{dw_2}{dt} &= w_1 - \frac{3}{10}w_2 + \frac{3}{10}w_5,\\
\frac{dw_3}{dt} &= -\frac{3}{5}w_3,\\
\frac{dw_4}{dt} &= \frac{1}{2}w_1 - \frac{1}{2}w_4 - w_5 + w_5w_7,\\
\frac{dw_5}{dt} &= \frac{1}{2}w_2 + w_4 - \frac{1}{2}w_5 - w_4w_7,\\
\frac{dw_6}{dt} &= -w_6,\\
\frac{dw_7}{dt} &= \frac{7}{120\pi^3}(14w_1w_5 - 14w_2w_4 - 1),
\end{aligned} \tag{39}$$

where $V > 0$ is a bifurcation parameter, representing the input voltage of the motor.

Setting $\frac{dw_i}{dt} = 0,\ \ i = 1, 2, \ldots, 7$, results in two equilibrium solutions (fixed points):

$$\begin{aligned}
w_1^{\pm} &= -\frac{3(-350V^2 + 15 \pm 10S)}{7630V},\\
w_2^{\pm} &= \frac{7315V^2 - 150 \pm 9S}{7360V},\\
w_3^{\pm} &= 0,\\
w_4^{\pm} &= -\frac{1}{14V},\\
w_5^{\pm} &= \frac{35V^2 \pm S}{70V},\\
w_6^{\pm} &= 0,\\
w_7^{\pm} &= \frac{-350V^2 + 124 \pm 10S}{109},
\end{aligned} \tag{40}$$

where

$$S = \sqrt{1225V^4 - 105V^2 - 25},$$

indicating that the equilibrium solutions exist when

$$V^2 \geq \frac{3 + \sqrt{109}}{70}.$$

The Jacobian of Equation (39) is given by

$$J(\boldsymbol{w}) = \begin{bmatrix}
-\frac{3}{10} & -1 & 0 & \frac{3}{10} & 0 & 0 & 0\\
1 & -\frac{3}{10} & 0 & 0 & \frac{3}{10} & 0 & 0\\
0 & 0 & -\frac{3}{5} & 0 & 0 & 0 & 0\\
\frac{1}{2} & 0 & 0 & -\frac{1}{2} & -1 + w_7 & 0 & w_5\\
0 & \frac{1}{2} & 0 & 1 - w_7 & -\frac{1}{2} & 0 & -w_4\\
0 & 0 & 0 & 0 & 0 & -1 & 0\\
\frac{49w_5}{60\pi^3} & -\frac{49w_4}{60\pi^3} & 0 & -\frac{49w_2}{60\pi^3} & \frac{49w_1}{60\pi^3} & 0 & 0
\end{bmatrix}. \tag{41}$$

The conditions for various singularities of the system have been obtained in [28], but the stability of the bifurcating limit cycles was not discussed. Here, we consider the stability of the limit cycles generated from an Hopf bifurcation. For Hopf bifurcation, the Jacobian of (41) has one pair of purely imaginary eigenvalues, which requires [28] that $V \geq V_0 = ((3 + \sqrt{109})/70)^{1/2} \approx 0.4381830425$. Since $\boldsymbol{w}^-$ is always unstable when $V > V_0$, we consider $\boldsymbol{w}^+$. It can be shown that $\boldsymbol{w}^+$ is stable when $0.4381830425 < V < 6.2395593195$ or $V > 7.75369242394$; unstable when $6.2395593195 < V < 7.35369242394$. The point $V_0 = 0.4381830425$ is a static critical point. Furthermore, we employ the criterion given in [28] to show that

$$V_h^1 = 6.2395593195 \ \text{ and } \ V_h^2 = 7.35369242394 \tag{42}$$

are two solutions at which Hopf bifurcations occur. When $V_h^1 = 6.2395593195$, the eigenvalues of $J(\boldsymbol{x})$ are

$$\begin{aligned}
&\pm 0.7905733366i, \quad -1, \quad -0.6, \quad -0.5630004665,\\
&-0.5184997667 \pm 1.0893171380i,
\end{aligned}$$

at which the equilibrium solution $\boldsymbol{w}^+$ becomes (see Equation (40))

$$\begin{aligned}
w_1^+ &= 0.0000063079, \quad w_2^+ = 6.2361231187,\\
w_4^+ &= -0.0114476949, \quad w_5^+ = 6.2361020924,\\
w_7^+ &= 0.9990816376, \quad w_3^+ = w_6^+ = 0.
\end{aligned} \tag{43}$$

To find the focus values associated with the Hopf critical point $V_h^1 = 6.2395593195$, introduce the following transformation:

$$\boldsymbol{w} = \boldsymbol{w}^+ + T\boldsymbol{x}, \tag{44}$$

where T is given by

$$T = \begin{bmatrix} 0.1086246665 & 0.4679587889 & 0 & 0 & -0.3435100286 & -0.0600500516 & 0.4185449984 \\ 0.5058443308 & -0.0007346807 & 0 & 0 & -0.1624709259 & 0.3390908038 & -0.0104422664 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0.5615866321 & 0.7517624035 & 0 & 0 & -0.2404254272 & -0.3457217123 & -0.5576926708 \\ 0.1456981721 & -0.2275738425 & 0 & 0 & 1.2874665260 & -0.0088875674 & -0.1562864724 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ -0.0589644813 & 0.0939157539 & 0 & 0 & 0.0301605532 & 0.1032562984 & -0.0923171911 \end{bmatrix},$$

under which the transformed system is given in the canonical form

$$\frac{d\boldsymbol{x}}{dt} = J\boldsymbol{x} + \boldsymbol{f}_2(\boldsymbol{x}), \tag{45}$$

where J is

$$J = \begin{bmatrix} 0 & 0.7905733365 & 0 & 0 & 0 & 0 & 0 \\ -0.7905733365 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -0.5630004665 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -0.5184997667 & 1.0893171380 \\ 0 & 0 & 0 & 0 & 0 & -1.0893171400 & -0.5184997874 \end{bmatrix}.$$

Then, applying the Maple program [9] results in the following focus values:

$$\begin{aligned} v_3 &= -0.1775337919 \times 10^{-2}, \\ v_5 &= -0.9320629120 \times 10^{-5}, \\ v_7 &= -0.1536975785 \times 10^{-6}, \end{aligned} \tag{46}$$

which implies that the limit cycles bifurcating from the critical point V_h^1 in the neighborhood of $\boldsymbol{w}^+$ are stable.

