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ORIGINAL ARTICLE

Nonlinear vibrations of a rotating shaft with broadband random variations of internal damping

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Abstract A simple Jeffcott rotor is considered with broadband temporal random variations of internal damping which are described using the theory of Markov processes. Transverse response of the rotor with stiffening nonlinearity either in external damping or in restoring force is studied by stochastic averaging method. This method reduces the problems to stochastic differential equations (SDEs) for which analytical solutions are obtained for the Fokker–Planck–Kolmogorov (FPK) equations for stationary probability density functions (PDFs) of the squared whirl radius of the shaft. These PDFs do exist beyond the dynamic instability threshold and they correspond to forward whirl of the rotor. At rotation speeds just slightly above the instability threshold, the response PDF has integrable singularity at zero which corresponds to intermittency in the response.

Keywords Rotordynamics · Dynamic instability · Random vibration · Intermittency

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1 Introduction

Internal or "rotating" damping is a well-known source of potential dynamic instability of shafts operating at supercritical speeds [1, 2]. This kind of destabilizing damping may be present due to energy dissipation in the shaft's material or rubbing between rotating components. Similar effect, in some cases, may also be a result of fluid flow in journal bearings, "slot effect" in fluid machinery, etc., with model of internal damping providing at least qualitatively adequate description for flow-induced dynamic instabilities [2]. In such cases, certain temporal random variations in flow conditions may be expected sometimes which may result in "smeared" instability threshold. The corresponding stochastic lateral response of the shaft should be studied then both for reliability predictions and for interpreting tests results.

The case where the variations are slow compared with the rotor's natural frequency had been studied in [3]. In that case, the shaft may be occasionally brought into the instability domain for brief periods of time even if it is nominally stable, i.e., if mean or expected value of the internal damping coefficient corresponds to stable rotation of the shaft – perhaps close to the instability threshold. Statistical analysis of the corresponding transient short-time sporadic outbreaks in response has been made in [3] using parabolic approximation for peaks of the coefficient of the internal damping [4].

The same rotor is considered in this paper but for different case of temporal random variations in the coefficient of internal damping. Namely, time-variant part of this coefficient is assumed to be a stationary random process which is broadband with respect to the shaft's natural bandwidth, however, has a negligibly small value of its power spectral density (PSD) at twice the natural frequency of the rotor. Thus, the theory of Markov processes [4–6] may be applied including the Fokker–Planck–Kolmogorov (FPK) equation for the response probability density function (PDF). This change in model of random excitation implies significant changes in the response pattern which require extension in the rotor's model. Namely, rapid temporal variations in the internal damping coefficient, as considered here, do not provide adequate time for transient instability to develop in case of nominally stable rotor; this can be clearly seen from the fact that the stationary response PDF degenerates into Dirac delta-function at zero which implies zero response in this case. Thus, analysis is made here for the rotor which is nominally unstable, i.e. one with mean value of the internal damping coefficient corresponding to steady rotation with frequency beyond the threshold for dynamic instability.

Thus, the above change in frequency content of the temporal variations considered could not but influence the adequate rotor's model, requiring some nonlinearity to be accounted for, which would restrict growth of the response. Thus, positive cubic nonlinearity is included here – either in external or "nonrotating" damping or in restoring force. The response analysis is made then with the use of stochastic averaging method which uncouples stochastic differential equations (SDEs) of forward-whirl motion from those of backward whirl (the latter is not excited at all). Furthermore, a single first-order SDE is derived for a squared radius of forward whirl so that the corresponding FPK equation can be solved analytically. This results in stationary PDF of the squared radius of whirl which is found to correspond to the gamma-distribution in case of nonlinearity in damping. In the vicinity of the instability threshold rotation speed, this PDF has a singularity at zero whirl radius. The singularity is nonintegrable for rotation speeds below the "nominal" instability threshold, i.e. one for the rotor without random variations in damping; this means that the radius of transverse response is identically zero for dynamically stable shaft in spite of the temporal random variations in the internal damping. On the other hand, the singularity in PDF is integrable in case of dynamic instability. Thus, it is found that the broadband random variations in the internal damping, as considered here, do not produce any shift in the threshold for dynamic instability: boundary for almost sure stability is the same as one for neutral stability of the "nominal" shaft. On the other hand, the rotor response within the instability domain may be very significantly influenced by the variations, especially near the instability threshold. In particular, the response may exhibit intermittency whereby very rare high-level outbreaks in response alternate with very long periods of almost zero response. The solution for response PDF can be used for probabilistic description of the shaft behaviour, particularly in the case of intermittency.

