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Accurate analytical approximate solutions to general strong nonlinear oscillators

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Abstract A new approach is presented for establishing the analytical approximate solutions to general strong nonlinear conservative single-degree-of-freedom systems. Introducing two odd nonlinear oscillators from the original general nonlinear oscillator and utilizing the analytical approximate solutions to odd nonlinear oscillators proposed by the authors, we construct the analytical approximate solutions to the original general nonlinear oscillator. These analytical approximate solutions are valid for small as well as large oscillation amplitudes. Two examples are presented to illustrate the great accuracy and simplicity of the new approach.

Keywords General nonlinear oscillator · Large amplitude · Odd nonlinearity · Newton method · Harmonic balance (HB) · Analytical approximation

1 Introduction

The harmonic balance (HB) method can be used to determine analytical approximate solutions to nonlinear oscillatory systems for which the nonlinear terms are "not small", i.e., no perturbation parameter needs to exist [1–3]. However, it is very difficult to construct analytical approximations of higher accuracy using such an approach because it requires analytical solutions to sets of complicated nonlinear algebraic equations.

As such, some authors developed various improved HB methods, such as: incremental harmonic balance in [4]; rational representations in [5]; combination of linearization with respect to incremental displacement only with the harmonic balance method in [6] and coupling of the Newton method with the harmonic balance method in [7]. In general, the success of these methods [3, 5–7] for conservative systems requires that the nonlinear restoring force $f(u)$ is an odd function of u [i.e. $f(-u) = -f(u)$], where u represents the displacement measured from the stable equilibrium position [8, 9]. If this condition is not satisfied, these various HB methods will leads to inconsistencies [10]. To overcome this deficiency, Gottlieb [11] and Wu et al. [12, 13] presented some methods to solve the nonlinear oscillators with a general nonlinear restoring force. For more accurate approximation, however, these methods result in a complex nonlinear algebraic equation(s) in terms of unknown frequency and analytical solution is difficult again.

In this paper, we present a new approach to establish accurate analytical approximate period and periodic solution to general strong nonlinear conservative single-degree-of-freedom oscillators. Based on the original general nonlinear oscillator, two new oscillators with odd nonlinearity are first addressed [13]. Utilizing the analytical approximate solutions to odd nonlinear oscillators developed by the authors [7], we construct new

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analytical approximate solutions to the original general nonlinear oscillator by piecing the approximate solutions corresponding to, respectively, the two new oscillators introduced. The interesting features of the proposed approach are its simplicity and excellent accuracy of both period and corresponding periodic solution for small as well as large oscillation amplitudes. Two examples are presented to validate the new approach.

2 Method of solution

Consider a conservative single-degree-of-freedom system governed by

$$\frac{d^2u}{dt^2}+f(u)=0,\quad u(0)=A,\quad \frac{du}{dt}(0)=0 \tag{1}$$

where $f(u)$ is a general nonlinear function of u. Let $V(u)=\int f(u)du$ be the potential energy of the system and suppose it reaches its minimum at $u=u_0$, called a centre. We assume $u_0=0$. Thus, the system will oscillate between asymmetric limits $[-B, A]$ where both $-B(B>0)$ and A have the same energy level, i.e.,

$$V(-B)=V(A) \tag{2}$$

Here, we combine the method of Wu et al. [7] with strategy of Wu and Lim [13] to construct analytical approximate expressions for the solution to the Equation (1). For the sake of convenience, the results established in [7], which is applicable only to the case of $f(-u)=-f(u)$, is briefly summarized as follows.

By coupling the Newton method with the method of harmonic balance, Wu et al. [7] obtained three analytical approximate periods and corresponding periodic solutions. The first analytical approximation to the period and periodic solution is

$$\begin{aligned}&T_1(A)=2\pi/\sqrt{\Omega_1(A)};\quad u_1(t)=A\cos\tau,\\&\quad\tau=\sqrt{\Omega_1(A)}t\end{aligned} \tag{3}$$

where

$$\begin{aligned}&\Omega_1(A)=a_1/A,\\&a_{2i-1}=\frac{4}{\pi}\int_0^{\pi/2}f\,(A\cos\tau)\cos[(2i-1)\tau]\,d\tau,\\&\quad i=1,2,\ldots\end{aligned} \tag{4}$$

The second analytical approximation to the period and periodic solution is

$$\begin{aligned}&T_2(A)=2\pi/\sqrt{\Omega_2(A)},\\&\Omega_2(A)=\Omega_1(A)+\Delta\Omega_1(A);\\&u_2(t)=X(A)\cos\tau+Y(A)\cos3\tau,\\&\quad\tau=\sqrt{\Omega_2(A)}t\end{aligned} \tag{5}$$

where

$$\begin{aligned}\Delta\Omega_1(A)&=a_3[2a_1-(b_0-b_4)A]/\Phi,\\X(A)&=A-2a_3A^2/\Phi,\quad Y(A)=2a_3A^2/\Phi,\\\Phi&=A[(b_2+b_4-b_0-b_6)A+18a_1],\\b_{2(i-1)}&=\frac{4}{\pi}\int_0^{\pi/2}f_u(A\cos\tau)\cos[2(i-1)\tau]d\tau,\\i&=1,\ 2,\ldots\end{aligned} \tag{6}$$

The third analytical approximation to the period and periodic solution is

$$\begin{aligned}&T_3(A)=2\pi/\sqrt{\Omega_3(A)},\\&\Omega_3(A)=\Omega_2(A)+\Delta\Omega_2(A)\\&u_3(t)=[X(A)+y_1(A)]\cos\tau+[Y(A)-y_1(A)\\&\qquad+y_2(A)]\cos3\tau-y_2(A)\cos5\tau,\\&\quad\tau=\sqrt{\Omega_3(A)}t\end{aligned} \tag{7}$$

where

$$\begin{aligned}\Delta\Omega_2(A)&=[c_5(\lambda_1\lambda_4-\lambda_2\lambda_3+\lambda_3\lambda_4)\\&\quad+(c_3-9Y\Omega_2)(\lambda_3\lambda_6-\lambda_1\lambda_5)\\&\quad+(c_1-X\Omega_2)(\lambda_2\lambda_5-\lambda_4\lambda_5-\lambda_4\lambda_6)]/\mathrm{E},\\y_1(A)&=[X(c_5\lambda_4-c_5\lambda_2+c_3\lambda_6)\\&\quad+9Y(c_5\lambda_1-c_1\lambda_6)]/\mathrm{E},\\y_2(A)&=[X(c_5\lambda_4-c_3\lambda_5)-9Y(c_5\lambda_3-c_1\lambda_5)]/\mathrm{E},\\\mathrm{E}&=X(\lambda_2\lambda_5-\lambda_4\lambda_5-\lambda_4\lambda_6)\\&\quad-9Y(\lambda_3\lambda_6-\lambda_1\lambda_5),\\\lambda_1&=(d_2-d_6)/2,\quad\lambda_2=(d_4-d_8)/2,\\\lambda_3&=(d_0-d_4)/2-\Omega_2,\\\lambda_4&=(d_2+d_4-d_6-d_0)/2+9\Omega_2,\end{aligned}$$