Simulation results for this example using system (39) with $V = 6.0$ and $V = 6.5$ are depicted in Fig. 2. The initial point is chosen as

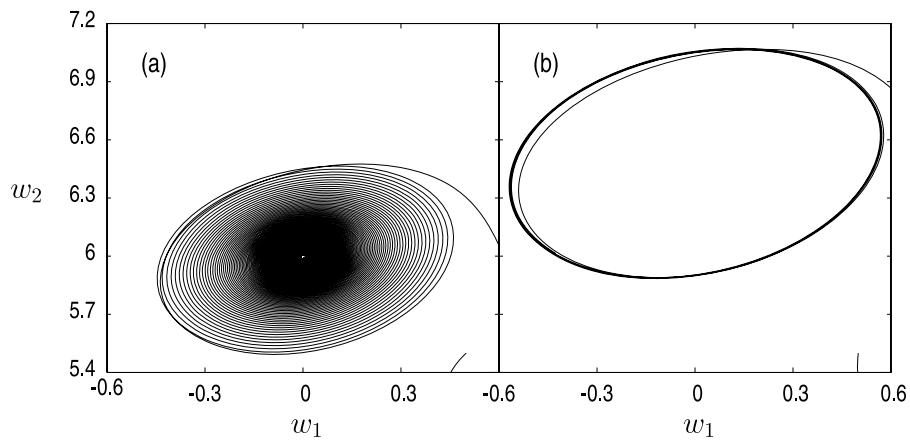

Fig. 2 Simulated trajectories of the induction machine model (39) projected on the w_1–w_2 plane with initial point $\boldsymbol{w}_0 = (0.5,\ 5.5,\ 2.0,\ -3.0,\ 1.0,\ 4.0,\ -5.0)^{\mathrm{T}}$. (a) Convergent to the stable equilibrium point $\boldsymbol{w}^+ = (0.0000070946,\ 5.9964264430,\ 0,\ -0.0119047619,\ 5.9964027942,\ 0,\ 0.9990067495)^{\mathrm{T}}$ when $V = 6.0$. (b) Convergent to a stable limit cycle when $V = 6.5$

$\boldsymbol{w}_0 = (0.5,\ 5.5,\ 2.0,\ -3.0,\ 1.0,\ 4.0,\ -5.0)^{\mathrm{T}}$. It can be seen from this figure, as expected, that when $V = 6.0 < V_h^1$, the trajectory converges to the stable equilibrium point

$$\begin{aligned}\boldsymbol{w}^+ = (&0.0000070946,\ 5.9964264430,\ 0,\\ &-0.0119047619,\ 5.9964027942,\\ &0,\ 0.9990067495)^{\mathrm{T}},\end{aligned}$$

as shown in Fig. 2(a). When $V = 6.5 > V_h^1$, the equilibrium point becomes unstable, and a supercritical Hopf bifurcation occurs, giving rise to a stable limit cycle (see Fig. 2(b)).

It should be pointed out that the perturbation method can be applied to the seven-dimensional system (45) without employing the center manifold theory (more precisely, the center manifold theory is incorporated in the unified perturbation approach), while the Poincaré method and the singular point value method cannot be directly applied to system (45).

Note that in the above two examples, except for the bifurcation parameter, all the other parameters are fixed. In the next section, we consider systems with free parameters and find the maximal number of limit cycles under appropriate choices of the parameters.

4 Application to Hilbert's 16th problem

The well-known 23 mathematical problems proposed by Hilbert in 1902 [21] has significant impacts on the twentieth-century mathematics. Out of 23 problems, 2 remain unsolved; one of them is the 16th problem. This problem includes two parts: the first part studies the relative positions of separate branches of algebraic curves, while the second part considers the upper bound of the number of limit cycles and their relative locations in polynomial vector fields. Generally, the second part of this problem is what is usually meant when talking about Hilbert's 16th problem. The recent developments on Hilbert's 16th problem may be found in the survey articles [29, 30]. A simplified version (Liénard equation) of the second part of Hilbert's 16th problem has recently been chosen by Smale [31] as one of the 18 challenging mathematical problems for the twenty-first century. Although the problem is still far away from being completely solved, related research has made great contributions to the development of modern mathematics.

Roughly speaking, the second part of Hilbert's 16th problem is to consider the planar vector fields, described by the following polynomial differential equations:

$$\frac{dx}{dt} = P_n(x, y), \quad \frac{dy}{dt} = Q_n(x, y), \tag{47}$$

where $P_n(x, y)$ and $Q_n(x, y)$ represent the nth-degree polynomials of x and y. The problem is to find the upper bound, defined by the so-called Hilbert number $H(n)$, on the number of limit cycles that the system can have. In general, this is a very difficult problem, particularly for determining global (large) limit cycles. Although it has not been possible to obtain a uniform upper bound for $H(n)$, tremendous efforts have been made in finding a large number of limit cycles and raising the lower bound of the Hilbert number $H(n)$ for general planar polynomial systems or for individual degree of systems, hoping to get close to the estimation of the upper bound of $H(n)$.

