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ORIGINAL ARTICLE

Lyapunov and LMI analysis and feedback control of border collision bifurcations

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Abstract Feedback control of piecewise smooth discrete-time systems that undergo border collision bifurcations is considered. These bifurcations occur when a fixed point or a periodic orbit of a piecewise smooth system crosses or collides with the border between two regions of smooth operation as a system parameter is quasistatically varied. The class of systems studied is piecewise smooth maps that depend on a parameter, where the system dimension n can take any value. The goal of the control effort in this work is to replace the bifurcation so that in the closed-loop system, the steady state remains locally attracting and locally unique ("nonbifurcation with persistent stability"). To achieve this, Lyapunov and linear matrix inequality (LMI) techniques are used to derive a sufficient condition for nonbifurcation with persistent stability. The derived condition is stated in terms of LMIs. This condition is then used as a basis for the design of feedback controls to eliminate border collision bifurcations in piecewise smooth maps and to produce the desirable behavior noted earlier. Numerical examples that demonstrate the effectiveness of the proposed control techniques are given.

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1 Introduction

Stabilizing feedback control laws for piecewise smooth discrete-time systems exhibiting border collision bifurcations (BCBs) are developed. The class of piecewise smooth systems considered in the theory of BCBs constitutes systems that are smooth everywhere except along borders separating regions of smooth behavior, where the system is only continuous. Border collision bifurcations are bifurcations that occur when a fixed point (or a periodic orbit) of such a piecewise smooth system crosses or collides with the border between two regions of smooth operation. The term border collision bifurcation was coined by Nusse and Yorke [26]. Border collision bifurcation had also been studied in the Russian literature, but under the name *C*-bifurcations, by Feigin [11, 12]. The results of Feigin were introduced to the Western literature in ref. [9].

Border collision bifurcations (BCBs) include bifurcations that are reminiscent of the classical bifurcations in smooth systems such as the fold and period doubling bifurcations. Despite such resemblances, border collision bifurcations present a much richer variety of possibilities than their smooth counterparts. Indeed, their classification is far from complete, and certainly very preliminary in comparison to the results available in the smooth case. In smooth maps, a bifurcation occurs from a one-parameter family of fixed points when a real eigenvalue or a complex conjugate pair of eigenvalues crosses the unit circle. In piecewise smooth (PWS) maps, however, a discontinuous change in the eigenvalues of the Jacobian matrix evaluated at the fixed point (or at a periodic point) occurs when the fixed point hits the border. As a result, border collision bifurcations for piecewise smooth systems in which the one-sided derivatives on the border are finite are classified based on the linearizations of the system on both sides of the border at criticality.

Border collision bifurcations have been observed both numerically and experimentally in applications such as power electronic devices [6, 21, 32] and in studies of grazing impact in mechanical oscillators [8, 24]. Other applications of PWS discrete-time systems that have been shown to exhibit BCBs include examples from computer networks (e.g., [14, 28, 30]), economics (e.g., [18]), biology (e.g., [29]). They have also been noted to arise in controlled linear discrete-time systems with PWS nonlinearity (e.g., [1]).

Border collision bifurcations in one-dimensional PWS maps were considered in refs. [3, 22, 27] and a complete classification was given. Nusse and Yorke [26] studied border collision bifurcations of two-dimensional piecewise smooth maps that occur when a parameter is varied, and derived a normal form for the corresponding piecewise-linear maps of any two-dimensional piecewise smooth map with one border and two regions. Furthermore, they presented many numerical examples of such border collision bifurcations and addressed the question of whether or not it would be possible to give a classification of the border collision bifurcations using the two-dimensional piecewise smooth normal form map (see ref. [26] for details). In ref. [25], the importance of such bifurcations for continuous-time systems was addressed. The authors of refs. [4, 31] proposed a classification for a class of two-dimensional maps undergoing border collision by exploiting the normal form. Other results on BCBs, including special phenomena that cannot be described here, are given in refs. [7, 10, 16].

It should be noted that work such as that in this paper focusing on maps has implications for continuous-time switched systems as well. Maps provide a concise representation that facilitates the investigation of system behavior and control design. They are also the natural models for many applications. Moreover, a control design derived using the map representation of a continuous-time PWS system can be reinterpreted so as to yield a continuous controller either analytically or numerically.

In the present paper, a sufficient condition is derived for nonbifurcation with persistent stability of piecewise smooth maps of any given finite dimension n that depend on a parameter. That is, a condition is found under which a PWS map possesses a locally asymptotically stable fixed point which is also the locally unique attractor for all values of the bifurcation parameter in a neighborhood of the critical value. This condition is derived using Lyapunov and linear matrix inequality (LMI) [5] methods. The derived condition is then used as a basis for the design of feedback controls that eliminate BCBs in piecewise smooth maps and produce desirable, locally stable behavior.

The paper proceeds as follows. In Section 2, brief background material on BCBs is given. In Section 3, Lyapunov analysis with the aid of LMIs is applied to PWS maps undergoing BCBs, resulting in a sufficient condition for nonbifurcation with persistent stability. In Section 4, the results of Section 3 are used in the synthesis of stabilizing feedback control laws and numerical examples that demonstrate the results are given. Concluding remarks are collected in Section 5.

2 Background on border collision bifurcations

Consider the one-parameter family of piecewise smooth maps

$$f(x,\mu) = \begin{cases} f_A(x,\mu), & x \in R_A \\ f_B(x,\mu), & x \in R_B \end{cases} \tag{1}$$

where $f\colon \mathbb{R}^{n+1} \to \mathbb{R}^n$ is piecewise smooth in x; f is smooth in x everywhere except on the border (hypersurface Γ) separating R_A and R_B where it is only continuous, f is smooth in μ and R_A, R_B are the two (nonintersecting) regions of smooth behavior. Of great interest is the study of the dynamics of f at a fixed point (or a periodic orbit) near or at the border Γ. If the fixed point (or periodic orbit) is in R_A (respectively, R_B) and is away from the border, then the local dynamics is determined by the single map f_A (respectively, f_B). If, however, the fixed point is close to the border, then jumps across the border can occur except in an extremely small neighborhood of the fixed point.

Therefore, for operation close to the border, both f_A and f_B are needed in the study of the possible behavior. For a fixed point at or near the border, the dynamics is determined by the linearizations of the map on both sides of the border.

Various types of BCBs occur in Equation (1) as the bifurcation parameter μ is varied through a critical value [2, 9, 25, 26]. Figure 1 is a schematic illustrating sample bifurcation diagrams for various border collision bifurcations. Such bifurcations occurring in the map (1) can be studied using the piecewise-linearized representation [9]

$$x(k+1) := F(x(k), \mu) = \begin{cases} Ax(k) + b\mu, & x_1(k) \le 0 \\ Bx(k) + b\mu, & x_1(k) > 0 \end{cases} \quad (2)$$

where A is the linearization of the PWS map f in R_A at a fixed point on the border approached from points in R_A near the border and B is the linearization of f at a fixed point on the border approached from points in R_B and b is the derivative of the map f with respect to μ. The coordinate system is chosen such that the sign of the first component of the vector x determines whether x is in R_A or R_B (a transformation to the form (2) is given in ref. [15]). If $x_1 = 0$, then x is on the border separating R_A and R_B. The continuity of F at the border implies that A and B differ only in their first columns.