$$\lambda_5 = (d_4 - d_2 + d_6 - d_8)/2,$$
$$\lambda_6 = (d_2 + d_8 - d_{10} - d_0)/2 + 25\Omega_2,$$
$$c_{2i-1} = \frac{4}{\pi}\int_0^{\pi/2} f\,[X(A)\cos\tau + Y(A)\cos 3\tau] \times \cos[(2i-1)\tau]\,d\tau, \quad i = 1,\,2,\,\ldots,$$
$$d_{2(i-1)} = \frac{4}{\pi}\int_0^{\pi/2} f_u\,[X(A)\cos\tau + Y(A)\cos 3\tau] \times \cos[2(i-1)\tau]\,d\tau, \quad i = 1,\,2,\,\ldots. \tag{8}$$

Note that, to obtain the analytical approximation solutions above, it is necessary to compute some of the coefficients $a_{2i-1}, b_{2(i-1)}, c_{2i-1}, d_{2(i-1)}, (i = 1,\,2,\,\ldots)$ for the case of $f(-u) = -f(u)$.

For the case of $f(u)$ being general nonlinear function of u, following the approach in [13], we introduce the two odd nonlinear oscillating systems:

$$\frac{d^2u}{dt^2} + g(u) = 0, \quad u(0) = A, \quad \frac{du}{dt}(0) = 0 \tag{9a}$$

and

$$\frac{d^2u}{dt^2} + h(u) = 0, \quad u(0) = B, \quad \frac{du}{dt}(0) = 0 \tag{9b}$$

where

$$g(u) = \begin{cases} f(u) & \text{if } u \geq 0, \\ -f(-u) & \text{if } u < 0, \end{cases} \tag{10a}$$

$$h(u) = \begin{cases} -f(-u) & \text{if } u \geq 0, \\ f(u) & \text{if } u < 0 \end{cases} \tag{10b}$$

In Equation 10(a) and (b), both $g(u)$ and $h(u)$ are odd functions of u. Hence, Equation 9(a) and (b) represent the two oscillating systems with odd nonlinearity, respectively. Substituting function g and h (also B instead of A for the latter) in Equation 10(a) and (b) with function f in Equations (4), (6) and (8), respectively, we may achieve the corresponding Fourier coefficients of the two systems with odd nonlinearity, as follows:

$$a_{(2i-1)g},\; b_{2(i-1)g},\; c_{(2i-1)g},\; d_{2(i-1)g},\; a_{(2i-1)h},\; b_{2(i-1)h},\; c_{(2i-1)h},\; d_{2(i-1)h}, \quad i = 1, 2, \ldots \tag{11}$$

Using the coefficients in Equation (11) and the analytical approximations in Equations (3), (5) and (7), we may obtain the first, second and third analytical approximate periods and the corresponding periodic solutions $T_{1g}(A)$, $u_{1g}(t)$, $T_{2g}(A)$, $u_{2g}(t)$, $T_{3g}(A)$, $u_{3g}(t)$ for system in Equation (9a), and $T_{1h}(B)$, $u_{1h}(t)$, $T_{2h}(B)$, $u_{2h}(t)$, $T_{3h}(B)$, $u_{3h}(t)$ for system in Equation (9b), respectively. The relation between the exact solutions of Equation 9(a) and (b), and that of Equation (1) can provide a definite guide for establishing analytical approximate solutions to Equation (1).

Let the exact period and periodic solution corresponding to Equation 9(a) and (b) be $T_{eg}(A)$ and $u_{eg}(t)$, and $T_{eh}(B)$ and $u_{eh}(t)$, respectively. Then the exact period $T_e(A)$ and the periodic solution $u_e(t)$ to Equation (1) may be obtained by piecing the two solutions above [13]:

$$T_e(A) = \frac{T_{eg}(A)}{2} + \frac{T_{eh}(B)}{2} \tag{12a}$$

and

$$u_e(t) = \begin{cases} u_{eg}(t), & \text{for } 0 \leq t \leq \frac{T_{eg}(A)}{4} \\ u_{eh}\left(t - \frac{T_{eg}(A)}{4} + \frac{T_{eh}(B)}{4}\right), & \text{for } \frac{T_{eg}(A)}{4} \leq t \leq \frac{T_{eg}(A)}{4} + \frac{T_{eh}(B)}{2} \\ u_{eg}\left(t + \frac{T_{eg}(A)}{2} - \frac{T_{eh}(B)}{2}\right), & \text{for } \frac{T_{eg}(A)}{4} + \frac{T_{eh}(B)}{2} \leq t \leq \frac{T_{eg}(A)}{2} + \frac{T_{eh}(B)}{2} \end{cases} \tag{12b}$$

Utilizing the analytical approximate solutions to Equation 9(a) and (b) and the relations in Equation 12(a) and (b), we can get the corresponding the kth ($k = 1, 2, 3$) analytical approximate period and periodic solution of Equation (1) as follows:

$$T_k(A) = \frac{T_{kg}(A)}{2} + \frac{T_{kh}(B)}{2} \tag{13a}$$

and

$$u_k(t) = \begin{cases} u_{kg}(t), & \text{for } 0 \leq t \leq \frac{T_{kg}(A)}{4} \\ u_{kh}\left(t - \frac{T_{kg}(A)}{4} + \frac{T_{kh}(B)}{4}\right), & \text{for } \frac{T_{kg}(A)}{4} \leq t \leq \frac{T_{kg}(A)}{4} + \frac{T_{kh}(B)}{2} \\ u_{kg}\left(t + \frac{T_{kg}(A)}{2} - \frac{T_{kh}(B)}{2}\right), & \text{for } \frac{T_{kg}(A)}{4} + \frac{T_{kh}(B)}{2} \leq t \leq \frac{T_{kg}(A)}{2} + \frac{T_{kh}(B)}{2} \end{cases} \tag{13b}$$

3 Illustrative examples

In this section, we will show through two examples that Equation 13(a) and (b) can provide excellent analytical approximations to the period and the corresponding periodic solutions of general nonlinear oscillators for small as well as large oscillation amplitudes.