If the problem is restricted to the neighborhoods of isolated fixed points, then the question is reduced to studying degenerate Hopf bifurcations, which gives rise to finding focus values. Alternatively, this is equivalent to computing the normal form of differential equations associated with an Hopf or a degenerate Hopf bifurcation. Suppose that the normal form associated with the Hopf singularity is given in polar coordinates (obtained using, say, the method given in [9]) described by Equation (19). The basic idea of finding k small limit cycles around the origin is as follows: first, find the conditions such that $v_1 = v_3 = \cdots = v_{2k-1} = 0$, but $v_{2k+1} \neq 0$, and then, perform appropriate small perturbations to prove the existence of k limit cycles. In 1952, Bautin [32] proved that a quadratic planar polynomial vector field can have at most three small limit cycles. For cubic systems, it has recently been proved that 12 small limit cycles can exist [33, 34, 35].

In the following, we consider two systems and show that the three methods reviewed in Section 2 for computing focus values have the same computational complexity. It should be pointed out that here the "computational complexity" is not the standard definition of computational complexity, since we do not consider computing issues such as CPU time, memory, etc. By "the same computational complexity," we merely mean how complex the results (or expressions for the focus values from symbolic computation) are obtained using

a method. Since in this paper we are more concerned with the expressions of the focus values, we use the length of the computer output (i.e., the number of the lines in output) to measure the "complexity" of the method itself.

The first example is a cubic-order system, which can have 12 limit cycles, and the second example is a symmetric Liénard system, which can have more limit cycles than one may expect.

4.1 A cubic-order system with Z_2 symmetry

To obtain a cubic-order system with Z_2 symmetry, apply the standard complex formulas [29]:

$$\frac{dz}{dt} = F_2(z,\overline{z}), \quad \frac{d\overline{z}}{dt} = \bar{F}_2(z,\overline{z}), \tag{48}$$

where

$$F_2(z,\overline{z}) = (A_0 + A_1|z|^2)z + (A_2 + A_3|z|^2)\overline{z} + A_4 z^3 + A_5\overline{z}^3. \tag{49}$$

Let $z = w_1 + iw_2$, $\overline{z} = w_1 - iw_2$, $A_j = a_j + ib_j$, where w_1, w_2 and a_j, b_j are all real. Then, system (48) is transformed to the following real form:

$$\begin{aligned}
\frac{dw_1}{dt} &= (a_0+a_2)w_1 - (b_0-b_2)w_2 + (a_1+a_3+a_4 \\
&\quad + a_5)w_1^3 - (b_1-b_3+3b_4-3b_5)w_1^2w_2 \\
&\quad + (a_1+a_3-3a_4-3a_5)w_1w_2^2 \\
&\quad - (b_1-b_3-b_4+b_5)w_2^3, \\
\frac{dw_2}{dt} &= (b_0+b_2)w_1 + (a_0-a_2)w_2 + (b_1+b_3 \\
&\quad + b_4+b_5)w_1^3 + (a_1-a_3+3a_4-3a_5)w_1^2w_2 \\
&\quad + (b_1+b_3-3b_4-3b_5)w_1w_2^2 \\
&\quad + (a_1-a_3-a_4+a_5)w_2^3.
\end{aligned} \tag{50}$$

The two eigenvalues of the Jacobian of Equation (50) evaluated at the origin $(w_1, w_2) = (0, 0)$ are $a_0 \pm \sqrt{a_2^2 + b_2^2 - b_0^2}$, indicating that the origin $(0, 0)$ is a saddle point or a node when $a_2^2 + b_2^2 - b_0^2 \geq 0$; a focus point or a center if $a_2^2 + b_2^2 - b_0^2 < 0$. When $a_2^2 + b_2^2 - b_0^2 = 0$, the origin is either a node or a double-zero singular point. By a parametric transformation, rename the coefficients of the resulting system to yield the following system:

$$\begin{aligned}
\frac{dx}{dt} &= ax + by + a_{30}x^3 + a_{21}x^2y + a_{12}xy^2 + a_{03}y^3, \\
\frac{dy}{dt} &= \pm bx + ay + b_{30}x^3 + b_{21}x^2y + b_{12}xy^2 + b_{03}y^3,
\end{aligned} \tag{51}$$

where $b > 0$.

For a vector field with Z_2-symmetry, naturally the best situation is to have two symmetric focus points about the origin. Thus, if N small limit cycles are found in the neighborhood of one focus point, the whole system would have $2N$ limit cycles. Without loss of generality, the two symmetric focus points are assumed to be located on the y-axis (or the x-axis), and further assumed to be precisely located at $(0, \pm 1)$ with a proper scaling, leading to the conditions of $a_{03} = -b$ and $b_{03} = -a$. Another condition, $a_{12} = a$, comes from making the two focus points be of Hopf type. Furthermore, by applying proper parameter scaling and time scaling, one obtains the following new system [35]:

$$\begin{aligned}
\frac{du}{d\tau} &= v + 2\bar{a}_{21}u^2 + 4auv - \frac{3}{2}v^2 + 4b\bar{a}_{30}u^3 \\
&\quad - 2\bar{a}_{21}u^2v - 2auv^2 + \frac{1}{2}v^3, \\
\frac{dv}{d\tau} &= -u - 4\bar{b}_{21}u^2 + 2(2a^2 \mp 2b^2 + 1)uv - 8\bar{b}_{30}u^3 \\
&\quad + 4\bar{b}_{21}u^2v - (2a^2 \mp b^2 + 1)uv^2,
\end{aligned} \tag{52}$$

where the coefficients $\bar{a}_{21}, \bar{b}_{21}, \bar{a}_{30}$, and $\bar{b}_{30}$ are expressed in terms of the original parameters $a, b, a_{21}, b_{21}, a_{30}, b_{30}$. Thus, based on Equation (52), one can compute the focus values and consider the existence of small limit cycles.

Note that system (52) contains 6 free parameters, which suggests that we may set 6 focus values zero thereby obtaining 7 small limit cycles for system (52), so that the original system may have 14 small limit cycles. However, it has been shown in [35] that the existence of 14 limit cycles is not possible. The maximal number of small limit cycles that a cubic-order system with Z_2 symmetry can have is 12. Since in this paper we are interested in the methods of computing focus values, we will not further discuss this interesting issue. (For details, readers are referred to [34, 35].)