The classification of BCBs depends on the eigenvalues of A and B [9]. A complete classification of BCBs is only available for one-dimensional PWS maps [3, 27]. For two-dimensional PWS maps, some results are available that only address a class of two-dimensional PWS maps [2, 16, 26, 31].

Although Feigin [9] studied general n-dimensional PWS maps exhibiting border collisions, only very general conditions for existence of a fixed point and period-2 solutions before and after the border were given. The classification scheme of ref. [9] does not give any information about stability or uniqueness of fixed points or period-2 orbits involved in the border collision bifurcation nor does it give information about higher period periodic orbits or chaos that might be involved in a border collision bifurcation. Therefore, in this paper, one of the main goals is to develop a sufficient condition for nonbifurcation with persistent stability that can be used in the design of stabilizing feedback control laws. This is done in the next section using Lyapunov and LMI techniques.

3 Lyapunov and LMI analysis of piecewise smooth maps

Recently, many researchers have studied stability of a fixed point of switched discrete-time linear systems (e.g., [13, 20, 23]) as well as continuous time switched systems (e.g., [19]). In all the referenced studies, Lyapunov techniques were used to obtain sufficient conditions for stability of the fixed point (or equilibrium point) of a piecewise linear system. For instance, in refs. [19, 23], quadratic as well as piecewise quadratic Lyapunov functions were used in the analysis of stability of switched systems and also in the synthesis of stabilizing controls. The authors are unaware of any previous study using Lyapunov methods to analyze the dynamics of switched systems depending on a parameter. Here, a quadratic Lyapunov function is used to study border collision bifurcations in PWS maps and to obtain sufficient conditions for nonbifurcation with persistent stability in such maps.

Consider the piecewise-linearized representation of PWS maps given in Equation (2). The sign of the first component of the vector x determines whether x is in R_A or in R_B. If $x_1 = 0$, then x is on the border separating R_A and R_B. The continuity of F at the border implies that A and B differ only in their first columns. That is, $a_{ij} = b_{ij}$, for $j \neq 1$, where $A = [a_{ij}]$ and $B = [b_{ij}]$.

Assume that 1 is not an eigenvalue of either A or B (so that both $I - A$ and $I - B$ are nonsingular). Formally solving for the fixed points of Equation (2) yields $x_A(\mu) = (I - A)^{-1} b\mu$ and $x_B(\mu) = (I - B)^{-1} b\mu$. For $x_A(\mu)$ to actually occur as a fixed point, the first component of $x_A(\mu)$ must be nonpositive. That is,

$$x_{A_1}(\mu) = (e^1)^{\mathrm{T}} \mu (I - A)^{-1} b \le 0 \quad (3)$$

where $(e^1)^{\mathrm{T}} = (1\ 0\ \cdots\ 0)$. Similarly, for $x_B(\mu)$ to actually occur, one needs

$$x_{B_1}(\mu) = (e^1)^{\mathrm{T}} \mu (I - B)^{-1} b > 0. \quad (4)$$

If, however, the first component of $x_A(\mu)$ is positive, then the fixed point is called a virtual fixed point. An analogous situation occurs if the first component of

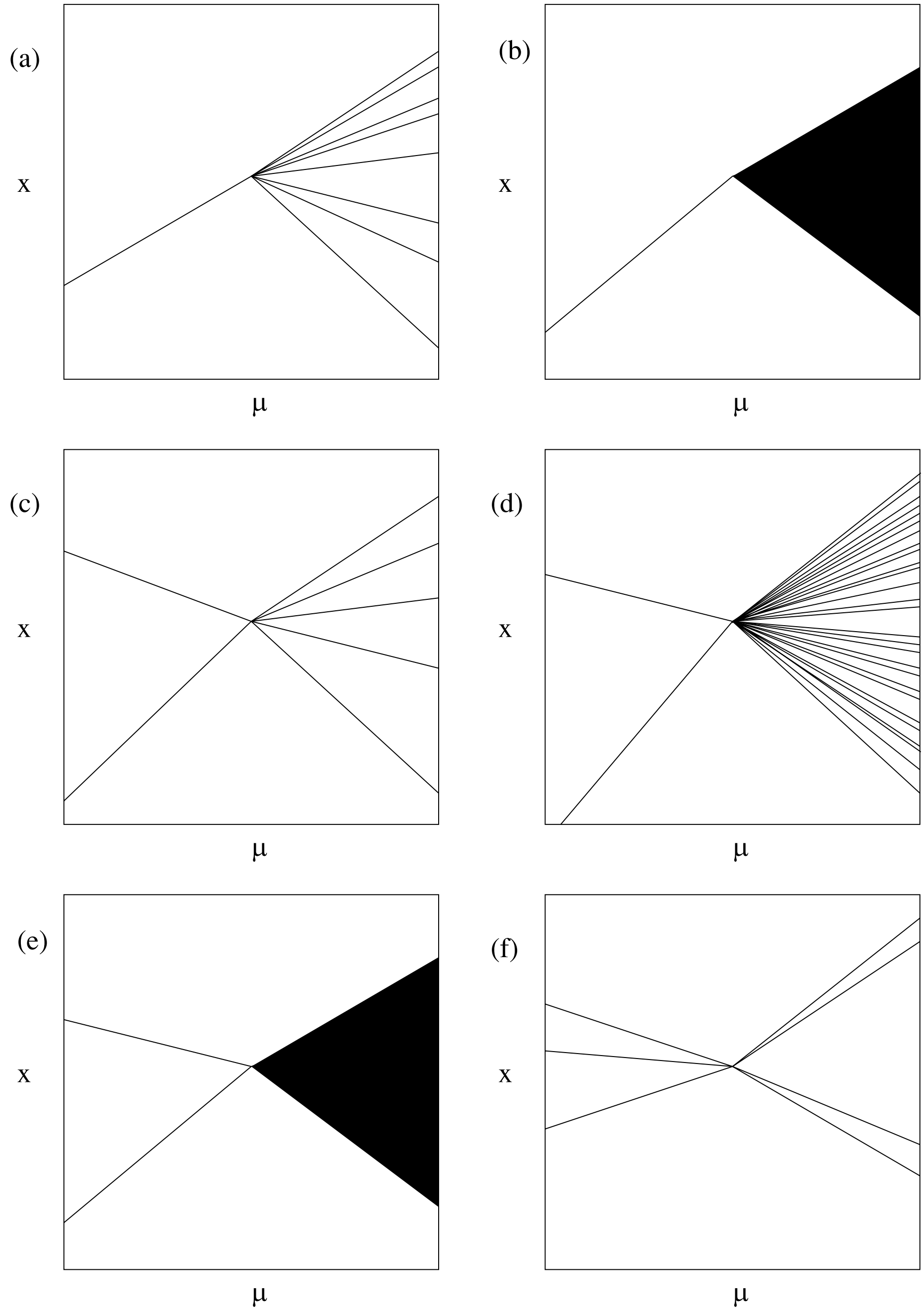

Fig. 1 Schematic illustrating sample border collision bifurcations. (a) BCB from a fixed point attractor to a period-8 attractor, (b) BCB from a fixed point attractor to a chaotic attractor, (c) BCB from a period-2 attractor to a period-5 attractor, (d) BCB from a period-2 attractor to a period-27 attractor, (e) BCB from a period-2 attractor to a chaotic attractor, (f) BCB from a period-3 attractor to a period-4 attractor

$x_B(\mu)$ is nonpositive. Virtual fixed points are important in studying the dynamics of a PWS map at or near the border.