Example 1. Consider the quadratic-cubic nonlinear oscillator governed by

$$\frac{d^2y}{dt^2} + y^2 + y^3 = 0, \quad y(0) = \hat{A}, \quad \frac{dy}{dt}(0) = 0 \tag{14a}$$

The potential energy function is $\hat{V}(y) = y^3/3 + y^4/4$, and it reaches its minimum at $y = -1$. Introducing a new variable, $u = y + 1$, we transform Equation (1) to

$$\frac{d^2u}{dt^2} + u - 2u^2 + u^3 = 0, \quad u(0) = A, \quad \frac{du}{dt}(0) = 0 \tag{14b}$$

where $A = \hat{A} + 1$.

Now, we study Equation (14b). For this example $f(u) = u - 2u^2 + u^3$, the corresponding potential energy function is

$$V(u) = u^2/2 - 2u^3/3 + u^4/4 \tag{15}$$

Using Equations (2) and (15), we can express $B(B > 0)$ in terms of A:

$$B = -\frac{2}{3} + \frac{1}{2}\sqrt{\frac{4}{9} + \frac{2(1-12K)}{3P} + \frac{2P}{3}} + \frac{1}{2}\sqrt{\frac{8}{9} - \frac{2(1-12K)}{3P} - \frac{2P}{3} + \frac{16}{27}\left(\frac{4}{9} + \frac{2(1-12K)}{3P} + \frac{2P}{3}\right)^{-1/2}} \tag{16}$$

where

$$K = A^2/2 - 2A^3/3 + A^4/4,$$

$$P = \left[1 - 12K + 2\sqrt{3K(1 - 24K + 144K^2)}\right]^{1/3}$$

For this example, using Equation (11), we obtain the corresponding Fourier coefficients as follow:

$$a_{1g} = A + \frac{3A^3}{4} - \frac{16A^2}{3\pi}, \quad a_{3g} = \frac{A^2(15A\pi - 64)}{60\pi},$$

$$b_{0g} = 2 + 3A^2 - \frac{16A}{\pi}, \quad b_{2g} = \frac{A(9\pi A - 32)}{6\pi},$$

$$b_{4g} = \frac{16A}{15\pi}, \quad b_{6g} = -\frac{16A}{35\pi},$$

$$c_{1g} = \frac{X_g}{4}\left(4 + 3X_g^2 + 3X_gY_g + 6Y_g^2\right) - \frac{16}{105\pi}\left(35X_g^2 + 14X_gY_g + 27Y_g^2\right),$$

$$c_{3g} = \frac{X_g^3}{4} - \frac{288X_gY_g}{35\pi} - \frac{4X_g^2}{\pi}\left(\frac{4}{15} - \frac{3\pi Y_g^2}{8}\right) + \frac{Y_g}{36\pi}\left(64Y_g + 36\pi - 27\pi Y_g^2\right),$$

$$c_{5g} = -\frac{144Y_g^2}{55\pi} + \frac{4X_g^2}{\pi}\left(\frac{4}{105} + \frac{3\pi Y_g}{16}\right) - \frac{4X_g}{\pi}\left(\frac{40Y_g}{63} - \frac{3\pi Y_g^2}{16}\right),$$

$$d_{0g} = 2 + 3X_g^2 + 3Y_g^2 + \frac{16(Y_g - 3X_g)}{3\pi},$$

$$d_{2g} = \frac{3X_g(X_g + 2Y_g)}{2} - \frac{16(9Y_g + 5X_g)}{15\pi},$$

$$d_{4g} = -\frac{48Y_g}{7\pi} + \frac{4X_g}{\pi}\left(\frac{4}{15} + \frac{3\pi Y_g}{4}\right),$$

$$d_{6g} = \frac{4}{\pi}\left(\frac{4Y_g}{9} - \frac{4X_g}{35} + \frac{3\pi Y_g^2}{8}\right),$$

$$d_{8g} = \frac{4}{\pi}\left(\frac{4X_g}{63} - \frac{12Y_g}{55}\right),$$

$$d_{10g} = \frac{4}{\pi}\left(\frac{12Y_g}{91} - \frac{4X_g}{99}\right), \tag{17a}$$

and

$$a_{1h} = B + \frac{3B^3}{4} + \frac{16B^2}{3\pi}, \quad a_{3h} = \frac{B^2(15B\pi + 64)}{60\pi},$$

$$b_{0h} = 2 + 3B^2 + \frac{16B}{\pi}, \quad b_{2h} = \frac{B(9\pi B + 32)}{6\pi},$$

$$b_{4h} = -\frac{16B}{15\pi}, \quad b_{6h} = \frac{16B}{35\pi},$$

$$
\begin{aligned}
c_{1h} &= \frac{X_h}{4}\left(4+3X_h^2+3X_hY_h+6Y_h^2\right)\\
&\quad + \frac{16}{105\pi}\left(35X_h^2+14X_hY_h+27Y_h^2\right),\\
c_{3h} &= \frac{X_h^3}{4}+\frac{288X_hY_h}{35\pi}+\frac{4X_h^2}{\pi}\left(\frac{4}{15}+\frac{3\pi Y_h^2}{8}\right)\\
&\quad + \frac{Y_h}{36\pi}\left(36\pi+27\pi Y_h^2-64Y_h\right),\\
c_{5h} &= \frac{144Y_h^2}{55\pi}+\frac{4X_h^2}{\pi}\left(\frac{3\pi Y_h}{16}-\frac{4}{105}\right)\\
&\quad + \frac{4X_h}{\pi}\left(\frac{40Y_h}{63}+\frac{3\pi Y_h^2}{16}\right),\\
d_{0h} &= 2+3X_h^2+3Y_h^2+\frac{16(3X_h-Y_h)}{3\pi},\\
d_{2h} &= \frac{3X_h(X_h+2Y_h)}{2}+\frac{16(9Y_h+5X_h)}{15\pi},\\
d_{4h} &= \frac{48Y_h}{7\pi}+\frac{4X_h}{\pi}\left(\frac{3\pi Y_h}{4}-\frac{4}{15}\right),\\
d_{6h} &= \frac{4}{\pi}\left(\frac{4X_h}{35}-\frac{4Y_h}{9}+\frac{3\pi Y_h^2}{8}\right),\\
d_{8h} &= \frac{4}{\pi}\left(\frac{12Y_h}{55}-\frac{4X_h}{63}\right),\\
d_{10h} &= \frac{4}{\pi}\left(\frac{4X_h}{99}-\frac{12Y_h}{91}\right)
\end{aligned}
\tag{17b}
$$

where

$$
\begin{aligned}
X_g &= Q_1(A), \quad Y_g = Q_2(A),\\
X_h &= -Q_1(-B), \quad Y_h = -Q_2(-B)\\
Q_1(X) &\equiv \frac{X(3360\pi-17152X+2415\pi X^2)}{40(84\pi-440X+63\pi X^2)},\\
Q_2(X) &\equiv \frac{7X^2(15\pi X-64)}{40(84\pi-440X+63\pi X^2)}
\end{aligned}
$$