We now apply the three methods reviewed in the previous section to compute the focus values of system (52) and obtain the following results.

```
The Poincare method:
b13:=1/2*a-2*a^2*b21b-2*a21b*b21b-a21b*a+2*b^2*b21b+3/2*b*a30b-1/2*b21b:
b15:=-11/36*a-7/3*a*a21b*b^2-25/9*b21b*a21b*a^2-1/6*a21b*b*a30b+92/9*b21b*b^2*a^2
      -11/9*b21b*a21b*b^2-8/9*b21b*a21b^2-14/3*b21b*a^4-50/9*b21b*b^4+11/9*a*a21b^2
      +2/3*b^3*a30b+11/36*b21b+1/9*a^3*a21b-5/3*a*b30b+2*b21b^2*a+23/3*b21b*b30b
      -2/9*a*b^4+4/9*a^3*b^2-2/9*a^5-40/9*b21b^3+152/9*a^3*b21b^2-2/3*b^5*a30b
      +40/9*b^6*b21b-160/9*a^2*b21b^3-16/3*a^3*b30b+2/9*a21b^2*a^3-1/9*a^3
      -6*b21b*a*b*a30b-8*a21b*b21b*b^2*a^2-1/3*a21b*a^2*b*a30b-8/9*a21b*a^3*b^2
      +4*a21b*b21b*a^4+4*a21b*b21b*b^4-2/9*a21b^2*b^2*a-4*a21b^2*b^2*b21b
      +4*a21b^2*a^2*b21b-2/3*a^4*b*a30b-152/9*b^2*b21b^2*a+4/3*a^2*b^3*a30b
      +40/3*b21b^2*b*a30b+10/3*a21b^2*b*a30b+8*a21b*b21b^2*a+4/9*a21b*a*b^4
      +52/3*a^2*b21b*b30b-40/3*a^2*b21b*b^4-52/3*b^2*b21b*b30b+16/3*b^2*a*b30b
      +40/3*b^2*b21b*a^4-3*b*a30b*b30b+1/3*a21b*b^3*a30b+52/3*a21b*b21b*b30b
      -10/3*a21b*a*b30b+23/18*a21b*b21b-11/12*b*a30b+11/9*b^2*a+5/9*b^2*b21b+5/9*a21b*a
      +7/9*a^2*b21b-40/9*a21b^3*b21b-20/9*a21b^3*a+4/9*a21b*a^5-40/9*a^6*b21b
      +160/9*b^2*b21b^3-160/9*a21b*b21b^3:
b17 := ...   (87 lines)
b19 := ...  (355 lines)
v111:= ... (1180 lines)

The perturbation method:
v3 := -2*a21b*b21b-a21b*a+1/2*a-1/2*b21b+3/2*b*a30b+2*b21b*b^2-2*b21b*a^2:
v5 := 4*a21b^2*b21b*a^2-7/3*a21b*a*b^2-8/9*a21b*b^2*a^3-11/9*a21b*b21b*b^2
      -2/9*a21b^2*a*b^2-11/36*a+16/3*a*b^2*b30b+4/3*b^3*a30b*a^2-25/9*a21b*b21b*a^2
      -4*a21b^2*b21b*b^2-1/6*a21b*b*a30b+40/3*b*a30b*b21b^2-152/9*a*b21b^2*b^2
      -10/3*a21b*a*b30b-40/3*b21b*b^4*a^2+4/9*a21b*a*b^4-2/3*b*a30b*a^4
      +92/9*b21b*b^2*a^2-3*b*a30b*b30b-11/12*b*a30b+11/36*b21b+4*a21b*b21b*b^4
      +4*a21b*b21b*a^4-52/3*b21b*b^2*b30b+52/3*a21b*b21b*b30b+52/3*b21b*a^2*b30b
      +1/3*a21b*b^3*a30b+40/3*b21b*b^2*a^4+160/9*b21b^3*b^2-16/3*a^3*b30b+23/3*b21b*b30b
      +11/9*a*b^2+4/9*b^2*a^3+5/9*a21b*a-40/9*a21b^3*b21b-20/9*a21b^3*a+40/9*b21b*b^6
      -40/9*b21b*a^6+152/9*b21b^2*a^3+1/9*a21b*a^3-5/3*a*b30b+2*a*b21b^2-8/9*a21b^2*b21b
      +11/9*a21b^2*a+7/9*b21b*a^2+5/9*b21b*b^2-2/3*b^5*a30b-160/9*a21b*b21b^3
      -160/9*b21b^3*a^2+2/9*a21b^2*a^3+2/3*b^3*a30b+4/9*a21b*a^5-2/9*a*b^4-14/3*b21b*a^4
      -50/9*b21b*b^4+23/18*a21b*b21b-1/3*a21b*b*a30b*a^2-8*a21b*b21b*b^2*a^2
      -6*a*b*a30b*b21b+10/3*a21b^2*b*a30b+8*a21b*a*b21b^2-40/9*b21b^3-1/9*a^3-2/9*a^5:
v7 := ...   (83 lines)
v9 := ...  (344 lines)
v11:= ... (1173 lines)

The singular point value method:
mu1 := I*(-3*b*a30b-a+b21b+4*b21b*a^2-4*b21b*b^2+4*a21b*b21b+2*a21b*a):
mu2 := -1/48*I*(736*b21b*b30b-17*a+1664*a21b*b21b*b30b+264*a21b*b*a30b-960*b21b^2*b*a30b
      -2544*a21b^2*b*a30b-352*a^3*a21b+64*b^2*a+1152*a^4*b21b-96*b^3*a30b-384*b^4*b21b
      -160*a*b30b-320*a*a21b*b30b+1664*a^2*b21b*b30b-1664*b^2*b21b*b30b-928*a^2*b*a30b
      -768*b^2*b21b*a^2+2976*b^3*a21b*a30b+864*b^2*a*a21b-384*b^4*a+768*b^2*a^3
      -1152*b^5*a30b+17*b21b-51*b*a30b-1152*a^4*b*a30b-1664*b21b^2*a+496*a21b^2*b21b
      -1024*a21b^2*a-20*a21b*b21b+122*a*a21b+1444*a^2*b21b+156*b^2*b21b+3392*a21b^3*b21b
      +1280*a^2*b21b^3+1280*b21b^3*a21b-1024*b^6*b21b+1024*a^6*b21b+768*a^5*a21b
      -2816*a^3*b21b^2+1984*a^3*a21b^2-512*a^3*b30b+1696*a*a21b^3-1280*b^2*b21b^3
      +2304*b^3*a30b*a^2-320*a^3-384*a^5+5760*a^4*a21b*b21b+512*a*b30b*b^2
      +5760*b^4*a21b*b21b+320*b21b^3+768*b^4*a*a21b-1536*b^2*a^3*a21b-3072*b^2*a^4*b21b
      +3072*b^4*a^2*b21b-512*b^2*a21b*b21b-288*b*a30b*b30b-11520*b^2*a^2*a21b*b21b
      +2752*a*b21b*b*a30b-2976*a^2*a21b*b*a30b-640*a^2*a21b*b21b-1984*b^2*a*a21b^2
      -8128*b^2*a21b^2*b21b-2176*a*b21b^2*a21b+8128*a^2*b21b*a21b^2+2816*b^2*a*b21b^2):
mu3:= ...   (85 lines)
mu4:= ...  (355 lines)
mu5:= ... (1156 lines)
```