Let $p_A(\lambda)$ and $p_B(\lambda)$ be the characteristic polynomials of A and B, respectively: $p_A(\lambda) = \det(\lambda I - A)$, $p_B(\lambda) = \det(\lambda I - B)$.

Using Cramer's rule, the fixed points can be written as

$$x_A(\mu) = (I - A)^{-1} b\mu = \frac{\operatorname{adj}(I - A) b\mu}{\det(I - A)} = \frac{\bar{b}_A}{p_A(1)}\mu, \quad (5)$$

and

$$x_B(\mu) = (I - B)^{-1} b\mu = \frac{\operatorname{adj}(I - B) b\mu}{\det(I - B)} = \frac{\bar{b}_B}{p_B(1)}\mu, \quad (6)$$

where $\bar{b}_A = \operatorname{adj}(I - A)b$ and $\bar{b}_B = \operatorname{adj}(I - B)b$. It can be shown that $\bar{b}_{A_1} = \bar{b}_{B_1} =: \bar{b}_1$ [9]. To see this, recall that A and B differ only in their first columns and $\operatorname{adj}(I - A) = (\operatorname{cof}(I - A))^{\mathrm{T}}$. Thus, the first row of $\operatorname{adj}(I - A)$ is equal to the first row of $\operatorname{adj}(I - B)$, which implies that $(e^1)^{\mathrm{T}} \operatorname{adj}(I - A)b = (e^1)^{\mathrm{T}} \operatorname{adj}(I - B)b =: \bar{b}_1$. Thus, the first component of $x_A(\mu)$ is $x_{A_1}(\mu) = \frac{\bar{b}_1}{p_A(1)}\mu$ and the first component of $x_B(\mu)$ is $x_{B_1}(\mu) = \frac{\bar{b}_1}{p_B(1)}\mu$. For the fixed point $x_A(\mu)$ to occur for $\mu \le 0$, it is required that $x_{A_1}(\mu) \le 0$, i.e., $\frac{\bar{b}_1}{p_A(1)}\mu \le 0 \iff \frac{\bar{b}_1}{p_A(1)} > 0$. Similarly, for the fixed point $x_B(\mu)$ to occur for $\mu > 0$, it is required that $x_{B_1}(\mu) > 0$, i.e., $\frac{\bar{b}_1}{p_B(1)}\mu > 0 \iff \frac{\bar{b}_1}{p_B(1)} > 0$. Therefore, a necessary and sufficient condition to have a fixed point for all μ is $p_A(1)p_B(1) > 0$ which is assumed to be in force in the remainder of the discussion.

Next, a change of variables is performed on Equation (2) to simplify the analysis.

Case (1): $\underline{\mu \le 0}$. The fixed point of F is $x_A(\mu) = (I - A)^{-1} b\mu$. Changing the state variable in Equation (2) to $z = x - x_A(\mu)$ yields after simplification

$$z(k+1) = \begin{cases} Az(k), & \text{if } z_1(k) \le -x_{A_1}(\mu) \\ Bz(k) + c\mu, & \text{if } z_1(k) > -x_{A_1}(\mu) \end{cases} \quad (7)$$

where $c = (B - A)(I - A)^{-1} b$. In the new coordinates, $z = 0$ is a fixed point for all $\mu \le 0$. (Note that the border $z_{\text{border}} = \{z\colon z_1 = -x_{A_1}(\mu)\}$, varies as a function of μ.) Note that since B and A differ only in their first columns, all elements of $B - A$ are zero except for the first column. Thus, $c\mu = (B - A)(I - A)^{-1} b\mu = (B - A)x_A(\mu) = x_{A_1}(\mu)(B^1 - A^1)$, where the notation A^i means the ith column of the matrix A.

Consider the quadratic Lyapunov function candidate

$$V(z) = z^{\mathrm{T}} P z, \quad \text{where } P = P^{\mathrm{T}} > 0. \quad (8)$$

Existence of a P such that $V(z)$ is a common quadratic Lyapunov function (CQLF) for the linear systems $z(k+1) = Az(k)$ and $z(k+1) = Bz(k)$ associated with System (2) will be the main condition in the results obtained below. The forward difference of V along trajectories of Equation (7) is $\Delta V(z(k)) = V(z(k+1) - V(z(k))$. Two cases need to be considered: $z_1(k) \le -x_{A_1}(\mu)$ and $z_1(k) > -x_{A_1}(\mu)$.

Case (1.1): $z_1(k) \le -x_{A_1}(\mu)$

$$\begin{aligned} \Delta V_L(z(k)) &= V(z(k+1)) - V(z(k)) \\ &= (Az(k))^{\mathrm{T}} PAz(k) - z(k)^{\mathrm{T}} Pz(k) \\ &= z(k)^{\mathrm{T}} (A^{\mathrm{T}} PA - P) z(k). \end{aligned} \quad (9)$$

Case (1.2): $z_1(k) > -x_{A_1}(\mu)$

$$\begin{aligned} \Delta V_R(z(k)) &= V(z(k+1)) - V(z(k)) \\ &= (Bz(k) + c\mu)^{\mathrm{T}} P (Bz(k) + c\mu) \\ &\quad - z(k)^{\mathrm{T}} Pz(k) = z(k)^{\mathrm{T}} (B^{\mathrm{T}} PB - P) z(k) \\ &\quad + 2\mu c^{\mathrm{T}} PBz(k) + \mu^2 c^{\mathrm{T}} Pc \\ &= z(k)^{\mathrm{T}} (B^{\mathrm{T}} PB - P) z(k) \\ &\quad + 2x_{A_1}(\mu)(B^1 - A^1)^{\mathrm{T}} PBz(k) \\ &\quad + x_{A_1}^2(\mu)(B^1 - A^1)^{\mathrm{T}} P (B^1 - A^1). \end{aligned} \quad (10)$$

Combining Equations (9) and (10) yields

$$\Delta V(z(k)) = \begin{cases} \Delta V_L(z(k)), & \text{if } z_1(k) \le -x_{A_1}(\mu) \\ \Delta V_R(z(k)), & \text{if } z_1(k) > -x_{A_1}(\mu) \end{cases} \quad (11)$$

From Equations (9) and (10), a necessary condition for $\Delta V(z(k))$ to be negative definite is that the following two matrix inequalities hold:

$$A^{\mathrm{T}} PA - P < 0, \quad (12)$$

$$B^{\mathrm{T}} PB - P < 0. \quad (13)$$

Moreover, the following claim, which asserts sufficiency of Equations (12) and (13) for negative definiteness of $\Delta V(z(k))$ is stated and proved.

Claim: (Sufficiency of LMIs (12)–(13) for a decreasing Lyapunov function)

If the matrix inequalities (12)–(13) are satisfied with $P = P^{\mathrm{T}} > 0$, then $\Delta V(z(k))$ given by Equation (11) is negative definite.