Substituting Equation 17(a) and (b) into Equations (3), (5) and (7) leads to the first three analytical approximate periods and the corresponding periodic solutions, respectively. They are:

$$
\begin{aligned}
&T_{1g}(A) = 2\pi/\sqrt{(-64A+12\pi+9\pi A^2)/(12\pi)},\\
&u_{1g}(t) = A\cos\tau, \quad \tau = \frac{2\pi t}{T_{1g}(A)},
\end{aligned}
\tag{18a}
$$

$$
\begin{aligned}
&T_{1h}(B) = 2\pi/\sqrt{(64B+12\pi+9\pi B^2)/(12\pi)},\\
&u_{1h}(t) = B\cos\tau, \quad \tau = \frac{2\pi t}{T_{1h}(B)};
\end{aligned}
\tag{18b}
$$

$$
\begin{aligned}
&T_{2g}(A) = \frac{2\pi}{\sqrt{\Omega_{2g}}}, \quad u_{2g}(t) = X_g\cos\tau+Y_g\cos 3\tau,\\
&\tau = \sqrt{\Omega_{2g}}t,
\end{aligned}
\tag{19a}
$$

$$
\begin{aligned}
&T_{2h}(B) = \frac{2\pi}{\sqrt{\Omega_{2h}}}, \quad u_{2h}(t) = X_h\cos\tau+Y_h\cos 3\tau,\\
&\tau = \sqrt{\Omega_{2h}}t;
\end{aligned}
\tag{19b}
$$

$$
\begin{aligned}
&T_{3g}(A) = \frac{2\pi}{\sqrt{\Omega_{3g}}}, \quad \Omega_{3g} = \Omega_{2g}+\Delta\Omega_{2g},\\
&u_{3g}(t) = (X_g+y_{1g})\cos\tau+(Y_g-y_{1g}+y_{2g})\cos 3\tau\\
&\quad - y_{2g}\cos 5\tau, \quad \tau = \sqrt{\Omega_{3g}}t,
\end{aligned}
\tag{20a}
$$

$$
\begin{aligned}
&T_{3h}(B) = \frac{2\pi}{\sqrt{\Omega_{3h}}}, \quad \Omega_{3h} = \Omega_{2h}+\Delta\Omega_{2h},\\
&u_{3h}(t) = (X_h+y_{1h})\cos\tau+(Y_h-y_{1h}+y_{2h})\cos 3\tau\\
&\quad - y_{2h}\cos 5\tau, \quad \tau = \sqrt{\Omega_{3h}}t
\end{aligned}
\tag{20b}
$$

where

$$
\begin{aligned}
\Omega_{2g} &= \mathrm{L}(A),\\
\Delta\Omega_{2g} &= [\mathrm{P}_1(X_g,Y_g,\Omega_{2g})+\mathrm{P}_2(X_g,Y_g,\Omega_{2g})]/\mathrm{NG},\\
\Omega_{2h} &= \mathrm{L}(-B),\\
\Delta\Omega_{2h} &= [\mathrm{P}_1(X_h,Y_h,\Omega_{2h})-\mathrm{P}_2(X_h,Y_h,\Omega_{2h})]/\mathrm{NH},\\
y_{1g} &= 1260\pi[\mathrm{P}_5(X_g,Y_g,\Omega_{2g})\\
&\quad +\mathrm{P}_6(X_g,Y_g,\Omega_{2g})]/\mathrm{NG},\\
y_{2g} &= 1260\pi[\mathrm{P}_7(X_g,Y_g,\Omega_{2g})\\
&\quad +\mathrm{P}_8(X_g,Y_g,\Omega_{2g})]/\mathrm{NG},\\
y_{1h} &= 1260\pi[\mathrm{P}_5(X_h,Y_h,\Omega_{2h})\\
&\quad -\mathrm{P}_6(X_h,Y_h,\Omega_{2h})]/\mathrm{NH},\\
y_{2h} &= 1260\pi[\mathrm{P}_7(X_h,Y_h,\Omega_{2h})\\
&\quad -\mathrm{P}_8(X_h,Y_h,\Omega_{2h})]/\mathrm{NH},\\
\mathrm{NG} &= \mathrm{P}_3(X_g,Y_g,\Omega_{2g})+\mathrm{P}_4(X_g,Y_g,\Omega_{2g}),\\
\mathrm{NH} &= \mathrm{P}_3(X_h,Y_h,\Omega_{2h})-\mathrm{P}_4(X_h,Y_h,\Omega_{2h}),
\end{aligned}
$$

$$\mathrm{L}(X) \equiv \frac{201600\pi^2 - 2131200L\pi - 1558080\pi L^3 + 108675\pi^2 L^4 + 64L^2(86656 + 4725\pi^2)}{2400\pi(84\pi - 440L + 63\pi L^2)},$$

$$\begin{aligned}
\mathrm{P}_1(X, Y, \Omega) \equiv{}& 330301440\pi[1299870X^2 \\
&- 1089506X^4Y + X^3(1330472 - 2995489Y^2) \\
&- 3XY^2(485976 - 121157Y^2) - XY^2(11528 \\
&- 155085Y^2) - Y^3(336088 + 546207Y^2)] \\
&+ 49165491375\pi^3[63X^7 - 99X^6Y \\
&+ 54X^5(4 - 3Y^2) - 24X^4Y(2 - 9Y^2) \\
&+ Y(4 + 3Y^2)^2(4 + 9Y^2) + 12X^2Y(4 + 15Y^2) \\
&+ 4X^3(52 + 57Y^2 + 54Y^4) + X(64 + 144Y^2 \\
&+ 180Y^4 + 81Y^6)] - 2642411520\pi\Omega(1765621X^3 \\
&+ 937871X^2Y - 2772081XY^2 - 1014411Y^3) \\
&- 196661965500\Omega\pi^3[558X^5 + 348X^4Y \\
&+ X^3(1064 - 591Y^2) + X^2(600Y + 981Y^3) \\
&+ X(560 + 936Y^2 + 693Y^4) + Y(560 + 1368Y^2 \\
&+ 711Y^4)] + 786647862000\pi^3\Omega^2(1036X \\
&+ 829X^3 + 1036Y + 1875X^2Y + 225XY^2 \\
&+ 1479Y^3) - 707983075800000\pi^3\Omega^3(X + Y),
\end{aligned}$$