The numbers given in the brackets denote the numbers of lines in the computer output files.

It is easy to see that

$$b_{13} = v_3 = \frac{i}{2}\,\mu_1,$$

which shows that the three different methods give the same first-order focus value (at most by a difference of a constant fact), as expected. For the second-order focus values, it can be shown that $b_{15} = v_5$, but μ_2 is not equal to v_5 by a difference of a constant fact. Further, for the third-order focus values, $b_{15} \neq v_5$. However, if setting $b_{13} = v_3 = \mu_1 = 0$, which results in

$$\bar{a}_{30} = \frac{a(2\bar{a}_{21} - 1) + \bar{b}_{21}(4\bar{a}_{21} + 4a^2 - 4b^2 + 1)}{3b},$$

then one has

$$b_{15} = v_5 = \frac{i}{2}\,\mu_2.$$

Further, if letting $b_{15} = v_5 = \mu_2 = 0$ (which yields $\bar{b}_{30} = \bar{b}_{30}(a, b, \bar{a}_{21}, \bar{b}_{21})$), then one obtains

$$b_{17} = v_7 = \frac{i}{2}\,\mu_3.$$

This process can be carried out to higher order focus values, i.e., if $b_{1(2i+1)} = v_{2i+1} = \mu_i = 0$, $i = 1, 2, \ldots, k-1$, then

$$b_{1(2k+1)} = v_{2k+1} = \frac{i}{2}\,\mu_k.$$

Remark: The above expressions of the focus values obtained by using different methods show that their computational complexity is in the same order (see the numbers of the computer output lines, given in the brackets).

4.2 A symmetric Liénard equation

The second example is the Liénard equation [36] described by

$$\frac{dx}{dt} = y, \quad \frac{dy}{dt} = -g(x) - f(x)y, \tag{53}$$

where $g(x)$ and $f(x)$ are polynomial functions of x. Here, we investigate a particular class of Liénard equations with Z_2 symmetry, in which $g(x)$ is a third-degree odd polynomial, while $f(x)$ is an even function of x. To be more specific, consider the following system:

$$\frac{dx}{dt} = y, \quad \frac{dy}{dt} = -\frac{1}{2}\,b^2 x(x^2 - 1) - y\sum_{i=0}^{m} a_i x^{2i}, \tag{54}$$

where $b \neq 0$ and a_i's are real coefficients. Equation (54) has three fixed points: $(0, 0)$ and $(\pm 1, 0)$. It is easy to use linear analysis to show that the origin $(0, 0)$ is a saddle point (with eigenvalues $\frac{1}{2}(-a_0 \pm \sqrt{a_0^2 + 2b^2})$). In order to have the two fixed points $(\pm 1, 0)$ being linear centers, the following condition must be satisfied:

$$\sum_{i=0}^{m} a_i = 0 \quad \text{or} \quad a_0 = -\sum_{i=1}^{m} a_i. \tag{55}$$

The eigenvalues of the Jacobian of system (54) evaluated at $(\pm 1, 0)$ are $\pm|b|i$. What we want to do is, for a given positive integer m, to choose appropriate values of a_i's such that system (54) has maximal number of limit cycles in the neighborhood of the two fixed points $(\pm 1, 0)$. This local analysis is based on the calculation of focus values or the normal form associated with the Hopf singularity.

