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ORIGINAL ARTICLE

Approximate symmetries and conservation laws of the geodesic equations for the Schwarzschild metric

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Abstract Approximate symmetries have been defined in the context of differential equations and systems of differential equations. They give approximately, conserved quantities for Lagrangian systems. In this paper, the exact and the approximate symmetries of the system of geodesic equations for the Schwarzschild metric, and in particular for the radial equation of motion, are studied. It is noted that there is an ambiguity in the formulation of approximate symmetries that needs to be clarified by consideration of the Lagrangian for the system of equations. The significance of approximate symmetries in this context is discussed.

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1 Introduction

Much of the development of geometry has been driven by its application to kinematics and dynamics [1, 2]. Of special relevance for our purposes here is the development of Lie symmetry methods [3]. In geometry they yield isometries [4]. They also provide methods for solving differential equations [5]. Symmetries are very useful because of their role in giving conservation laws through Noether's theorem [6]. The system of geodesic equations inherits the symmetries of the manifold and has an additional dilation algebra [7]. Often a manifold does not possess some symmetry but *nearly* does so. These "approximate symmetries" of manifolds should give valuable information about them. Methods have been developed for determining the approximate symmetries of ordinary differential equations ODEs [8, 9] and systems of differential equations (DEs) [10]. Symmetries have also been extensively used in the general theory of relativity [11]. The Schwarzschild metric has much fewer symmetries (four generators) than the Minkowski metric. However, one would expect that in the limit of small gravitational mass we should recover the "lost" symmetries. Hence, we should expect to find some "approximate isometries". They can be looked for by "crossing the bridge" provided by [7] from geometry to differential equations and looking at the approximate symmetries of the system of geodesic equations.

In this paper, we look at the approximate symmetries of the system of geodesic equations for the Schwarzschild metric. Interesting insights emerge from looking at a single "reduced" equation from the system, i.e. the radial orbital equation. In the rest of this section, a very brief review of symmetries and approximate symmetries for systems of differential equations is given to establish the notation used. In Section 2, isometries are discussed as symmetries in the geometrical context and as the symmetries of the Lagrangian for the arc length. This leads to the consideration of approximate isometries. Then the Schwarzschild metric is discussed in Section 3 and a summary and discussion is given in Section 4.

The isometries are given by the Killing equation

$$\mathcal{L}_{\mathbf{k}}\mathbf{g} = 0, \tag{1}$$

which can be written in component form as

$$g_{ab,c}k^c + g_{ac}k^c{}_{,b} + g_{bc}k^c{}_{,a} = 0, \tag{2}$$

where g_{ab} are the components of the metric tensor $\mathbf{g}$ and k^a are the components of the Killing vector field $\mathbf{k}$, which gives the isometry and the Einstein summation convention, where repeated indices are summed over, has been used.

On the other hand, the symmetries of a system of ODEs

$$\mathbf{E}\big(s; \mathbf{x}(s), \mathbf{x}'(s), \mathbf{x}''(s), \ldots, \mathbf{x}^{(n)}(s)\big) = 0, \tag{3}$$

under point transformations

$$(s, \mathbf{x}) \longrightarrow (\xi(s, \mathbf{x}), \eta(s, \mathbf{x})), \tag{4}$$

are given by operators

$$\mathbf{X} = \xi\frac{\partial}{\partial s} + \eta.\frac{\partial}{\partial \mathbf{x}} + \eta^{(1)}.\frac{\partial}{\partial \mathbf{x}'} + \cdots + \eta^{(n)}.\frac{\partial}{\partial \mathbf{x}^{(n)}}, \tag{5}$$

such that on the solution of $\mathbf{E} = 0$ we have

$$\mathbf{X}\mathbf{E} = 0. \tag{6}$$

If

$$\mathbf{E} = \mathbf{E}_0 + \epsilon\mathbf{E}_1 + O(\epsilon^2) \tag{7}$$

and $\exists$ an $\mathbf{X_0}$ such that

$$\mathbf{X}_0\mathbf{E}_0 = 0, \tag{8}$$

and we can define $\mathbf{X_1}$ such that

$$\mathbf{X}_1\mathbf{E}_0 = -\mathbf{X}_0\mathbf{E}_1, \tag{9}$$

so that

$$\mathbf{X}\mathbf{E} := (\mathbf{X}_0 + \epsilon\mathbf{X}_1)\mathbf{E} = O(\epsilon^2), \tag{10}$$

then $\mathbf{X_0} + \epsilon\mathbf{X_1}$ is called a first-order *approximate symmetry* of $\mathbf{E} = 0$ (see [8–10]).