Proof: Assume that there is a $P = P^{\mathrm{T}} > 0$ such that Equations (12)–(13) are satisfied. Then, $\Delta V_L = z^{\mathrm{T}}(A^{\mathrm{T}}PA - P)z < 0 \ \forall z \neq 0$. It remains to show that $\Delta V_R < 0$. Let $z = (z_1, z_2)^{\mathrm{T}}$, where $z_1 \in \mathbb{R}$ and $z_2 \in \mathbb{R}^{n-1}$. Note that ΔV is continuous for all z. Continuity of ΔV follows from the continuity of V and continuity of the map (7). Since $\Delta V_L < 0$ ($\Delta V_L = 0$ if and only if $z = 0$) and ΔV is continuous for all z, it follows that $\Delta V_R < 0$ at the border $\{z_1 = -x_{A_1}(\mu)\}$ (since $\lim_{(z_1,z_2)\to(-x_{A1}^-(\mu),z_2)} \Delta V_L = \lim_{(z_1,z_2)\to(-x_{A1}^+(\mu),z_2)} \Delta V_R$). It remains to show that $\Delta V_R < 0$ for all z in the region $z_1 > -x_{A_1}(\mu)$ (note that $-x_{A_1}(\mu) > 0$). Completing the squares in Equation (10), $\Delta V_R(z)$ can be rewritten as follows:

$$\begin{aligned}\Delta V_R(z) &= z^{\mathrm{T}}(B^{\mathrm{T}}PB - P)z + 2x_{A_1}(\mu)(B^1 - A^1)^{\mathrm{T}}PBz \\ &\quad + x_{A_1}^2(\mu)(B^1 - A^1)^{\mathrm{T}}P(B^1 - A^1) \\ &= (z-\alpha)^{\mathrm{T}}(B^{\mathrm{T}}PB - P)(z-\alpha) \\ &\quad -\alpha^{\mathrm{T}}(B^{\mathrm{T}}PB - P)\alpha \\ &\quad + x_{A_1}^2(\mu)(B^1 - A^1)^{\mathrm{T}}P(B^1 - A^1) \end{aligned} \tag{14}$$

where $\alpha = -x_{A_1}(\mu)(B^{\mathrm{T}}PB - P)^{-1}B^{\mathrm{T}}P(B^1 - A^1)$. Let $\mathcal{N} \subset \mathbb{R}^n$ such that $\mathcal{N}$ is convex and contains the origin (for example, a ball). Since the fixed point $x_A(\mu)$ is close to the origin for small μ, the hyperplane $z_1 = -x_{A_1}(\mu)$ slices the neighborhood $\mathcal{N}$. Consider $\Delta V_R(z)$ restricted to $\mathcal{N}$. The second derivative of $\Delta V_R(z)$ with respect to z (i.e., its Hessian matrix) is $\nabla^2 \Delta V_R = 2(B^{\mathrm{T}}PB - P) < 0$. Thus, $\Delta V_R(z)$ is strictly concave on $\mathcal{N}$, i.e., for every $z, y \in \mathcal{N}$, and $\theta \in (0, 1)$, $\Delta V_R(\theta z + (1-\theta)y) > \theta \Delta V_R(z) + (1-\theta)\Delta V_R(y)$. Note that $\Delta V_R(0) = x_{A_1}^2(\mu)(B^1 - A^1)^T P(B^1 - A^1) > 0$. Next, it is shown that $\Delta V_R < 0 \ \forall z \in \mathcal{N}$ with $z_1 > -x_{A_1}(\mu)$. By way of contradiction, suppose there is a $y \in \mathcal{N}$, with $y_1 > -x_{A1}(\mu)$, such that $\Delta V_R(y) > 0$. Since $\Delta V_R(z)$ is strictly concave, it follows that $\Delta V_R(z)$ is positive along the line segment connecting 0 and y: $\Delta V_R(\theta \cdot 0 + (1-\theta)y) > \theta \underbrace{\Delta V_R(0)}_{>0} + (1-\theta)\underbrace{\Delta V_R(y)}_{>0} > 0, \quad \forall \theta \in (0, 1)$. But, along the line connecting $z = 0$ with $z = y$, there is a point z^* with $z_1^* = -x_{A_1}(\mu)$ where $\Delta V_R(z^*) < 0$, which is a contradiction. Thus, $\Delta V_R(z) < 0$ for all $z \in \mathcal{N}$ with $z_1 > -x_{A_1}(\mu) > 0$. □

The following proposition summarizes the results so far.

Proposition 1. *The forward difference of $V = z^{\mathrm{T}}Pz$, with $P = P^{\mathrm{T}} > 0$, along trajectories of Equation (7) with $\mu \leq 0$ is negative definite (i.e., $\Delta V(z) < 0$) if and only if the following matrix inequalities hold:*

$$A^{\mathrm{T}}PA - P < 0, \tag{15}$$

$$B^{\mathrm{T}}PB - P < 0. \tag{16}$$

Case (2): $\underline{\mu > 0}$. The fixed point of F is $x_B(\mu) = (I - B)^{-1}b\mu$. Changing the state variable in Equation (2) to $z = x - x_B(\mu)$ yields after simplification

$$z(k+1) = \begin{cases} Az(k) + c\mu, & \text{if } z_1(k) \leq -x_{B_1}(\mu) \\ Bz(k), & \text{if } z_1(k) > -x_{B_1}(\mu) \end{cases} \tag{17}$$

where $c = (A - B)(I - B)^{-1})b$. In the new coordinates, $z = 0$ is a fixed point for all $\mu > 0$. (Note that the border $z_{\text{border}} = \{z : z_1 = -x_{B_1}(\mu)\}$, varies as a function of μ.) Note that since B and A differ only in their first columns, all elements of $A - B$ are zero except for the first column. Thus, $c\mu = (A - B)(I - B)^{-1}b\mu = (A - B)x_B(\mu) = x_{B_1}(\mu)(A^1 - B^1)$.

Consider the same quadratic Lyapunov function candidate as in Equation (8) earlier:

$$V(z) = z^{\mathrm{T}}Pz, \qquad \text{where } P = P^{\mathrm{T}} > 0.$$

The forward difference of V along trajectories of Equation (17) is $\Delta V(z(k)) = V(z(k+1) - V(z(k))$. There are two cases: $z_1(k) \leq -x_{B_1}(\mu)$ and $z_1(k) > -x_{B_1}(\mu)$. (Note that $x_{B1}(\mu) > 0$ from Equation (4).)

Case (2.1): $z_1(k) \leq -x_{B_1}(\mu)$

$$\begin{aligned}&\Delta V_L(z(k)) \\ &\quad = V(z(k+1)) - V(z(k)) \\ &\quad = (Az(k) + c\mu)^{\mathrm{T}}P(Az(k) + c\mu) - z(k)^{\mathrm{T}}Pz(k)\end{aligned}$$

$$= z(k)^{\mathrm{T}}(A^{\mathrm{T}}PA - P)z(k) + 2\mu c^{\mathrm{T}}PAz(k) + \mu^2 c^{\mathrm{T}}Pc$$
$$= z(k)^{\mathrm{T}}(A^{\mathrm{T}}PA - P)z(k) + 2x_{B_1}(\mu)(A^1 - B^1)^{\mathrm{T}}PAz(k) + x_{B_1}^2(\mu)(A^1 - B^1)^{\mathrm{T}}P(A^1 - B^1). \quad (18)$$

Case (2.2): $z_1(k) > -x_{B_1}(\mu)$

$$\begin{aligned}\Delta V_R(z(k)) &= V(z(k+1)) - V(z(k)) \\ &= (Bz(k))^{\mathrm{T}}PBz(k) - z(k)^{\mathrm{T}}Pz(k) \\ &= z(k)^{\mathrm{T}}(B^{\mathrm{T}}PB - P)z(k). \end{aligned} \quad (19)$$

Combining Equations (18) and (19) yields

$$\Delta V(z(k)) = \begin{cases} \Delta V_L(z(k)), & \text{if } z(k) \le -x_{B_1}(\mu) \\ \Delta V_R(z(k)), & \text{if } z(k) > -x_{B_1}(\mu) \end{cases} \quad (20)$$

Proposition 2. (Necessary and sufficient conditions for a decreasing Lyapunov function)
The forward difference of $V = z^{\mathrm{T}}Pz$, with $P = P^{\mathrm{T}} > 0$, along trajectories of Equation (17) with $\mu \ge 0$ is negative definite (i.e., $\Delta V(z) < 0$) if and only if the following matrix inequalities hold:

$$A^{\mathrm{T}}PA - P < 0, \quad (21)$$
$$B^{\mathrm{T}}PB - P < 0. \quad (22)$$

Proof: Necessity follows from Equations (18) and (19), and the proof for sufficiency is similar to that for the case $\mu \le 0$ earlier. □

By combining Proposition 1 and Proposition 2, the main result of this paper is obtained.