2 Analysis by stochastic averaging – case of nonlinearity in damping

Consider a simple Jeffcott rotor with weightless shaft of a stiffness K rotating with angular velocity ν. The horizontal shaft carries a disk of mass m at its midspan and possesses external or "nonrotating" damping and internal or "rotating" damping with corresponding damping coefficients c_n and c_r, respectively; cubic nonlinearity in the nonrotating damping will also be accounted for. Furthermore, stationary zero-mean temporal random variations in the coefficient of the internal damping will be included in the equation of motion.

Let $X(t)$ and $Y(t)$ be lateral horizontal and vertical displacements, respectively, of the disk's centre in the inertial frame with origin at the undeformed shaft's axis. Then, neglecting gravity force for sufficiently high rotation speeds, one can write the following single equation of motion for complex displacement $Z = X + iY$, $i = \sqrt{-1}$ [1, 2]

$$\begin{aligned} &\ddot{Z} + 2(\alpha + \beta(1 + \xi(t)) + \gamma_d |Z^2|)\dot{Z} \\ &\quad + \Omega^2 Z - 2i\beta(1 + \xi(t))\nu Z = 0 \end{aligned} \tag{1}$$

Here, $\Omega^2 = K/m$, $\alpha = c_n/2m$, $\beta = c_r/2m$ and $\gamma_d > 0$, whereas $\xi(t)$ is a stationary zero-mean broadband random process with PSD $\Phi_{\xi\xi}(\omega)$. This PSD is assumed to be a decreasing function which is broadband in the vicinity of zero frequency with respect to the shaft's natural bandwidth, as defined by its total damping factor $\alpha + \beta$; however, its values at $\omega \geq 2\Omega$ are assumed to be negligibly small.

In the absence of any temporal random variations of parameters, Equation (1) – with $\xi(t) \equiv 0$ – clearly has a trivial solution $Z(t) \equiv 0$. This solution is stable if $\nu < \nu_*$, and unstable, if $\nu > \nu_*$, where $\nu_* = \Omega \cdot (1 + \alpha/\beta)$ is instability threshold of the shaft; at this rotation speed, Equation (1) with $\xi(t) \equiv 0$ has a neutrally stable periodic solution with period $2\pi/\Omega$ [1, 2]. The stochastic system (1) will be studied here for the case where the shaft operates at steady rotation speed. It will be assumed to be lightly damped, with damping coefficients α, β, γ_d being proportional to a small parameter.

Solution to the SDE (1) may, now, be sought in the form

$$
\begin{aligned}
Z &= Z_+ \exp(i\Omega t) + Z_- \exp(-i\Omega t), \\
\dot{Z} &= i\Omega[Z_+ \exp(i\Omega t) - Z_- \exp(-i\Omega t)]
\end{aligned} \tag{2}
$$