2 Isometries and approximate isometries

Isometries are vector fields along which the metric is left invariant. In other words, the metric transported along a curve to which the isometry is tangent, remains unchanged. The transport is provided by using Taylor's theorem with the directional derivative in the exponential. However, the derivative used has to be the *geometrical*, Lie derivative [11] and *not* the usual intrinsic derivative (which is the contraction of the vector field with the covariant derivative). Thus, for an isometry $\mathbf{k}$, we require that

$$\mathbf{g}(\mathbf{a} + \mathbf{k}) = \exp(\mathcal{L}_{\mathbf{k}})\mathbf{g}(\mathbf{x})|_{\mathbf{x}=\mathbf{a}} = \mathbf{g}(\mathbf{a}). \tag{11}$$

This requirement is equivalent to the Killing Equation (1) which can be written in component form as (2).

The metric tensor $\mathbf{g}$ defines the arc length along a curve

$$ds^2 = \mathbf{dx} \cdot \mathbf{g} \cdot \mathbf{dx}, \tag{12}$$

so that we have

$$1 = g_{ab}\dot{x}^a\dot{x}^b. \tag{13}$$

This can be regarded as an equation for a Lagrangian dependent on x^a and its derivative $\dot{x}^a$, $L[x^a, \dot{x}^a] = 1$, as g_{ab} is a function of x^a, which yields the geodesic equations

$$\ddot{x}^a + \Gamma^a_{bc}\dot{x}^b\dot{x}^c = 0, \tag{14}$$

as the Euler–Lagrange equations, where

$$\Gamma^a_{bc} = \frac{1}{2} g^{ad}(g_{bd,c} + g_{cd,b} - g_{bc,d}). \tag{15}$$

Consequently, all isometries are symmetries of the geodesic equations. Since Noether symmetries yield conservation laws, isometries give quantities that are conserved under the motion.

We can follow the procedure for defining approximate symmetries of differential equations to define approximate isometries. Writing $\mathbf{K} = \mathbf{k} + \epsilon \mathbf{l}$ and $\mathbf{G} = \mathbf{g} + \epsilon \mathbf{h}$, where $\mathbf{K}$ and $\mathbf{G}$ are the isometry generator and the metric for the space under consideration and $\mathbf{k}$ and $\mathbf{g}$ are the exact isometries and the metric for the space whose isometries are known. If Equation (1) holds, we can define the *approximate isometry* $\mathbf{l}$ by the requirement that it also hold to $O(\epsilon^2)$ with $\mathbf{G}$ and $\mathbf{K}$ replacing $\mathbf{g}$ and $\mathbf{k}$. This yields to the equation in component form,

$$\begin{aligned} &l^c g_{ab,c} + l^c_{,b} g_{ab} + l^c_{,a} g_{bc} \\ &\quad = -\big(k^c h_{ab,c} + k^c_{,b} h_{ac} + k^c_{,a}\big) h_{bc}. \end{aligned} \tag{16}$$

On the other hand, we could have used the approximate symmetries of the geodesic equations, excluding the re-parameterization symmetries, to define the approximate isometries. This procedure could, in principle, bring in non-Noether approximate symmetries due to the possibility of the re-parameterization symmetry of geodesics "mixing with" the approximate symmetry in much the same manner as they did for the system of geodesic equations in flat spaces. It is not entirely clear that this would, indeed, occur or what significance (if any) should be attached to such symmetries. For the Noether symmetries we expect that the approximate symmetries should yield approximate conservation laws and hence, we would get approximately conserved quantities. Consequently, it would be useful to apply the latter procedure to a specific case and check what occurs. Here, we use it for the Schwarzschild metric.