Proposition 3. (Sufficient condition for nonbifurcation with persistent stability in n-dimensional PWS maps)
The PWS map (2) has a globally asymptotically stable fixed point for all $\mu \in \mathbb{R}$ if there is a $P = P^{\mathrm{T}} > 0$ such that

$$A^{\mathrm{T}}PA - P < 0,$$
$$B^{\mathrm{T}}PB - P < 0.$$

Corollary 1. *If at $\mu = 0$ the origin of the map (2) is quadratically stable, i.e., can be shown to be stable using a quadratic Lyapunov function $V = x^{\mathrm{T}}Px$ with $P > 0$, then the fixed point depending on μ on both sides of the border is attracting and no bifurcation occurs from the origin as μ is varied through zero.*

Later, a numerical example is given to demonstrate how the Lyapunov and LMI techniques considered in this section can be applied.

Example 1. Consider the three-dimensional PWS map

$$x(k+1) = \begin{cases} Ax(k) + b\mu, & x_1(k) \le 0 \\ Bx(k) + b\mu, & x_1(k) > 0 \end{cases} \quad (23)$$

where

$$A = \begin{pmatrix} 0.4192 & 0.3514 & 0.3473 \\ 0.2840 & -0.2733 & -0.3107 \\ 0.1852 & -0.2224 & -0.3974 \end{pmatrix},$$

$$B = \begin{pmatrix} -0.60 & 0.3514 & 0.3473 \\ 0.56 & -0.2733 & -0.3107 \\ -0.90 & -0.2224 & -0.3974 \end{pmatrix} \text{ and } b = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

The eigenvalues of A and B are $\{0.5653, -0.7413, -0.0755\}$ and $\{0.0395, -0.6551 \pm j\,0.4246\}$, respectively. Although both A and B are Schur stable matrices, it cannot be concluded that no bifurcation for Equation (23) occurs at $\mu = 0$.

A common quadratic Lyapunov function $V = x^{\mathrm{T}}Px$, with $P = P^{\mathrm{T}} > 0$ that satisfies the conditions of Proposition 3 exists for this example. To wit:

$$P = \begin{pmatrix} 1.6304 & 0.1559 & -0.1313 \\ 0.1559 & 1.3200 & 0.4436 \\ -0.1313 & 0.4436 & 1.3266 \end{pmatrix}$$

is obtained using the MATLAB LMI toolbox. Thus, the PWS map (23) has a unique attracting fixed point for all μ (see Fig. 2).

4 Feedback control design

In this section, the results of Section 3 are used in the design stabilizing feedback control laws. It is important to emphasize that for this approach to apply, the control action should not introduce discontinuity in the map. This is because, as discussed in the Introduction,

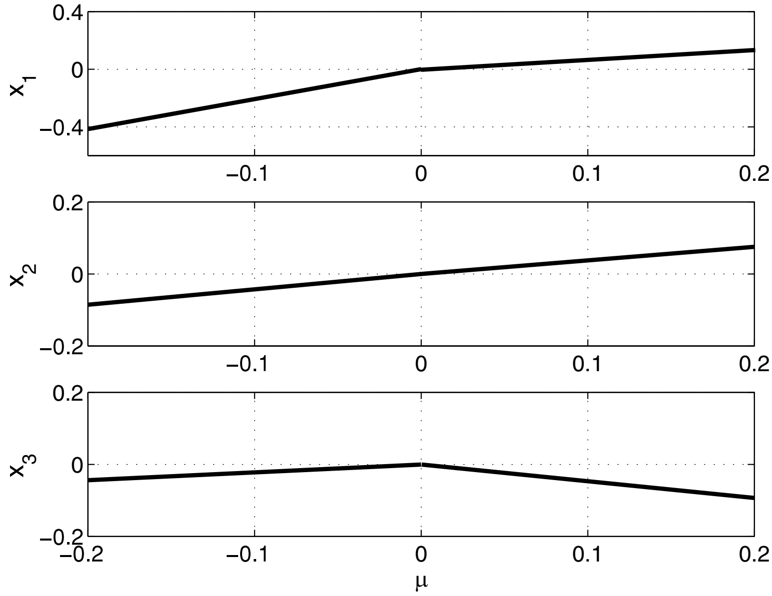

Fig. 2 Bifurcation diagram for Example 1. Each *solid line* represents a path of stable fixed points

the definition of BCBs requires that the system map be continuous at the border, and thus the results presented here on nonbifurcation with persistent stability also apply only under this condition. Therefore, to maintain continuity of the map after control is applied, it is assumed that the input vectors on both sides of the border are equal. In this work, the input vectors are both taken to be equal to b (the derivative of the map with respect to the bifurcation parameter.)

Simultaneous feedback control is considered first, followed by switched feedback control design.

4.1 Simultaneous feedback control design

In this control method, the same control is applied on both sides of the border. The purpose of pursuing stabilizing feedback acting on both sides of the border is to ensure robustness with respect to modeling uncertainty. Moreover, transformation to the normal form is not required when simultaneous control is used. All that is needed is a good estimate of the Jacobian matrices on both sides of the border.

Consider the closed-loop system using static linear state feedback

$$x(k+1) = \begin{cases} Ax(k) + b\mu + bu(k), & \text{if } x_1(k) \le 0 \\ Bx(k) + b\mu + bu(k), & \text{if } x_1(k) > 0 \end{cases} \tag{24}$$

$$u(k) = gx(k) \tag{25}$$

where g is the control gain (row) vector.

The following proposition gives stabilizability condition for the border collision bifurcation with this type of control policy.

Proposition 4. *If there exist a $P = P^{\mathrm{T}} > 0$, and a feedback gain (row) vector g such that*

$$P - (A + bg)^{\mathrm{T}} P (A + bg) > 0 \tag{26}$$

$$P - (B + bg)^{\mathrm{T}} P (B + bg) > 0 \tag{27}$$

then, any border collision bifurcation that occurs in the open-loop system ($u \equiv 0$) of Equation (24) can be eliminated and persistent stability is guaranteed using simultaneous feedback (25). Equivalently, if there exists a Q and y such that

$$\begin{pmatrix} Q & AQ + by \\ (AQ + by)^{\mathrm{T}} & Q \end{pmatrix} > 0, \tag{28}$$

$$\begin{pmatrix} Q & BQ + by \\ (BQ + by)^{\mathrm{T}} & Q \end{pmatrix} > 0, \tag{29}$$

then any border collision bifurcation that occurs in Equation (24) can be eliminated using simultaneous feedback (25). Here $Q = P^{-1}$ and the feedback gain is given by $g = yP$.