Here, Z_+ and Z_- are complex amplitudes of the forward and backward whirl, respectively, which should be slowly varying functions of time because of the adopted assumptions. Therefore, the stochastic averaging method can be applied after the basic equation is reduced to a form with small parameter on the right-hand side [4–6]. This reduction can be implemented by resolving the relations (2) for Z_+ and Z_- and differentiating as described in [7] so that

$$
\begin{aligned}
\dot{Z}_+ &= \frac{1}{2}\frac{d}{dt}\left[\frac{Z + \dot{Z}}{i\Omega}\exp(-i\Omega t)\right] \\
&= \frac{1}{2i\Omega}(\ddot{Z} + \Omega^2 Z)\exp(-i\Omega t) \\
&= \frac{1}{2i\Omega}[-2(\alpha + \beta(1 + \xi(t)) + \gamma_d|Z^2|)\dot{Z} \\
&\quad + 2i\nu\beta(1 + \xi(t))Z]\exp(-i\Omega t) \\
&= -[\alpha + \beta(1 + \xi(t)) + \gamma_d\{|Z_+^2| + |Z_-^2| \\
&\quad + Z_+ Z_-^* \exp(2i\Omega t) + Z_- Z_+^* \exp(-2i\Omega t)\}] \\
&\quad \times [Z_+ \exp(i\Omega t) - Z_- \exp(-i\Omega t)]\exp(-i\Omega t) \\
&\quad + \left(\beta\frac{\nu}{\Omega}\right)(1 + \xi(t))[Z_+ \exp(i\Omega t) \\
&\quad + Z_- \exp(-i\Omega t)] \cdot \exp(-i\Omega t) \\
&\cong \left[(\alpha + \beta)\left(\frac{\nu}{\nu_*} - 1\right) - \gamma_d|Z_+^2|\right]Z_+ \\
&\qquad + \beta\left(\frac{\nu}{\Omega - 1}\right)Z_+\varsigma(t)
\end{aligned} \tag{3}
$$

where a star in superscript denotes complex conjugate quantities; similarly,

$$
\begin{aligned}
\dot{Z}_- \cong &-\left[(\alpha + \beta)\cdot\left(1 + \frac{\nu}{\nu_*}\right) + \gamma_d|Z_-^2|\right]Z_- \\
&-\beta\left(\frac{\nu}{\Omega + 1}\right)Z_-\varsigma(t)
\end{aligned} \tag{4}
$$

The last, approximate equalities in the SDEs (3) and (4) are actually obtained by applying principle of stochastic averaging according to the Stratonovich–Khas'minsky theorem [4–6]. For terms without random excitation, the corresponding averaging over the period resulted just in neglecting terms with complex exponent factor $\exp(2i\Omega t)$ and $\exp(-2i\Omega t)$. As for the terms with random process $\xi(t)$, only those without the above complex exponent factors are retained in the limiting white-noise approximation of the stochastic averaging method [4–6] since these factors bring quantity $\Phi_{\xi\xi}(2\Omega)$ into the corresponding drift coefficients, and this quantity is assumed to be negligibly small; in the retained terms, $\varsigma(t)$ may be regarded as an equivalent white noise with intensity factor $\sigma^2 = 2\pi\Phi_{\xi\xi}(0)$.

Thus, the limiting SDEs (3) and (4) are found to be completely uncoupled. The latter of these, which describes backward whirl of the rotor, has the obvious trivial solution $Z_- \equiv 0$ which is almost surely stable at any value of rotation speed. Therefore, analysis of the complex SDE (3), which describes forward whirl, may provide complete description of motion. Thus, two equivalent SDEs for real and imaginary parts Z_{+R} and Z_{+I} of Z_+ may be written accordingly and transformed to a single SDE for squared radius of forward whirl

$$
V = |Z^2| = |Z_+^2| = Z_{+R}^2 + Z_{+I}^2 = X^2 + Y^2 \tag{5}
$$

where the last equality is obtained from the condition $Z_- \equiv 0$ which implies $\dot{X} = -\Omega Y$, $\dot{Y} = \Omega X$. As long as the SDE (3) is a "physical" one, i.e. in Stratonovich sense [5, 6], common rule for transformation of variables can be used rather than Ito's formula [6] so that

$$
\dot{V} = 2(Z_{+R}\dot{Z}_{+R} + Z_{+I}\dot{Z}_{+I}) = -f(V) + h(V)\varsigma(t) \tag{6}
$$