3 Approximate symmetries for the Schwarzschild metric

The isometries of Minkowski space have the Poincarè algebra, $so(1,3) \oplus_s \mathbb{R}^4$, where $\oplus_s$ stands for the semidirect sum (denoting that the operations in the sub-algebras do not commute). Note that $so(1,3)$ is isomorphic to $so(3) \oplus so(3)$. This algebra corresponds to the conservation of angular momentum (one of the $so(3)$s), "spin angular momentum" (the other $so(3)$) and the (linear) energy-momentum ($\mathbb{R}^4$). The symmetry generators are

$$\begin{aligned} \mathbf{X}_0 &= \partial/\partial t, \\ \mathbf{X}_1 &= \cos\phi\,\partial/\partial\theta - \cot\theta\sin\phi\,\partial/\partial\phi, \\ \mathbf{X}_2 &= \sin\phi\,\partial/\partial\theta + \cot\theta\cos\phi\,\partial/\partial\phi, \\ \mathbf{X}_3 &= \partial/\partial\phi, \end{aligned} \tag{17}$$

with the symmetry algebra $\mathbb{R} \oplus so(3)$ corresponding to the conservation of energy and angular momentum; and

$$\begin{aligned} \mathbf{X}_4 = {}& \sin\theta\cos\phi\frac{\partial}{\partial r} + \frac{\cos\theta\cos\phi}{r}\frac{\partial}{\partial\theta} \\ & - \frac{\csc\theta\sin\phi}{r}\frac{\partial}{\partial\phi}, \end{aligned} \tag{18}$$

$$\begin{aligned} \mathbf{X}_5 = {}& \sin\theta\sin\phi\frac{\partial}{\partial r} + \frac{\cos\theta\sin\phi}{r}\frac{\partial}{\partial\theta} \\ & + \frac{\csc\theta\cos\phi}{r}\frac{\partial}{\partial\phi}, \end{aligned} \tag{19}$$

$$\mathbf{X}_6 = \cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}, \tag{20}$$

which give the conservation of linear momentum and

$$\begin{aligned} \mathbf{X}_7 = {}& \frac{r\sin\theta\cos\phi}{c}\frac{\partial}{\partial t} + ct\bigg(\sin\theta\cos\phi\frac{\partial}{\partial r} \\ & + \frac{\cos\theta\cos\phi}{r}\frac{\partial}{\partial\theta} - \frac{\csc\theta\sin\phi}{r}\frac{\partial}{\partial\phi}\bigg), \end{aligned} \tag{21}$$

$$\begin{aligned} \mathbf{X}_8 = {}& \frac{r\sin\theta\sin\phi}{c}\frac{\partial}{\partial t} + ct\bigg(\sin\theta\sin\phi\frac{\partial}{\partial r} \\ & + \frac{\cos\theta\sin\phi}{r}\frac{\partial}{\partial\theta} + \frac{\csc\theta\cos\phi}{r}\frac{\partial}{\partial\phi}\bigg), \end{aligned} \tag{22}$$

$$\mathbf{X}_9 = \frac{r\cos\theta}{c}\frac{\partial}{\partial t} + ct\left(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right), \tag{23}$$

which give the conservation of spin angular momentum due to Lorentz invariance.

The symmetry algebra for the geodesic equations is $sl(4,\mathbb{R})$, which has many symmetries that do not correspond to conservation laws, arising from the "mixing" of the geodesic re-parameterization generators with the Noether symmetry generators. On account of the extra symmetries of the geodesic equations one would expect that it should be the direct definition of approximate isometries, by Equation (16), which would be useful.

The field of a point gravitational source at rest at the origin is given by the Schwarzschild metric [12]

$$ds^2 = e^{\nu(r)}dt^2 - e^{-\nu(r)}dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2, \tag{24}$$

where

$$e^{\nu(r)} = 1 - 2GM/c^2 r, \tag{25}$$

and G is Newton's gravitational constant, M is the mass of the point gravitational source and c is the speed of light in vacuum. This metric has four isometries $\mathbf{X}_0$, $\mathbf{X}_1$, $\mathbf{X}_2$ and $\mathbf{X}_3$, corresponding to the conservation of energy and angular momentum. The geodesic equations have the symmetries given by the above isometry algebra added to the dilation algebra d_2, generated by the re-parameterization allowed for the geodetic parameter. Note that conservation of linear momentum is "lost" as a test particle put at a finite distance from the gravitational source will start to move. Further, the "spin angular momentum" conservation is also "lost", as the motion of the test particle will no longer be Lorentz invariant in the field of the gravitating source.