Proof: The closed-loop system is given by

$$x(k+1) = \begin{cases} (A+bg)x(k)+\mu b, & \text{if } x_1(k) \leq 0 \\ (B+bg)x(k)+\mu b, & \text{if } x_1(k) > 0 \end{cases} \tag{30}$$

Using Proposition 3, a sufficient condition to eliminate the BCB is the existence of a $P = P^{\mathrm{T}} > 0$ such that

$$P - (A+bg)^{\mathrm{T}} P (A+bg) > 0 \tag{31}$$

$$P - (B+bg)^{\mathrm{T}} P (B+bg) > 0 \tag{32}$$

where g is the control gain to be chosen.

Next, inequalities (31)–(32) are shown to be equivalent to Equations (28)–(29) using the Schur complement [5, 23]. It is straightforward to show that

$$\begin{aligned} & P - (A+bg)^{\mathrm{T}} P (A+bg) > 0 \\ \Longleftrightarrow \quad & P^{-1} - (A+bg) P^{-1} (A+bg)^{\mathrm{T}} > 0, \end{aligned}$$

and

$$\begin{aligned} & P - (B+bg)^{\mathrm{T}} P (B+bg) > 0 \\ \Longleftrightarrow \quad & P^{-1} - (B+bg) P^{-1} (B+bg)^{\mathrm{T}} > 0. \end{aligned}$$

The nonlinear matrix inequalities shown earlier are transformed into LMIs using the Schur complement [5, 23]:

$$\begin{aligned} & P^{-1} - (A+bg)P^{-1}(A+bg)^{\mathrm{T}} \\ & = P^{-1} - (A+bg)P^{-1} P P^{-1} (A+bg)^{\mathrm{T}} \\ & = P^{-1} - (AP^{-1} + bgP^{-1}) P (AP^{-1} + bgP^{-1})^{\mathrm{T}} > 0 \\ & \Longleftrightarrow \begin{pmatrix} P^{-1} & AP^{-1}+by \\ (AP^{-1}+by)^{\mathrm{T}} & P^{-1} \end{pmatrix} > 0. \end{aligned}$$

Similarly,

$$\begin{aligned} & P - (B+bg)^{\mathrm{T}} P (B+bg) > 0 \\ \Longleftrightarrow \quad & \begin{pmatrix} P^{-1} & BP^{-1}+by \\ (BP^{-1}+by)^{\mathrm{T}} & P^{-1} \end{pmatrix} > 0 \end{aligned}$$

by similar reasoning. □

The following proposition states that if a CQLF exists in one coordinate system, another CQLF exists in a different coordinate system arrived at using a simultaneous similarity transformation applied to both A and B.

Proposition 5. (CQLF and similarity transformations) *Suppose $V = x^{\mathrm{T}} P x$ (with $P = P^{\mathrm{T}} > 0$) is a common quadratic Lyapunov function for both of the matrices A and B (i.e., $A^{\mathrm{T}} P A - P < 0$ and $B^{\mathrm{T}} P B - P < 0$). Then $\tilde{V} = x^{\mathrm{T}} \tilde{P} x$ with $\tilde{P} = (T^{-1})^{\mathrm{T}} P T^{-1} = \tilde{P}^{\mathrm{T}} > 0$ is a common quadratic Lyapunov function for $\tilde{A} = TAT^{-1}$ and $\tilde{B} = TBT^{-1}$ (i.e. $\tilde{A}^{\mathrm{T}} \tilde{P} \tilde{A} - \tilde{P} < 0$ and $\tilde{B}^{\mathrm{T}} \tilde{P} \tilde{B} - \tilde{P} < 0$). In other words, if a CQLF exists in one coordinate system, another CQLF exists if a simultaneous change of coordinates is applied to both A and B.*

Proof: See ref. [15]. □

Remark 1. The switching control design presented earlier does not depend on the border separating the two regions of smooth behavior. Thus, transformation to the normal form is not required before the control design.

4.2 Switched feedback control design

Consider the closed-loop system using static piecewise linear state feedback

$$f_\mu(x(k)) = \begin{cases} Ax(k) + b\mu + bu(k), & \text{if } x_1(k) \leq 0 \\ Bx(k) + b\mu + bu(k), & \text{if } x_1(k) > 0 \end{cases} \tag{33}$$

where

$$u(k) = \begin{cases} g_1 x(k), & x_1(k) \leq 0 \\ g_2 x(k), & x_1(k) > 0 \end{cases} \tag{34}$$

with the restriction that g_1 and g_2 may only differ in their first component, i.e., $g_{1i} = g_{2i}$, $i = 2, 3, \ldots, n$. This condition is imposed to maintain continuity along the border $\{x\colon x_1 = 0\}$.

Proposition 6. *If there exists a $P = P^{\mathrm{T}} > 0$, and feedback gains g_1 and g_2 with $g_{1i} = g_{2i}$, $i = 2, 3, \ldots, n$ such that*

$$P - (A + bg_1)^{\mathrm{T}} P (A + bg_1) > 0, \tag{35}$$

$$P - (B + bg_2)^{\mathrm{T}} P (B + bg_2) > 0, \tag{36}$$

then any border collision bifurcation that occurs in the open-loop system ($u \equiv 0$) of Equation (33) can be eliminated using switching feedback (34). Equivalently, if *there exists a Q, y_1 and $\alpha \in \mathbb{R}$ such that*

$$\begin{pmatrix} Q & AQ + by_1 \\ (AQ + by_1)^{\mathrm{T}} & Q \end{pmatrix} > 0, \tag{37}$$

$$\begin{pmatrix} Q & BQ + by_1 \\ (BQ + by_1)^{\mathrm{T}} & Q \end{pmatrix} - \alpha \begin{pmatrix} 0 & b(e^1)^{\mathrm{T}} Q \\ Qe^1 b^{\mathrm{T}} & 0 \end{pmatrix} > 0, \tag{38}$$

then any border collision bifurcation that occurs in Equation (33) can be eliminated using switching feedback (34). Here, $Q = P^{-1}$ and the feedback gains are given by $g_1 = y_1 P$ and $g_2 = g_1 - \alpha(e^1)^{\mathrm{T}}$.

Proof: The closed-loop system is given by

$$x(k+1) = \begin{cases} (A + bg_1)x(k) + \mu b, & \text{if } x_1(k) \le 0 \\ (B + bg_2)x(k) + \mu b, & \text{if } x_1(k) > 0 \end{cases} \tag{39}$$

Using Proposition 3, a sufficient condition to eliminate the BCB is the existence of a $P = P^{\mathrm{T}} > 0$ such that

$$P - (A + bg_1)^{\mathrm{T}} P(A + bg_1) > 0 \tag{40}$$

$$P - (B + bg_2)^{\mathrm{T}} P(B + bg_2) > 0 \tag{41}$$

where g_1, g_2 are the control gains to be chosen. Inequalities (40), (41) are equivalent to

$$\begin{pmatrix} Q & AQ + by_1 \\ (AQ + by_1)^{\mathrm{T}} & Q \end{pmatrix} > 0 \tag{42}$$

$$\begin{pmatrix} Q & BQ + by_2 \\ (BQ + by_2)^{\mathrm{T}} & Q \end{pmatrix} > 0 \tag{43}$$

respectively, where $Q = P^{-1}$, $g_1 = y_1 P$, and $g_2 = y_2 P$. This equivalence can be shown using similar reasoning as that used in the proof of Proposition 4.