By introducing the following scalings:

$$a_i \Longrightarrow b\,a_i, \quad i = 0, 1, \ldots, m,$$

the transformation

$$x = \pm(1 + u), \quad y = \pm bv,$$

and the time scaling $\tau = bt$, we obtain the following canonical form:

$$\frac{du}{d\tau} = v,$$
$$\frac{dv}{d\tau} = -u - \frac{3}{2}u^2 - \frac{1}{2}u^3 - v\sum_{i=0}^{m} a_i(1 + u)^{2i}. \tag{56}$$

Now, consider the case of $m = 5$. We apply the three methods to compute the focus values (normal forms) of system (56) and obtain the following results.

```
The Poincare method:
b13 := -3/4*a3-2*a4+1/4*a1-15/4*a5:
b15 := 1/8*a1+1/4*a2+7/8*a3+3/2*a4+5/8*a5-160/9*a4^3-15/4*a3^3-1000/9*a4*a5^2
      -25/36*a1^2*a5+5/9*a1^2*a2+5/12*a1^2*a3-20/3*a1*a4^2-5/4*a1*a3^2+5/9*a1*a2^2
      -625/36*a1*a5^2-5/3*a2^2*a3-40/9*a2^2*a4-25/3*a2^2*a5-160/9*a2*a4^2-5*a2*a3^2
      -125/3*a2*a5^2-20*a3^2*a4-125/4*a3^2*a5-100/3*a3*a4^2-875/12*a3*a5^2-20/9*a1*a2*a4
      -50/9*a1*a2*a5-20/3*a1*a3*a4-25/2*a1*a3*a5-200/9*a1*a4*a5-20*a2*a3*a4
      -100/3*a2*a3*a5-500/9*a2*a4*a5-100*a3*a4*a5-625/12*a5^3+5/36*a1^3-700/9*a4^2*a5:
b17 := ...  (40 lines)
b19 := ... (156 lines)
b111:= ... (507 lines)

The perturbation method:
v3 := -3/4*a3-2*a4-15/4*a5+1/4*a1:
v5 := -5*a3^2*a2+5/12*a3*a1^2-875/12*a3*a5^2-25/3*a5*a2^2-25/36*a5*a1^2+5/9*a2*a1^2
      -5/3*a3*a2^2-40/9*a4*a2^2-20*a4*a3^2+5/9*a2^2*a1-625/36*a5^2*a1-125/3*a5^2*a2
      -160/9*a4^2*a2-100/3*a4^2*a3-700/9*a4^2*a5-125/4*a3^2*a5-5/4*a3^2*a1
      -1000/9*a4*a5^2-20/3*a4^2*a1-160/9*a4^3-500/9*a4*a5*a2-15/4*a3^3-625/12*a5^3
      -200/9*a4*a5*a1-50/9*a5*a2*a1-25/2*a3*a5*a1-100/3*a3*a5*a2+5/36*a1^3-20/9*a4*a2*a1
      -20*a4*a3*a2-20/3*a4*a3*a1-100*a4*a3*a5+1/8*a1+1/4*a2+7/8*a3+3/2*a4+5/8*a5:
v7 := ...  (40 lines)
v9 := ... (156 lines)
v11:= ... (501 lines)

The singular point value method:
mu1 := Complex(-1/2)*(a1-3*a3-8*a4-15*a5):
mu2 := Complex(1/96)*(81*a1-48*a2-1128*a4-483*a3-1695*a5-768*a2^2*a3-2048*a2^2*a4
      -3840*a2^2*a5-8000*a5^2*a1-33600*a5^2*a3-51200*a5^2*a4-15360*a4^2*a3
      -35840*a4^2*a5-8192*a4^2*a2+192*a1^2*a3-320*a1^2*a5-576*a3^2*a1-9216*a3^2*a4
      -14400*a3^2*a5+256*a2^2*a1-3072*a4^2*a1-9216*a3*a2*a4-46080*a3*a4*a5
      -25600*a4*a2*a5-2560*a1*a2*a5-19200*a5^2*a2+64*a1^3-1728*a3^3-8192*a4^3
      -24000*a5^3-10240*a1*a4*a5-3072*a1*a4*a3-5760*a1*a3*a5-1024*a1*a2*a4
      -15360*a3*a2*a5+256*a1^2*a2-2304*a3^2*a2):
mu3:= ...  (40 lines)
mu4:= ... (162 lines)
mu5:= ... (542 lines)
```

Similar to the first example, the first-order focus values are

$$b_{13} = v_3 = \frac{i}{2}\,\mu_1,$$

which again indicates that the three different methods give the same first-order focus value (at most by a difference of a constant fact). For the second-order focus values, we have $b_{15} = v_5$, while μ_2 is not equal to v_5 by a difference of a constant fact. Similarly, for the third-order focus values, $b_{17} \neq v_7$. Setting $b_{13} = v_3 = \mu_1 = 0$ yields

$$a_1 = 3a_3 + 8a_4 + 15a_5,$$

and

$$b_{15} = v_5 = \frac{i}{2}\,\mu_2.$$

Further letting $b_{15} = v_5 = \mu_2 = 0$ results in

$$a_2 = -5a_3 - 10a_4 - 10a_5,$$

and

$$b_{17} = v_7 = \frac{i}{2}\,\mu_3.$$

Continuing this process yields that

$$b_{1(2i+1)} = v_{2i+1} = \mu_i = 0 \ \ (i = 1, 2, \ldots, k-1)$$
$$\Longrightarrow \quad b_{1(2k+1)} = v_{2k+1} = \frac{i}{2}\,\mu_k.$$

This example again shows that the three different methods have the same computational complexity.

To end this section, we present a numerical example with 10 small limit cycles. We have the following focus values:

$$
\begin{aligned}
v_1 &= -(a_0 + a_1 + a_2 + a_3 + a_4 + a_5),\\
v_3 &= \frac{1}{4}(a_1 - 3a_3 - 8a_4 - 15a_5),\\
&\quad \text{when } v_1 = 0,\\
v_5 &= \frac{1}{4}(a_2 + 5a_3 + 10a_4 + 10a_5),\\
&\quad \text{when } v_1 = v_3 = 0,\\
v_7 &= -\frac{5}{16}(a_3 - 14a_5),\\
&\quad \text{when } v_1 = v_3 = v_5 = 0,\\
v_9 &= -\frac{7}{16}(a_4 + 9a_5),\\
&\quad \text{when } v_1 = v_3 = v_5 = v_7 = 0,\\
v_{11} &= \frac{21}{32}\,a_5,\\
&\quad \text{when } v_1 = v_3 = v_5 = v_7 = v_9 = 0.
\end{aligned}
$$