where

$$f(V) = 2\left[-(\alpha+\beta)\left(\frac{\nu}{\nu_* - 1}\right) + \gamma_d \Omega^2 V\right] V$$

and

$$h(V) = 2\beta\left(\frac{\nu}{\Omega - 1}\right) \tag{7}$$

Concluding this section, it seems relevant to present some comments on behaviour of shafts with more general types of damping nonlinearity. Thus, let the state-dependent part of the damping coefficient $\gamma_d|Z^2|$ in the governing Equation (1) be replaced by $\gamma_d \cdot g(|Z^2|, |\dot{Z}^2|)$. The resulting approximate SDEs (3) and (4) would then be coupled in general. For example, in case $g = \gamma_d|\dot{Z}^2|$, the corresponding nonlinear terms in the SDEs (3) and (4) would be $\gamma_d\Omega^2(|Z_+^2| + 2|Z_-^2|)Z_+$ and $\gamma_d\Omega^2(|Z_-^2| + 2|Z_+^2|)Z_-$, respectively. It may be expected, however, that the above analysis of the SDE (3) with $Z_- \equiv 0$ would still be valid – at least if the intensity of random excitation is not too high – as long as in case $\varsigma(t) \equiv 0$, the solution $Z_- \equiv 0$ would be the only one for the extended SDE set (3), (4) with nonlinear parametric coupling as described. Of course, stochastic stability analysis of the response with $Z_- \equiv 0$ for this set requires a separate study.

3 Solution for the PDF of squared whirl radius

As long as the problem is reduced now to a single first-order Stratonovich SDE (6) for the squared whirl radius $V(t)$, the analytical solution is available to the FPK equation for its stationary PDF $p(V)$ [5]. Namely

$$p(V) = \frac{C}{h(V)} \exp\left[-\frac{2}{\sigma^2}\int \frac{f(V)}{h^2(V)}\, dV\right] \tag{8}$$

where C is a constant that should be obtained from normalization condition of $p(V)$ on $[0, \infty)$. Substituting expressions (7) into the general solution (8) and evaluating integrals yields the following explicit expression for the PDF of scaled whirl radius which is normalized in $[0, \infty)$ for the case where $\nu - \nu_*$ is positive:

$$p(z) = (\Gamma(\varepsilon))^{-1} z^{\varepsilon - 1} e^{-z}$$

where

$$z = \frac{\gamma_d \Omega^2 V}{[\beta\sigma(\nu/\Omega - 1)]^2}$$

and

$$\varepsilon = \frac{(\alpha+\beta)(\nu/\nu_* - 1)}{[\beta\sigma(\nu/\Omega - 1)]^2} \tag{9}$$

Here, Γ is the gamma-function. Thus, it can be seen that at rotation speeds beyond instability threshold, the squared radius of (forward) whirl $V(t)$ is a gamma-distributed stationary random process. Below this threshold, that is for $\varepsilon < 0$, the function $p(V)$ as presented in relation (8) still satisfies the FPK equation; however, its singularity at $V = 0$ is not integrable. This solution which, actually, is degenerated into the Dirac delta-function $\delta(V)$ implies the absence of any response in the stochastically stable system (6).

Thus, it can be seen that present broadband random variations with restricted PSD in the coefficient of internal damping of a two-degrees-of-freedom (TDOF) rotor do not produce any shift in the instability boundary. In this respect, the present case is different from that of random variations in natural frequency of a single-degree-of-freedom (SDOF) system which may produce such a shift in the boundary for negative-damping-type instability [4]; that shift, due to stochastic parametric instability, is proportional to value of variations' PSD at twice the natural frequency.