$$\begin{aligned}
\mathrm{P}_2(X, Y, \Omega) \equiv{}& -2199023255552(429X^4 - 264X^3Y \\
&- 1206X^2Y^2 - 224XY^3 + 289Y^4) \\
&- 69854400\pi^2[917826X^6 - 1019268X^5Y \\
&+ 459X^4(4264 - 4309Y^2) - 96X^3Y(3235 \\
&- 11003Y^2) + 8XY(54740 + 103626Y^2 \\
&- 40635Y^4) - Y^2(419248 + 492624Y^2 \\
&+ 133641Y^4) + 2X^2(466856 + 2916Y^2 \\
&+ 549513Y^4)] + 558835200\pi^2\Omega[2535663X^4 \\
&+ 1408764X^3Y + X^2(2533076 - 2703195Y^2) \\
&- 2Y^2(475958 + 120819Y^2) + 2XY(834260 \\
&+ 498129Y^2)] - 1117670400\pi^2\Omega^2(4091061X^2 \\
&+ 6572650XY - 1520539Y^2),
\end{aligned}$$

$$\begin{aligned}
\mathrm{P}_3(X, Y, \Omega) \equiv{}& 42278584320\pi(3718X^3 \\
&+ 17743X^2Y - 79148XY^2 + 23067Y^3) \\
&+ 196661965500\pi^3[18X^5 + 9X^4Y \\
&+ 36X^2Y(7 + 24Y^2) + 9X^3(4 - 57Y^2) \\
&+ X(16 - 384Y^2 - 459Y^4) + 9Y(16 + 48Y^2 \\
&+ 27Y^4)] - 786647862000\pi^3\Omega[129X^3 \\
&+ 1023X^2Y + 468Y(2 + 3Y^2) \\
&+ 8X(17 - 144Y^2)] \\
&+ 707983075800000\pi^3\Omega^2(X + Y),
\end{aligned}$$

$$\begin{aligned}
\mathrm{P}_4(X, Y, \Omega) \equiv{}& -2235340800\pi^2[21294X^4 \\
&+ 66414X^3Y - 27Y^2(6692 + 7087Y^2) \\
&+ 30XY(5134 + 16817Y^2) + 7X^2(2912 \\
&- 75315Y^2) - 4(80184X^2 + 572225XY \\
&- 508491Y^2)\Omega],
\end{aligned}$$

$$\begin{aligned}
\mathrm{P}_5(X, Y, \Omega) \equiv{}& 468242775\pi^2X^6 - 9364855500\pi^2X^5Y \\
&+ 420426547200Y^4 + 110X^2Y^2[3478126592 \\
&+ 8513505\pi^2(16 + 15Y^2)] + 3003X^4[4194304 \\
&+ 51975\pi^2(4 - 39Y^2)] - 77X^3Y[6809452544 \\
&+ 6081075\pi^2(56 + 117Y^2)] \\
&- 15X[40682651648Y^3 + 10405395\pi^2Y(128 \\
&+ 396Y^2 + 225Y^4)] - 624323700\pi^2X[25X^3 \\
&- 498X^2Y - 648XY^2 - 25Y(32 + 51Y^2)]\Omega,
\end{aligned}$$

$$\begin{aligned}
\mathrm{P}_6(X, Y, \Omega) \equiv{}& -221760\pi[23322X^5 - 620862X^4Y \\
&- 9477Y^3(44 + 45Y^2) + 9X^3(1144 + 18691Y^2) \\
&- 17X^2Y(54028 + 121977Y^2) + XY^2(528992 \\
&+ 872643Y^2) - 52(5478X^3 - 210375X^2Y \\
&- 108472XY^2 - 200475Y^3)]\Omega,
\end{aligned}$$

$$\begin{aligned}
\mathrm{P}_7(X, Y, \Omega) \equiv{}& -143(X + Y)[3274425\pi^2X^5 \\
&- 81860625\pi^2X^4Y + 764411904Y^3 \\
&+ 528X^3(131072 - 297675\pi^2Y^2) \\
&- 2091909120X^2Y - 4131389440XY^2 \\
&- \pi^2(117879300X^2Y + 13097700XY^2 \\
&- 68762925XY^4 + 9823275XY^4) \\
&+ 117879300\pi^2XY(X + Y)]\Omega,
\end{aligned}$$

Table 1 Comparison of approximate periods with exact period for Example 1

A	B	T_e	T_P/T_e	T_1/T_e	T_2/T_e	T_3/T_e
0.1	0.0881633	6.35618	1.08190	0.999808	1.00002	0.999998
0.5	0.284142	8.02019	1.16892	0.988442	1.00124	0.999683
0.9	0.332812	18.4918		0.734453	0.958175	0.947356
1.5	0.414335	11.2716		0.948117	0.963956	1.00211
2.0	0.760768	6.23016		0.953896	0.988732	0.999434
5.0	3.67468	1.73240		0.975543	0.998600	0.999903
8.0	6.66944	1.01572		0.977319	0.999075	0.999921
10.0	8.66837	0.796755		0.977686	0.999169	0.999924
40.0	38.6668	0.188579		0.978244	0.999310	0.999929
70.0	68.6667	0.106971		0.978267	0.999315	0.999929
100.0	98.6667	0.0746626		0.978272	0.999317	0.999929

$$
\begin{aligned}
\mathrm{P}_8(X, Y, \Omega) \equiv\ & 2882880\pi[1408X^5 - 42415X^4Y \\
& - 729Y^3(28 + 9Y^2) - 4XY^2(10189 - 3825Y^2) \\
& - 12X^2Y(1727 + 4107Y^2) + 12X^3(11-9642Y^2) \\
& - 36(33X^3 - 517X^2Y - 1117XY^2 - 567Y^3)]\Omega
\end{aligned}
$$

Applying the L-P perturbation method, Mickens [14] obtained the second-order analytical approximate period $T_P(A)$ and the periodic solution $u_P(t)$ as follows:

$$T_P(A) = 2\pi/\omega_P(A), \quad \omega_P(A) = 1 - 31A^2/24 \tag{21a}$$

and

$$
\begin{aligned}
u_P(t) =\ & A\cos[\omega_P(A)t] - \frac{A^2}{3}\{-3 + 2\cos[\omega_P(A)t] \\
& + \cos[2\omega_P(A)t]\} + \frac{A^2}{3} \\
& \times\left\{-4 + \frac{223}{96}\cos[\omega_P(A)t] + \frac{4}{3}\cos[2\omega_P(A)t]\right. \\
& \left. + \frac{11}{32}\cos[3\omega_P(A)t]\right\}
\end{aligned} \tag{21b}
$$

One the other hand, the exact period $T_e(A)$ is

$$
\begin{aligned}
T_e(A) =\ & \int_0^{\pi/2} \frac{2dt}{\sqrt{1 - (4A(1 + \sin t + \sin^2 t))/(3(1 + \sin t)) + A^2(1 + \sin^2 t)/2}} \\
& + \int_0^{\pi/2} \frac{2dt}{\sqrt{1 + (4B(1 + \sin t + \sin^2 t))/(3(1 + \sin t)) + B^2(1 + \sin^2 t)/2}}
\end{aligned} \tag{22}
$$

where B is given in terms of A, in Equation (16).