Let $a_5 = 0.002$, and select the other parameters as

$$
\begin{aligned}
a_5 &= 0.002,\\
a_4 &= -9a_5 + \epsilon_1,\\
a_3 &= 14a_5 - \epsilon_2,\\
a_2 &= -5a_3 - 10a_4 - 10a_5 - \epsilon_3,\\
a_1 &= 3a_3 + 8a_4 + 15a_5 + \epsilon_4,\\
a_0 &= -a_1 - a_2 - a_3 - a_4 - a_5 + \epsilon_5,
\end{aligned}
$$

where the perturbations are chosen as

$$
\epsilon_1 = 0.1 \times 10^{-4},\ \ \epsilon_2 = 0.1 \times 10^{-7},
$$
$$
\epsilon_3 = 0.1 \times 10^{-11},\ \ \epsilon_4 = 0.1 \times 10^{-16},
$$
$$
\epsilon_5 = 0.1 \times 10^{-23}.
$$

Then, we have the perturbed focus values

$$
\begin{aligned}
v_{11} &= 0.13092188621480516884116851907 1 \times 10^{-2},\\
v_9 &= -0.43729668984998167395805239021 6 \times 10^{-5},\\
v_7 &= 0.31248592319463145473698632407 \times 10^{-8},\\
v_5 &= -0.24999874858487645346314 \times 10^{-12},\\
v_3 &= 0.25 \times 10^{-17},\\
v_1 &= -0.1 \times 10^{-23},
\end{aligned}
$$

and the perturbed parameters

$$
\begin{aligned}
a_0 &= -0.00199000999990000009999999,\\
a_1 &= -0.02992002999999999,\\
a_2 &= 0.019900049999,\\
a_3 &= 0.02799999,\\
a_4 &= -0.01799,\\
a_5 &= 0.002,
\end{aligned}
$$

under which the approximate amplitudes of the five small limit cycles are obtained as

$$
r_1 = 0.000646, \quad r_2 = 0.003344, \quad r_3 = 0.008842,
$$
$$
r_4 = 0.029766, \quad r_5 = 0.048554.
$$

The simulated phase portrait for this example is shown in Fig. 3, where 3 large limit cycles, which enclose all the 10 small limit cycles, are also obtained. It should be pointed out that the large limit cycles can be numerically simulated, while the small limit cycles cannot be obtained by numerical simulation. The existence of the small limit cycles must be proved theoretically.

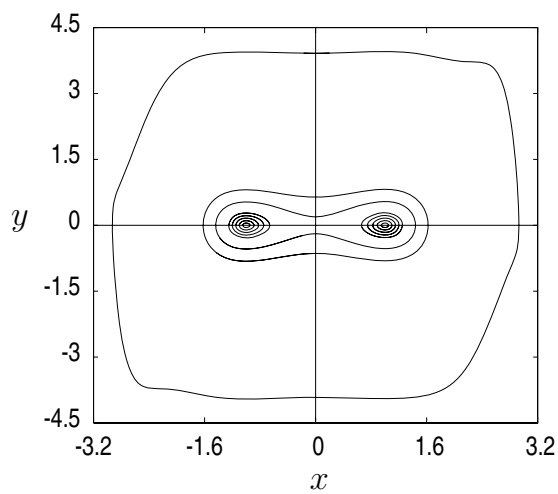

Fig. 3 Simulated phase portrait of Leénard equation (54) showing 10 small limit cycles around the fine focus points $(\pm 1, 0)$ and 3 large limit cycles under the perturbed parameter values: $b = 1$, $a_0 = -0.00199000999990000009999999$, $a_1 = -0.02992002999999999$, $a_2 = 0.019900049999$, $a_3 = 0.02799999$, $a_4 = -0.01799$, and $a_5 = 0.002$

5 Remarks on the minimal singular point value

In this section, we propose the concept of "the minimal singular point value" based on the definition of singular point value. This generalization is similar to the generalization of the normal form theory to the unique normal form theory, which is also called the minimal normal theory or the simplest normal form theory (e.g., see [37–40].

In order to illustrate the concept, we re-investigate the second example considered in the previous section. Recall that the basic idea in computing the singular point value is to simplify the computation of the focus values. Let us look at the relation between the singular point values and the focus point values, given by Equation (31):

$$v_{2k+1}(2\pi) = i\pi\left(\mu_k + \sum_{j=1}^{k-1}\xi_k^{(j)}\mu_j\right), \quad k = 1, 2, \ldots,$$

where $\xi_m^{(j)}$ $(j = 1, 2, \ldots, k-1)$ are polynomial functions. The main question here is how to choose the polynomial functions $\xi_k^{(j)}$ so that μ_k becomes the simplest. In other words, when we are trying to find μ_k, we should use the previous information of $\xi_k^{(j)}$ as much as possible.

Definition 1. The kth-order minimal singular point value, $\bar{\mu}_k$, is the one such that the expression of μ_k in Equation (31) becomes the simplest under proper choices of the polynomial functions $\xi_k^{(j)}$, and then $\bar{\mu}_k = \mu_k$.