Considering the definition of approximate symmetries by Equation (16), with ϵ defined to be $2GM/c^2$, we obtain a system of 10 linear first-order partial differential equations for $\mathbf{l}$, of which 7 are nonhomogeneous. Due to this non-homogeneity there is no guarantee that there is *any* (even the trivial) solution. However, if there is, then the "00" part of Equation (16) is

$$l^0_{,0} = -\frac{1}{2r^2}(k^1 - 2rk^0, 1). \tag{26}$$

Here the k^0 and k^1 are, respectively, the time and radial components of a general linear combination of the full set of $\mathbf{X}$s given above. As is easily seen, there will be six arbitrary parameters in the expression for $l^0_{,0}$. This corresponds to a 6-parameter energy re-scaling symmetry (for the flat 3-space) that is not contained in the "lost" symmetries. Though it would be interesting to explore the significance of the resulting set of symmetries, it is clear that they will not be what we were looking for here. These symmetries are for the metric tensor itself and not for the Lagrangian constructed from it. For the Lagrangian giving the equations of motion, which yield the conserved quantities, we must look for the (Noether) symmetries of the geodesic equations.

To avoid dealing with the full system of equations, we can first use the above symmetries to reduce the system to a single orbital equation

$$\frac{d^2u}{d\phi^2} + u = \frac{GM}{h^2} + \frac{3GM}{c^2}u^2, \tag{27}$$

where h is the classical angular momentum per unit mass and $u = 1/r$. In the classical limit $c \to \infty$ it gives the classical orbital equation. This has only one symmetry, corresponding to the conservation of azimuthal angular momentum, $\partial/\partial\phi$ apart from the symmetry of the dilation group. We can now look at the approximate symmetries of this equation, with the small parameter $\epsilon = 2GM/c^2$. This equation has two stable approximate symmetries

$$\begin{aligned}\mathbf{X}_{a1} &= \sin\phi\partial/\partial u + \epsilon(2\sin\phi\partial/\partial\phi + u\cos\phi\partial/\partial u),\\ \mathbf{X}_{a2} &= \cos\phi\partial/\partial u - \epsilon(2\cos\phi\partial/\partial\phi - u\sin\phi\partial/\partial u).\end{aligned} \tag{28}$$

At best, only some of the exact symmetries "lost" in going from Minkowski to Schwarzschild space have been recovered. Since the orbital equation had been derived by using the symmetries to restrict the motion to an (arbitrarily chosen) equatorial plane, it could be expected that *all* of them will reappear as approximate symmetries in the full system of geodesic equations

$$\ddot{t} + \nu'\dot{t}\dot{r} = 0, \tag{29}$$

$$\ddot{r} + \frac{1}{2}(e^\nu)'(e^\nu c^2\dot{t}^2 - e^{-\nu}\dot{r}^2) - re^\nu(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2) = 0, \tag{30}$$

$$\ddot{\theta} + \frac{2}{r}\dot{r}\dot{\theta} - \sin\theta\cos\theta\dot{\phi}^2 = 0, \tag{31}$$

$$\ddot{\phi}^2 + \frac{2}{r}\dot{r}\dot{\phi} + 2\cot\theta\dot{\theta}\dot{\phi} = 0. \tag{32}$$

Here ν' is given by

$$\nu' = \frac{2GM/c^2r^2}{1 - 2GM/c^2r}. \qquad (33)$$

A problem arises in setting up the equations for the approximate symmetries for the system of Equations (29)–(32). In Equation (33) we can write

$$\nu' = \epsilon/r^2 + O(\epsilon^2), \qquad (34)$$

as the lowest order approximation, instead of the correct expression

$$\nu' e^{\nu} = \epsilon/r^2. \qquad (35)$$

One would then tend to cancel off the extra e^{ν} left in the equation. The result is *not* the same as taking the correct expression! The problem is still worse. There is another term with $\nu' e^{-\nu}$. Which way should this term be included? We could have taken the approximation to just be $-\nu'$, or as that term with $e^{-2\nu}$, or keep it with the $e^{-\nu}$. To get the "correct" one we need to go back to the first principles in the formulation of geodesic equations as the Euler–Lagrange equations for extremising the arc length with the Lagrangian $g_{ij}\dot{x}^i\dot{x}^j$, requiring that the metric be static and spherically symmetric. In that case, we must treat the e^{ν} as one function and $e^{-\nu}$ as a totally distinct function, as they are varied separately. In that case, we get Equation (30) as

$$\ddot{r} + \frac{\epsilon}{2r^2}c^2\dot{t}^2 - \frac{\epsilon}{2r^2}\dot{r}^2 - r\left(1 - \frac{\epsilon}{r}\right)(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2) = 0. \qquad (36)$$

With the "correct" expression used, one recovers the previous conservation laws as approximate conservation laws. The new approximate (variational) symmetry generators are exactly the same as the exact symmetry generators that were "lost" due to the gravitational field. Note that Lorentz invariance is recovered as an approximate symmetry in the gravitational field. So the 'trivial' (in the sense that they are epsilon multiples of the exact symmetries) approximate symmetries provide all the known conservation laws approximately. That is, the conservation laws are approximately inherited by the perturbed geodesic equations of the Schwarzschild metric.