The exact period $T_e(A)$ obtained by Equation (22) and the approximate periods T_1, T_2, T_3, and T_P computed, respectively, by Equations (13a) and (21a) are listed in Table 1. For this oscillator, it is required that the oscillation amplitude $A \neq 1$, since Equation (14b) has a homoclinic orbit with period $+\infty$ for $A = 1$. The incomplete columns T_P in Table 1 are due to the perturbation solution giving meaningless results. Furthermore, we have

$$
\begin{aligned}
& \lim_{A\to 0^+} T_e = \lim_{A\to 0^+} T_P = \lim_{A\to 0^+} \\
& \qquad T_1 = \lim_{A\to 0^+} T_2 = \lim_{A\to 0^+} T_3 = 2\pi,
\end{aligned} \tag{23a}
$$

$$
\begin{aligned}
& \lim_{A\to+\infty} \frac{T_1}{T_e} = 0.978277, \quad \lim_{A\to+\infty} \frac{T_2}{T_e} = 0.999318, \\
& \lim_{A\to+\infty} \frac{T_3}{T_e} = 0.999930
\end{aligned} \tag{23b}
$$

From Table 1 and Equation 23(a) and (b), we conclude that the proposed approach yields highly accurate analytical approximate periods for whole range of oscillation amplitudes except a small interval containing $A = 1$.

For purpose of comparison, the exact periodic solutions $u_e(t)$ achieved by integrating (14b) and the analytical approximate periodic solutions $u_P(t)$, $u_1(t)$, $u_2(t)$, and $u_3(t)$ computed by Equations (21b) and (13b), respectively, are plotted in Figs. 1–3 for the time

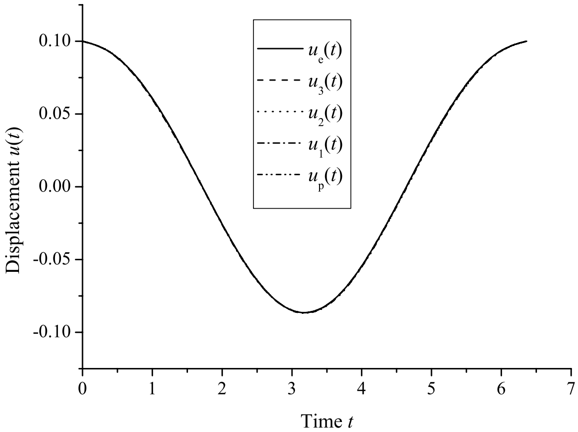

Fig. 1 Comparison of approximate periodic solutions with exact periodic solution for $A = 0.1$ in Example 1

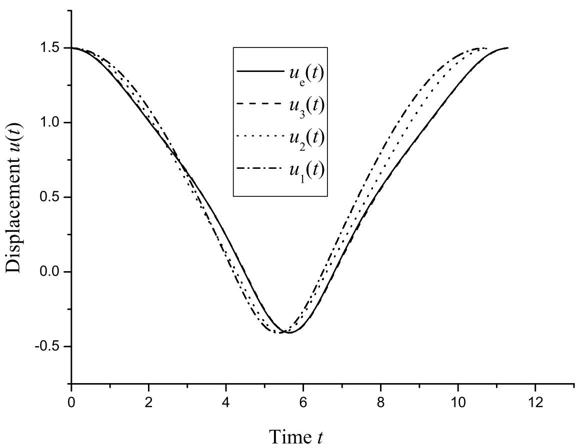

Fig. 2 Comparison of approximate periodic solutions with exact periodic solution for $A = 1.5$ in Example 1

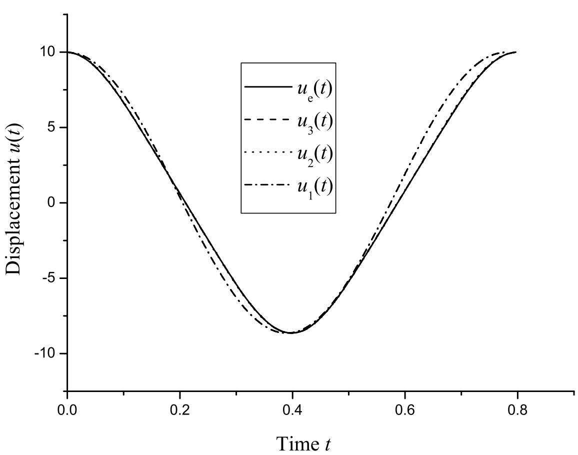

Fig. 3 Comparison of approximate periodic solutions with exact periodic solution for $A = 10$ in Example 1

in one exact period. These figures correspond to, three different amplitudes of oscillation $A = 0.1$, $A = 1.5$ and $A = 10$, respectively.

These figures show that the proposed third analytical approximate periodic solutions, provide the most excellent approximations with respect to the exact periodic solutions for small as well as large oscillation amplitudes. The proposed first two approximations are generally acceptable.

Example 2. Consider the nonlinear oscillator [13] governed by

$$\frac{d^2u}{dt^2} + f(u) = 0, \quad u(0) = A, \quad \frac{du}{dt} = 0 \tag{24a}$$

where

$$f(u) = \begin{cases} u^3, & \text{if } u \geq 0, \\ -u^2, & \text{if } u < 0. \end{cases} \tag{24b}$$

For this problem, the corresponding potential energy function is

$$V(u) = \begin{cases} \dfrac{u^4}{4}, & \text{if } u \geq 0, \\ -\dfrac{u^3}{3}, & \text{if } u < 0 \end{cases} \tag{25}$$