The restriction $g_{1i} = g_{2i}$, $i = 2, 3, \ldots, n$, can be written as

$$g_2 = g_1 - \alpha(e^1)^{\mathrm{T}} \tag{44}$$

where $\alpha \in \mathbb{R}$. Therefore,

$$\begin{aligned} y_1 - y_2 &= g_1 Q - g_2 Q \\ &= (g_1 - g_2)Q \\ &= \alpha(e^1)^{\mathrm{T}} Q. \end{aligned} \tag{45}$$

Substituting $y_2 = y_1 - \alpha(e^1)^{\mathrm{T}} Q$ in Equation (43) yields Equation (38). This completes the proof. □

Note that if $\alpha = 0$ in Equation (38), then the switching feedback control (34) becomes simultaneous control.

Remark 2. The switching control design shown earlier (with no restriction on feedback gains) was used in ref. [23] for stabilization of the origin of discrete-time switching systems; bifurcation control was not an objective in that work.

4.3 Numerical examples

We now present numerical examples that demonstrate the proposed feedback control methods.

Example 2. (Fixed point attractor bifurcating to instantaneous chaos)

Consider the three-dimensional PWS map

$$x(k+1) = \begin{cases} Ax(k) + b\mu, & x_1(k) \le 0 \\ Bx(k) + b\mu, & x_1(k) > 0 \end{cases} \tag{46}$$

where

$$A = \begin{pmatrix} 0.0334 & 1.7874 & -0.1705 \\ -0.4588 & -0.4430 & -0.8282 \\ 0.0474 & -0.0416 & 0.8000 \end{pmatrix},$$

$$B = \begin{pmatrix} 0.8384 & 1.7874 & -0.1705 \\ -0.8180 & -0.4430 & -0.8282 \\ 0.6602 & -0.0416 & 0.8000 \end{pmatrix} \text{ and } b = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

The eigenvalues of A and B are $\{0.766, -0.1878 \pm j\,0.8389\}$ and $\{-0.1157, 0.6555 \pm j\,1.0987\}$, respectively. Note that A is Schur stable, but B is unstable. Simulation results show that Equation (46) undergoes a border collision bifurcation from a fixed point attractor to instantaneous chaos at $\mu = 0$ (see Fig. 3).

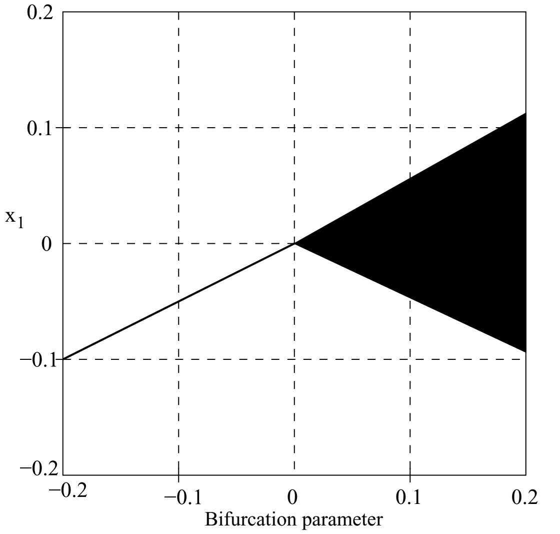

Fig. 3 Bifurcation diagram for Example 2. The *solid line* represents a path of stable fixed points and the *shaded region* represents a one-piece chaotic attractor growing out of the fixed point at $\mu = 0$

Feedback control design: Using the results of Proposition 4, a symmetric and positive definite matrix Q and a feedback control gain vector g that satisfy the LMIs (28)–(29) are sought. A solution to Equations (28)–(29) is obtained using the MATLAB LMI toolbox. To wit:

$$Q = \begin{pmatrix} 0.4753 & -0.0428 & -0.1694 \\ -0.0428 & 0.8821 & -0.1647 \\ -0.1694 & -0.1647 & 0.5041 \end{pmatrix},$$

$$\begin{aligned} y &= (-0.1601 \quad -1.4937 \quad 0.3356), \\ g &= yQ^{-1} \\ &= (-0.5193 \quad -1.7324 \quad -0.0747). \end{aligned}$$

The closed-loop matrices are given by

$$\begin{aligned} A_c &= A + bg \\ &= \begin{pmatrix} -0.4859 & 0.0550 & -0.2452 \\ -0.4588 & -0.4430 & -0.8282 \\ 0.0474 & -0.0416 & 0.8000 \end{pmatrix}, \\ B_c &= B + bg \\ &= \begin{pmatrix} 0.3191 & 0.0550 & -0.2452 \\ -0.8180 & -0.4430 & -0.8282 \\ 0.6602 & -0.0416 & 0.8000 \end{pmatrix}. \end{aligned}$$

The eigenvalues of A_c and B_c are $\{0.8141, -0.4715 \pm j\,0.1409\}$ and $\{-0.4507, 0.5634 \pm j\,0.3498\}$, respectively. The bifurcation diagram of the closed-loop system is depicted in Fig. 4.

Example 3. (Saddle-node border collision bifurcation) Consider the three-dimensional PWS map

$$x(k+1) = \begin{cases} Ax(k) + b\mu, & x_1(k) \le 0 \\ Bx(k) + b\mu, & x_1(k) > 0 \end{cases} \tag{47}$$

where

$$A = \begin{pmatrix} 0.0350 & -0.2280 & -0.9385 \\ -0.3123 & -0.0029 & 0.9191 \\ -0.3825 & -0.5107 & 0.5553 \end{pmatrix},$$

$$B = \begin{pmatrix} 3.3000 & -0.2280 & -0.9385 \\ -0.6299 & -0.0029 & 0.9191 \\ 0.3705 & -0.5107 & 0.5553 \end{pmatrix} \text{ and } b = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

The eigenvalues of A and B are $\{-0.2921, 0.4397 \pm j\,0.3470\}$ and $\{3.1739, 0.3392 \pm j\,0.4756\}$, respectively. Note that A is Schur stable, but B is unstable. Simulation results show that Equation (47) undergoes a saddle-node border collision bifurcation where a stable and an unstable fixed point collide and disappear as μ is increased through zero (see Fig. 5).

Feedback control design: A simultaneous stabilizing feedback control based on Proposition 4 does not exist for this example. Therefore, a stabilizing switched feedback control using Proposition 6 is sought. Using the LMI toolbox in MATLAB, a symmetric and positive definite matrix Q, and feedback control gain vectors g_1 and g_2 that satisfy the LMIs (37)–(38) are obtained:

$$\alpha = 3.0972$$

$$Q = \begin{pmatrix} 25.3606 & 4.5507 & 7.9810 \\ 4.5507 & 43.0961 & 9.8713 \\ 7.9810 & 9.8713 & 30.8840 \end{pmatrix},$$

$$\begin{aligned} y_1 &= (5.7709 \quad 14.8260 \quad 34.4887), \\ g_1 &= y_1 Q^{-1} \\ &= (-0.1436 \quad 0.1024 \quad 1.1211), \\ g_2 &= g_1 - \alpha (e^1)^{\mathrm{T}} \\ &= (-3.2408 \quad 0.1024 \quad 1.1211). \end{aligned}$$