For the singular point values given for the Liénard system (see the μ_j's given in the previous section), let us check if these singular point values are the simplest. In other words, we want to see if we can further simplify the expressions of the singular point values. For simplicity, let $\bar{\mu}_j = \frac{i}{2}\mu_j$. Then, Equation (31) becomes

$$v_{2k+1} = \bar{\mu}_k + \sum_{j=1}^{k-1}\xi_k^{(j)}\bar{\mu}_j, \quad k = 1, 2, \ldots, \tag{57}$$

Now, consider the focus values v_j. It has been shown that

$$v_3 = \bar{\mu}_1 = \frac{1}{4}(a_1 - 3a_3 - 8a_4 - 15a_5). \tag{58}$$

Then, we obtain

$$\begin{aligned} v_5 = {} & \frac{1}{4}(a_2 + 5a_3 + 10a_4 + 10a_5) \\ & + \frac{1}{72}(a_1 - 3a_3 - 8a_4 - 15a_5) \\ & \times \big[9 + 10\big(a_1^2 + 4a_2^2 + 9a_3^2 + 16a_4^2 + 25a_5^2\big) \\ & + 20a_1(2a_2 + 3a_3 + 4a_4 + 5a_5) + 40a_2(3a_3 \\ & + 4a_4 + 5a_5) + 60a_3(4a_4 + 5a_5) + 400a_4a_5\big], \end{aligned} \tag{59}$$

which indicates that we may choose

$$\bar{\mu}_2 = \frac{1}{4}(a_2 + 5a_3 + 10a_4 + 10a_5), \tag{60}$$

and

$$\begin{aligned} \xi_1^{(1)} = {} & \frac{1}{18}\big[9 + 10\big(a_1^2 + 4a_2^2 + 9a_3^2 + 16a_4^2 + 25a_5^2\big) \\ & + 20a_1(2a_2 + 3a_3 + 4a_4 + 5a_5) + 40a_2(3a_3 \\ & + 4a_4 + 5a_5) + 60a_3(4a_4 + 5a_5) + 400a_4a_5\big]. \end{aligned} \tag{61}$$

Thus, we have a simple expression for v_5:

$$v_5 = \bar{\mu}_2 + \xi_1^{(1)}\bar{\mu}_1. \tag{62}$$

For v_7, after a lengthy manipulation, we similarly have

$$\begin{aligned} v_7 = {} & -\frac{5}{16}(a_3 - 14a_5) \\ & + \frac{1}{82944}(a_1 - 3a_3 - 8a_4 - 15a_5) \\ & \times \{(a_1 - 3a_3 - 8a_4 - 15a_5) \\ & \times \big[(a_1 - 15a_5 - 3a_3 - 8a_4)\big(7540(a_1 - 3a_3 \\ & - 8a_4 - 15a_5)(a_1 - 19a_3 - 40a_4 - 15a_5) \\ & + 723840a_3^2 + 2895360a_3a_4 + 24489 \\ & + 2895360a_4^2\big) + 160(a_2 + 5a_3 + 10a_4 + 10a_5) \\ & \times\big(377(a_1 - 3a_3 - 8a_4 - 15a_5)(a_1 - 15a_3 \\ & -32a_4 - 15a_5) + 377(a_2 + 5a_3 + 10a_4 + 10a_5) \\ & \times (3a_1 + 4a_2 - 13a_3 - 32a_4 - 5a_5) \end{aligned}$$

$$
\begin{aligned}
&+477+18096a_3^2+72384a_4^2+72384a_3a_4\big)\\
&-\big(55296a_3-36864a_4-737280a_5\\
&+11581440a_4a_3^2+23162880a_4^2a_3\\
&+15441920a_4^3+1930240a_3^3\big)\big]\\
&+16(a_2+5a_3+10a_4+10a_5)\\
&\times\big[(a_2+5a_3+10a_4+10a_5)\big(7540(a_2+5a_3\\
&+10a_4+10a_5)(a_2-3a_3-6a_4+10a_5)+5643\\
&+180960a_3^2+723840a_3a_4+723840a_4^2\big)\\
&-\big(7452a_3-3528a_4-92160a_5+1930240a_4^3\\
&+241280a_3^3+1447680a_4a_3^2+2895360a_4^2a_3\big)\big]\\
&+\big(16605+61767680a_4^3a_3-1080576a_4a_3\\
&+46325760a_4^2a_3^2-5898240a_4a_5\\
&+15441920a_4a_3^3-2949120a_3a_5-1670400a_4^2\\
&-122688a_3^2+1930240a_3^4+30883840a_4^4\big)\big\}\\
&+\frac{1}{576}(a_2+5a_3+10a_4+10a_5)\big[328(a_2+5a_3\\
&+10a_4+10a_5)(a_2+a_3+2a_4+10a_5)+81\\
&+1312a_3^2+5248a_3a_4+5248a_4^2\big]\\
\equiv\ &\bar{\mu}_3+\xi_2^{(1)}\bar{\mu}_1+\xi_2^{(2)}\bar{\mu}_2, \qquad (63)
\end{aligned}
$$

where $\bar{\mu}_3=-\frac{5}{16}(a_3-14a_5)$.

Similarly, we can find $\bar{\mu}_4$ and $\bar{\mu}_5$. In summary, we have

$$
\begin{aligned}
\bar{\mu}_1&=\frac{1}{4}(a_1-3a_3-8a_4-15a_5),\\
\bar{\mu}_2&=\frac{1}{4}(a_2+5a_3+10a_4+10a_5),\\
\bar{\mu}_3&=-\frac{5}{16}(a_3-14a_5),\\
\bar{\mu}_4&=-\frac{7}{16}(a_4+9a_5),\\
\bar{\mu}_5&=\frac{21}{32}a_5. \qquad (64)
\end{aligned}
$$

The above singular point values are much simpler than those obtained by using the singular point method, given in the previous section. These simple values are merely very simple linear functions of the coefficients, and believed to be the minimal or the simplest singular point values. In fact, note that these simpler singular point values are exactly the ones obtained in the previous section for the second example, i.e., $\bar{\mu}_k=v_{2k+1}$ when v_i $(i=1,3,\ldots,2k-1)$ is set zero. This is certainly not surprising for this example, since each solution from $v_i=0$ is just a simple linear expression.

However, no method has been developed so far for computing the minimal singular point values for general nonlinear dynamical systems. This is an important research topic for future study of nonlinear dynamical systems.

6 Conclusions

In this paper, we have considered three typical methods for computing focus values with a detailed comparison. Applications of these methods to some practical problems and Hilbert's 16th problem have shown that the different methods have the same complexity, but none of them gives the simplest (or minimal) singular point value. Further research is needed to develop an efficient approach for computing focus values.

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