Using Equations (2) and (25), we can express $B (B > 0)$ in terms of A as

$$B = \sqrt[3]{\frac{3A^4}{4}} \tag{26}$$

For this example, according to Equation (11), we obtain the corresponding Fourier coefficients as follows:

$$a_{1g} = \frac{3A^3}{4}, \quad a_{3g} = \frac{A^3}{4}, \quad b_{0g} = 3A^2, \quad b_{2g} = \frac{3A^2}{2},$$
$$b_{4g} = b_{6g} = 0, \quad c_{1g} = \frac{6371A^3}{9216}, \quad c_{3g} = \frac{959A^3}{3456},$$
$$c_{5g} = \frac{23A^3}{768}, \quad d_{0g} = \frac{265A^2}{96}, \quad d_{2g} = \frac{575A^2}{384},$$
$$d_{4g} = \frac{23A^2}{192}, \quad d_{6g} = \frac{A^2}{384}, \quad d_{8g} = d_{10g} = 0 \tag{27a}$$

and

$$a_{1h}=\frac{8B^2}{3\pi},\quad a_{3h}=\frac{8B^2}{15\pi},\quad b_{0h}=\frac{8B}{\pi},\quad b_{2h}=\frac{8B}{3\pi},$$
$$b_{4h}=-\frac{8B}{15\pi},\quad b_{6h}=\frac{8B}{35\pi},\quad c_{1h}=\frac{2904488B^2}{1134375\pi},$$
$$c_{3h}=\frac{188104B^2}{309375\pi},\quad c_{5h}=-\frac{10484872B^2}{262040625\pi},$$
$$d_{0h}=\frac{6376B}{825\pi},\quad d_{2h}=\frac{11224B}{4125\pi},\quad d_{4h}=-\frac{1784B}{4125\pi},$$
$$d_{6h}=\frac{1576B}{7875\pi},\quad d_{8h}=-\frac{107336B}{952875\pi},$$
$$d_{10h}=\frac{25496B}{353925\pi} \tag{27b}$$

Substitution of Equation 27(a) and (b) into Equations (3), (5) and (7) yields the first three analytical approximate periods and the corresponding periodic solutions, respectively. They are

$$T_{1g}(A)=\frac{4\pi}{\sqrt{3A}}\approx\frac{7.255197}{A},\quad u_{1g}(t)=A\cos\tau,\quad \tau=\frac{2\pi t}{T_{1g}(A)}, \tag{28a}$$

$$T_{1h}(B)=\pi\sqrt{\frac{3\pi}{2B}}\approx\frac{6.819781}{\sqrt{B}},\quad u_{1h}(t)=B\cos\tau,\quad \tau=\frac{2\pi t}{T_{1h}(B)}; \tag{28b}$$

$$T_{2g}(A)=\frac{8\pi\sqrt{2/23}}{A}\approx\frac{7.411241}{A},$$
$$u_{2g}(t)=\frac{23A}{24}\cos\tau+\frac{A}{24}\cos3\tau,$$
$$\tau=\frac{2\pi t}{T_{2g}(A)}, \tag{29a}$$

$$T_{2h}(B)=\frac{5\pi}{2}\sqrt{\frac{165\pi}{677B}}\approx\frac{6.872464}{\sqrt{B}},$$
$$u_{2h}(t)=\frac{268B}{275}\cos\tau+\frac{7B}{275}\cos3\tau,$$
$$\tau=\frac{2\pi t}{T_{2h}(B)}; \tag{29b}$$

$$T_{3g}(A)=\frac{8\pi}{5A}\sqrt{\frac{273033543}{125441879}}\approx\frac{7.41578}{A},$$
$$u_{3g}(t)=\frac{21904831241A}{22934817612}\cos\tau+\frac{987420271A}{22934817612}\cos3\tau+\frac{3547175A}{1911234801}\cos5\tau,\quad \tau=\frac{2\pi t}{T_{3g}(A)}, \tag{30a}$$

$$T_{3h}(B)=2\pi\Big/\sqrt{\frac{35469710905878705454464B}{134933066613244217587 5\pi}}\approx\frac{6.86887}{\sqrt{B}},$$
$$u_{3h}(\tau)=\frac{278408876063746916 9B}{2855726277528978150}\cos\tau+\frac{110210200787959534B}{4283589416293467225}\cos3\tau-\frac{20028548732335B}{31153377573043398}\cos5\tau,$$
$$\tau=\frac{2\pi t}{T_{3h}(B)} \tag{30b}$$

For this problem, Wu and Lim [13] got two analytical approximate periods, and the first one is same as the first one computed in the present paper. Their second analytical approximate period is

$$T_{WL}(A)=\frac{1}{2}\left[\frac{24\pi}{\sqrt{62+2\sqrt{421}}A}+2\pi\sqrt{\frac{945\pi}{\left(1448+\sqrt{1064512}\right)B}}\right]\approx\frac{1}{2}\left(\frac{7.42789}{\mathrm{A}}+\frac{6.87490}{\sqrt{\mathrm{B}}}\right) \tag{31}$$

The L-P perturbation method can not directly be used, since no linear term in u exist. The exact period $T_e(A)$ is

$$T_e(A)=\frac{2}{A}\int_0^{\pi/2}\sqrt{\frac{2}{1+\sin^2 t}}dt+\frac{2}{\sqrt{B}}\int_0^{\pi/2}\sqrt{\frac{3(1+\sin t)}{2\left(1+\sin t+\sin^2 t\right)}}dt\approx\frac{3.70815}{A}+\frac{3.43463}{\sqrt{B}} \tag{32}$$

where B is given in terms of A in Equation (26).

Table 2 Comparison of approximate periods with exact period for Example 2

A	T_e	T_1/T_e	T_{WL}/T_e	T_2/T_e	T_3/T_e
0.1	53.8066	0.982791	1.00133	0.999675	0.999934
0.4	15.9078	0.984336	1.00125	0.999797	0.999935
0.7	9.86793	0.985002	1.00122	0.999850	0.999936
1.0	7.31147	0.985433	1.00120	0.999884	0.999936
4.0	2.35702	0.987086	1.00111	1.00001	0.999938
7.0	1.51444	0.987718	1.00108	1.00006	0.999938
10.0	1.14713	0.988103	1.00106	1.00010	0.999938
40.0	0.400784	0.989438	1.00099	1.00020	0.999940
70.0	0.265121	0.989896	1.00097	1.00024	0.999940
100.0	0.204333	0.990162	1.00096	1.00026	0.999940
400.0	0.0756442	0.991017	1.00091	1.00033	0.999941
700.0	0.0510031	0.991289	1.00090	1.00035	0.999941
1000.0	0.0397414	0.991442	1.00089	1.00036	0.999942

The exact period $T_e(A)$ computed by Equation (32) and the approximate periods T_1, T_2, T_3 and T_{WL} calculated, respectively, by Equations (13a) and (31) are listed in Table 2. In addition, we have

$$\begin{aligned}
&\lim_{A\to 0^+}\frac{T_1}{T_e}=0.978277, \quad \lim_{A\to 0^+}\frac{T_2}{T_e}=0.999318,\\
&\lim_{A\to 0^+}\frac{T_3}{T_e}=0.999929, \quad \lim_{A\to 0^+}\frac{T_{WL}}{T_e}=1.00156,\\
&\lim_{A\to +\infty}\frac{T_1}{T_e}=0.992880, \quad \lim_{A\to +\infty}\frac{T_2}{T_e}=1.00047,\\
&\lim_{A\to +\infty}\frac{T_3}{T_e}=0.999943, \quad \lim_{A\to +\infty}\frac{T_{WL}}{T_e}=1.00082
\end{aligned} \tag{33}$$

From Table 2 and Equation (33), we may conclude that Equation (13a) is capable of providing excellent analytical approximations to the period for the whole range of values of oscillation amplitude.