Furthermore, mean value of the squared radius of whirl is found to be insensitive to the level of random variations and equal to the squared whirl radius V_0 of the rotor with constant internal damping, the latter being defined as a nonzero root of $f(V)$ that is of drift or "nonstochastic" part of the right-hand side of the SDE (7). Indeed, using property of gamma-function for evaluating the integral of expression (8), one can obtain the following after substituting expressions (9):

$$\langle V\rangle = \frac{[\beta\sigma(\nu/\Omega - 1)]^2\langle z\rangle}{\gamma_d\Omega^2} = \frac{(\alpha+\beta)(\nu/\nu_* - 1)}{\gamma_d\Omega^2} = V_0,$$

where

$$\langle z\rangle = \int_0^\infty z p(z)\, dz = \frac{\Gamma(\varepsilon+1)}{\Gamma(\varepsilon)} = \varepsilon \tag{10}$$

and angular brackets denote expectation operator.

On the other hand, influence of damping variations on the response pattern may be very significant especially in case of operation just slightly beyond the instability threshold that is for small $\nu - \nu_*$. In particular, the response may be of an intermittent nature within range of small ε whereby very rare high-level outbreaks in response alternate with very long periods of almost zero response. Actually, the gamma-distributed response with integrable singularity in its stationary PDF has been derived originally in [4] for a nonlinearly damped SDOF system with random parametric excitation. However, correlation between this type of PDF and intermittent nature of the response has been established for that system much later [8]. Such a correlation has also been established in [9], where solution in the form of product of gamma-distributions had been obtained for population sizes of two interacting species described by the Lotka–Volterra model as used in population dynamics. It had been suggested in [5, 9] to describe the degree of intermittency for a stationary gamma-distributed process by ratio of stay times above and below its mean value. Thus, for the present process, $V(t)$ with PDF (9), this ratio is found to be

$$\lambda = \frac{\text{Prob}\,\{V > V_0\}}{\text{Prob}\,\{V < V_0\}} = \frac{\int_{V_0}^{\infty} p(V)\,dV}{\int_0^{V_0} p(V)\,dV} = \frac{\Gamma(\varepsilon,\varepsilon)}{\Gamma(\varepsilon) - \Gamma(\varepsilon,\varepsilon)} \tag{11}$$

where function Γ that depends on two arguments is the incomplete gamma-function. Both asymptotic formulae for complete and incomplete gamma-functions and direct numerical calculations show that $\lambda \to 0$ with $\varepsilon \to 0$ [9].

Yet another potential indicator of intermittency in a stationary random process is provided by condition that its ratio of standard deviation to mean value is very large compared with unity. Evaluating relevant weighted integral of $p(V)$ to find the mean square of $V(t)$ and using expression (9), one can obtain this ratio as

$$\frac{[\langle V^2\rangle - \langle V\rangle^2]^{1/2}}{\langle V\rangle} = \frac{[\langle z^2\rangle - \langle z\rangle^2]^{1/2}}{\langle z\rangle} = \frac{1}{\sqrt{\varepsilon}}$$

where

$$\langle z^2\rangle = \int_0^{\infty} z^2 p(z), dz = \frac{\Gamma(\varepsilon+2)}{\Gamma(\varepsilon)} = \varepsilon(\varepsilon+1) \tag{12}$$

This indicator of intermittency can be used in cases where the analytical solution for the response PDF is not available but method of moments as based on direct SDE calculus can be applied. Thus, in present case, expression (10) can be derived by equating to zero the expectation of the right-hand side of the Stratonovich SDE (6) after transformation of variable $u = \ln V$ is applied. And expression for $\langle V^2\rangle$ can be obtained by direct equating to zero the expectation of the right-hand side of the Stratonovich SDE (6) and using formula for Wong–Zakai correction [6] to calculate the expectation $\langle V\xi(t)\rangle$.