4 Summary and discussion

In this paper, we used the connection between symmetries of geodesics and the underlying spaces [7] to propose that one could usefully define approximate isometries by carrying the approximate symmetries of the system of geodesic equations over to the manifold on which the geodesics lie. This was applied to the Schwarzschild metric and the approximate symmetries were interpreted.

One might have expected that the same method that was used for defining approximate symmetries of differential equations could have been used to define approximate isometries directly. However, the resulting approximate isometry will *not*, in general, provide an approximately conserved quantity. The approximate isometry, when applied to the Lagrangian giving the geodesic equation, will *not* give a conservation to $O(\epsilon^2)$ due to the action of the approximate symmetry generator on the $\dot{\mathbf{x}}$ term in the Lagrangian

$$L[\mathbf{x}, \dot{\mathbf{x}}] = \mathbf{g}.\dot{\mathbf{x}}\dot{\mathbf{x}} := g_{ij}\dot{x}^i\dot{x}^j. \qquad (37)$$

For the direct definition it would only be provided by some application of the variational principle, which is provided by the Einstein field equations for the exact isometry but is not provided by any consideration taken for the approximate isometry. (It is possible that the problem noted here is related to the fact that the Schwarzschild metric, regarded as a perturbation of the Minkowski is unstable in the limit of zero perturbation.) *This* is why we need to "cross the bridge" to carry the concept of approximate symmetries of geodesic equations to the approximate isometry.

The first point to note is that the exact isometries "lost" due to the gravitational mass, are recovered as approximate isometries of those metrics with the "correct" definition. This is reasonable. If the gravitational field were made negligible, then to the lowest order the isometries *should* come up *approximately*. From the radial orbital equation we were already able to see that the linear momentum in the equatorial plane is approximately conserved. This strongly indicated that the full linear momentum would be approximately conserved but that needed to be seen by computing the approximate variational symmetries of the full set of geodesic equations. Further, the exact form of the conserved linear momentum could not be seen in the orbital equation and the conservation of the spin angular momentum was not seen there at all. Obtaining the approximate symmetries from the full set of geodesic equations is a difficult task and it would be useful if one could find some means to obtain more information from the reduced equation.

The problem of ambiguity of the form of equation to be used for approximate symmetries was noted and it was pointed out that the "correct" form comes from consideration of the Lagrangian for the geodesic equations. Only if that form is used, do we obtain the approximate conservation laws.

A point worth marking is the difference between the conservation laws obtained for the perturbed system of equations and the perturbed single, orbital, equation. In the system we get the exactly conserved quantities as the approximately conserved quantities. However, in the reduced form, with the orbital equation, we see that the conservation law gets modified, so that the approximately conserved quantity has a part related to the unperturbed conserved quantity. It is not clear that this separation will hold for other spacetimes, like gravitational wave spacetimes.

It is worth mentioning that the interplay of physics and mathematics in the link between the geometry and the differential equations, is also useful in the reverse direction. Physics told us to expect the approximate symmetries of the geodesic equations. As such, from purely physical considerations we should, in principle, be able to identify the approximate symmetries for the system of geodesic equations for a given metric. This could help us to formulate more general criteria for the existence of approximate symmetries of geodesic equations in a manner similar to that in which geometric considerations helped in formulating the criteria for linearizability of second-order quadratically semilinear systems of ODEs [13].

Finally, we stress that the insights obtained here encourages one to hope that further insights could be obtained for other metrics. One direction to proceed would be to take well-understood metrics to further establish the link that has been indicated here. The other would be to use that link to better understand metrics whose physics is not so well understood. In particular, gravitational wave spacetimes would be worth examining for their approximate symmetries. It would, particularly, be interesting to see if the approximate conservation obtained in relativity corresponds to what would be expected on the basis of the pseudo-Newtonian [14] and the extended pseudo-Newtonian [15] formalisms. Since the latter gives the momentum imparted to test particles by gravitational waves [16], the insights from the approximate symmetry analysis would provide a useful check.

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