For comparison, the exact periodic solutions $u_e(t)$ achieved by integrating Equation 24(a) and (b) and the analytical approximate periodic solutions $u_1(t)$, $u_2(t)$, and $u_3(t)$ computed by Equation (13b), are plotted in Figs. 4–6 for the time in one exact period. These figures correspond to three different amplitudes of oscillation $A = 0.1$, $A = 1$ and $A = 10$, respectively.

These figures show that the proposed analytical approximate periodic solutions in Equation (13b) are very accurate. Especially, the proposed second and third

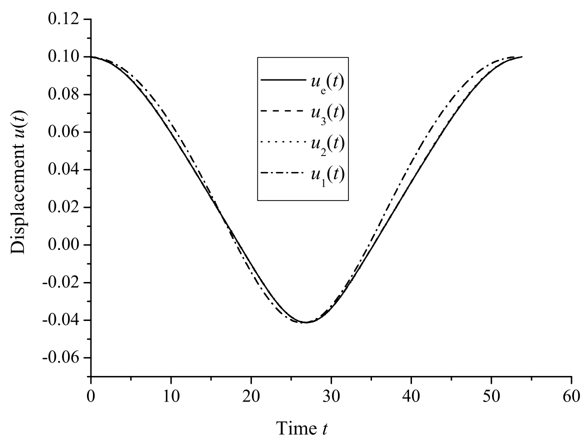

Fig. 4 Comparison of approximate periodic solutions with exact periodic solution for $A = 0.1$ in Example 2

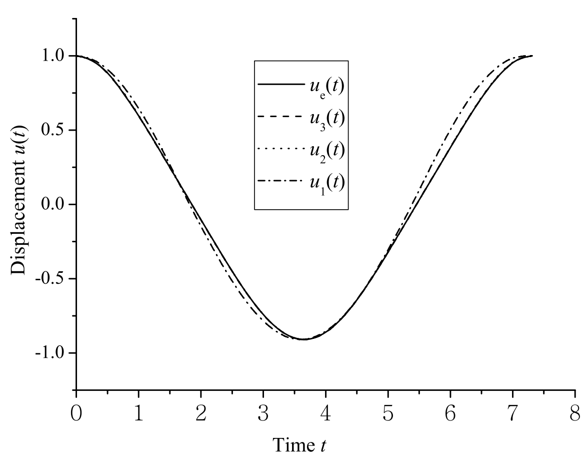

Fig. 5 Comparison of approximate periodic solutions with exact periodic solution for $A = 1$ in Example 2

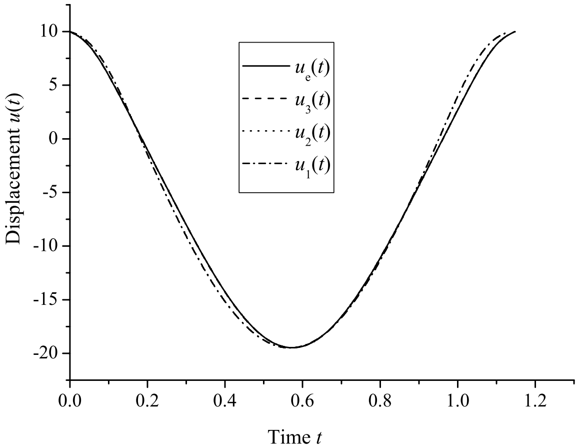

Fig. 6 Comparison of approximate periodic solutions with exact periodic solution for $A = 10$ in Example 2

analytical approximations provide the most excellent solutions with respect to the exact periodic solutions for small as well as large amplitude of oscillation. The proposed first analytical approximations are generally acceptable.

4 Conclusions

A new approach has been presented for establishing the analytical approximate solutions to general strong nonlinear conservative single-degree-of-freedom systems. By introducing two odd nonlinear oscillators from the original general nonlinear oscillator and utilizing the analytical approximate solutions to odd nonlinear oscillators proposed by the authors, we have constructed the analytical approximate solutions to the original general nonlinear oscillator. These analytical approximate solutions are valid for small as well as large amplitudes of oscillation. Two examples have shown the great accuracy and simplicity of the new approach.

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References

1. Nayfeh, A.H., Mook, D.T.: Nonlinear Oscillations. Wiley, New York (1979)
2. Hagedorn, P.: Nonlinear Oscillations. Clarendon, Oxford (1988)
3. Mickens, R.E.: Oscillations in Planar Dynamic Systems. World Scientific, Singapore (1996)
4. Lau, S.L., Cheung, Y.K.: Amplitude incremental variational principle for nonlinear vibration of elastic system. ASME J. Appl. Mech. **48**, 959–964 (1981)
5. Mickens, R.E.: A generalization of the method of harmonic balance. J. Sound Vib. **111**, 515–518 (1986)
6. Wu, B.S., Li, P.S.: A method for obtaining approximate analytic periods for a class of nonlinear oscillators. Meccanica **36**, 167–176 (2001)
7. Wu, B.S., Sun, W.P., Lim, C.W.: An analytical approximate technique for a class of strongly non-linear oscillators. Int. J. Non-Linear Mech. **41**, 766–774 (2006)
8. Mickens, R.E.: Comments on the method of harmonic-balance. J. Sound Vib. **94**, 456–460 (1984)
9. Yuste, S.B.: Comments on the method of harmonic-balance in which Jacobi elliptic functions are used. J. Sound Vib. **145**, 381–390 (1991)
10. Rao, A.V., Rao, B.N.: Some remarks on the harmonic-balance method for mixed-parity nonlinear oscillations. J. Sound Vib. **170**, 571–576 (1994)
11. Gottlieb, H.P.W.: On the harmonic balance method for mixed-parity nonlinear oscillations. J. Sound Vib. **152**, 189–191 (1992)
12. Wu, B.S., Li, P.S.: A new approach to nonlinear oscillations. ASME J. Appl. Mech. **68**, 951–952 (2001)
13. Wu, B.S., Lim, C.W.: Large amplitude nonlinear oscillations of a general conservative system. Int. J. Non-Linear Mech. **39**, 859–870 (2004)
14. Mickens, R.E.: Construction of a perturbation solution to a mixed parity system that satisfies the correct initial conditions. J. Sound Vib. **167**, 564–567 (1993)