4 The case of nonlinearity in restoring force

In this section, rotor with "stiffening" nonlinearity in restoring force rather than in damping is considered with its model being the same otherwise. Thus, the equation of motion is written as

$$\begin{aligned}&\ddot{Z} + 2\,[\alpha + \beta + \beta\cdot\xi(t)]\,\dot{Z} + \Omega^2(1+\gamma_f|Z^2|)Z\\ &\quad - 2i\beta[1+\xi(t)]\nu Z = 0\end{aligned} \tag{13}$$

Response analysis follows similar lines as for the case of nonlinear damping but with an important modification which is dictated by properties of the periodic solution for the shaft without any random variations of parameters [2]. Solution to Equation (13) with $\xi(t) \equiv 0$ for the case $\nu > \nu_*$ may be represented in the form

$$Z(t) = R\exp(i\Lambda t)$$

where

$$\Lambda = \frac{\nu\beta}{\alpha+\beta} = \frac{\Omega\nu}{\nu_*}$$

and

$$\gamma_f R^2 = \left(\frac{\nu}{\nu_*}\right)^2 - 1$$

so that

$$\Lambda^2 = \Omega^2(1+\gamma_f V)$$

where

$$V = R^2 \tag{14}$$

Expressions (14) for Λ and radius of whirl R has been obtained here by substituting expression for $Z(t)$ into Equation (12) and equating to zero the imaginary and real parts, respectively, of the resulting coefficients of $Z(t)$. The solution (14) clearly describes forward whirl of the shaft operating at rotation speed beyond the instability threshold.

The solution to the full SDE (13) is now represented in the form similar to expression (2) but with frequency of oscillations being dependent, now, on the squared whirl radius according to relation (14):

$$\begin{aligned} Z &= Z_+ \exp(i\Lambda t) + Z_- \exp(-i\Lambda t), \\ \dot{Z} &= i\Lambda[Z_+ \exp(i\Lambda t) - Z_- \exp(-i\Lambda t)], \\ \Lambda &= \Omega(1+\gamma_f V)^{1/2} \end{aligned} \tag{15}$$

The whole procedure for analysis by stochastic averaging is now repeated with Ω replaced by $\Lambda(V)$, where $V(t)$ is slowly varying function which is held fixed during averaging over the period. This results in the SDE (6) with

$$f(V) = -2(\alpha+\beta)\left[\frac{\nu}{\nu_*\sqrt{1+\gamma_f V}} - 1\right] V$$

and

$$h(V) = 2\beta\left(\frac{\nu}{\Omega\sqrt{1+\gamma_f V}} - 1\right) V \tag{16}$$

The corresponding general solution (8) may once again be qualified to represent the stationary PDF $p(V)$ of $V(t)$ provided it satisfies conditions for integrability both at zero and at infinity. The former of these conditions is clearly seen to be the same as for the case of nonlinearity in damping – the shaft without random variations in damping should be unstable in the linear approximation; moreover, intermittent behaviour may be expected once again for small $\nu - \nu_*$. The latter condition may be satisfied for the present case of stiffening nonlinearity in restoring force which means that this nonlinearity may restrict growth of whirl radius in case of instability (in the linear approximation). This fact is well known for shafts without random variations in damping [2]; however, the variations may bring in a destabilizing effect due to nonlinearity in the expression (16) for $h(V)$, thereby reducing domain where steady-state response does exist.

5 Conclusions

Transverse response of a Jeffcott rotor with nonlinear external damping and broadband temporal random variations of internal damping has been studied using theory of Markov processes and asymptotic stochastic averaging method. The latter leads to two reduced SDEs for complex amplitudes of forward and backward whirl. These SDEs are found to be completely uncoupled as long as PSD of the randomly varying coefficient of internal damping has a negligibly small level at twice the natural frequency of the rotor. The SDE of backward whirl has the almost surely stable trivial (zero) solution at any rotation speed whereas nontrivial solution to the SDE of the forward whirl does exist if rotation speed of the shaft exceeds its threshold for dynamic instability. Analytical solution is obtained for the FPK equation for stationary PDF of the squared whirl radius of the shaft which corresponds to forward whirl of the rotor. At rotation speeds just slightly above the instability threshold, this response PDF has integrable singularity at zero which corresponds to intermittent nature in the response and can be described using the analytical solution for its PDF. Similar results have also been obtained for the case of nonlinearity in restoring force.

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