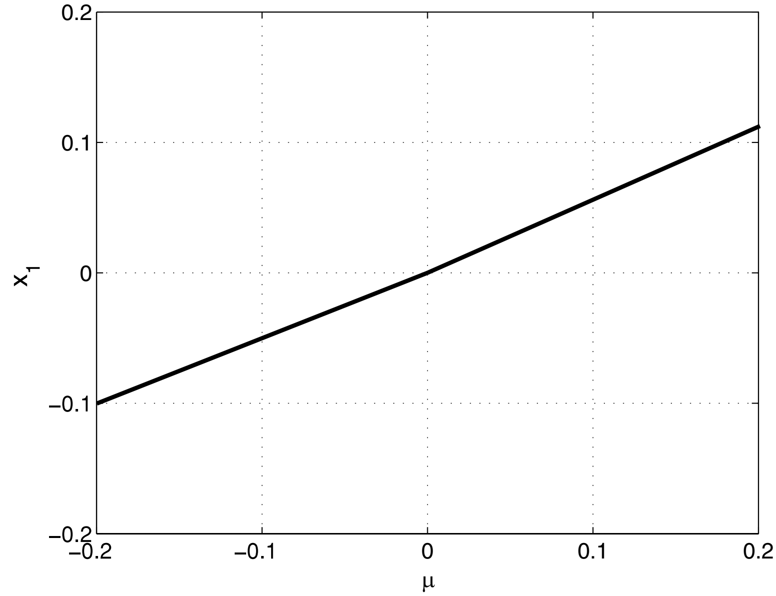

Fig. 4 Bifurcation diagram for Example 2 with simultaneous feedback control $u(k) = gx(k)$. The *solid lines* represent a path of stable fixed points

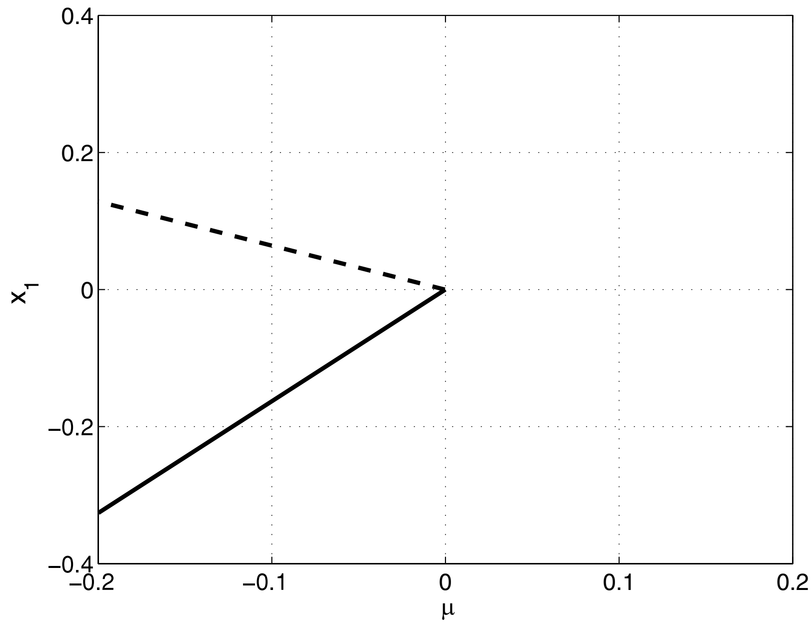

Fig. 5 Bifurcation diagram for Example 3 without control. The *solid line* represents a path of stable fixed points whereas the *dashed line* represents a path of unstable fixed points

The closed-loop matrices are given by

$$A_c = A + bg_1 = \begin{pmatrix} -0.1086 & -0.1256 & 0.1826 \\ -0.3123 & -0.0029 & 0.9191 \\ -0.3825 & -0.5107 & 0.5553 \end{pmatrix},$$

$$B_c = B + bg_2 = \begin{pmatrix} 0.0592 & -0.1256 & 0.1826 \\ -0.6299 & -0.0029 & 0.9191 \\ 0.3705 & -0.5107 & 0.5553 \end{pmatrix}.$$

The eigenvalues of A_c and B_c are $\{0.0011, 0.2213 \pm j\,0.6236\}$ and $\{-0.0002, 0.3059 \pm j\,0.5102\}$, respectively. The bifurcation diagram of the closed-loop system is similar to Fig. 4.

5 Concluding remarks

Lyapunov and LMI analysis of piecewise smooth discrete-time systems that undergo border collision bifurcations have been considered. One of the main contributions of this paper is the derivation of a sufficient condition for nonbifurcation with persistent stability in PWS maps of dimension n that depend on a parameter. This condition was used in the design of stabilizing feedback control laws to eliminate border collision bifurcations in PWS maps and produce desirable locally stable behavior as the border is crossed. Numerical examples were given to demonstrate the efficacy of the proposed control techniques.

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References

1. Alvarez-Ramirez, J.: Dynamics of controlled-linear discrete time systems with a dead-zone nonlinearity. In: Proceedings of the American Control Conference, Baltimore, MD, Vol. 1, pp. 761–765 (1994)
2. Banerjee, S., Grebogi, C.: Border collision bifurcations in two-dimensional piecewise smooth maps. Phys. Rev. E **59**(4), 4052–4061 (1999)
3. Banerjee, S., Karthik, M.S., Yuan, G.H., Yorke, J.A.: Bifurcations in one-dimensional piecewise smooth maps – theory and applications in switching systems. IEEE Trans. Circuits Syst. I **47**(3), 389–394 (2000)
4. Banerjee, S., Ranjan, P., Grebogi, C.: Bifurcations in two-dimensional piecewise smooth maps – theory and applications in switching circuits. IEEE Trans. Circuits Syst. I **47**(5), 633–643 (2000)
5. Boyd, S., Ghaoui, L.El., Feron, E., Balakrishnan, V.: Linear Matrix Inequalities in System and Control Theory, Studies in Applied Mathematics, Vol. 15. SIAM, Philadelphia, PA (1994)
6. Deane, J.H.B., Hamill, D.C.: Instability, subharmonics, and chaos in power electronic systems. IEEE Trans. Power Electron. **5**(3), 260–268 (1990)
7. Di Bernardo, M., Budd, C.J., Champneys, A.R.: Corner collision implies border-collision bifurcation. Physica D **154**, 171–194 (2001)
8. Di Bernardo, M., Budd, C.J., Champneys, A.R.: Grazing and border-collision in piecewise-smooth systems: A unified analytical framework. Phys. Rev. Lett. **86**(12), 2553–2556 (2001)
9. Di Bernardo, M., Feigin, M.I., Hogan, S.J., Homer, M.E.: Local analysis of C-bifurcations in n-dimensional piecewise smooth dynamical systems. Chaos Solitons Fractals **10**(11), 1881–1908 (1999)
10. Dutta, M., Nusse, H.E., Ott, E., Yorke, J.A., Yuan, G.H.: Multiple attractor bifurcations: A source of unpredictability in piecewise smooth systems. Phys. Rev. Lett. **83**(21), 4281–4284 (1999)
11. Feigin, M.I.: Doubling of the oscillation period with C-bifurcations in piecewise continuous systems. Prikl. Mat. Mekh. **34**, 861–869 (1970)
12. Feigin, M.I.: The increasingly complex structure of the bifurcation tree of a piecewise-smooth system. J. Appl. Math. Mech. **59**(6), 853–863 (1995)
13. Feng, G.: Stability analysis of piecewise discrete-time linear systems. IEEE Trans. Autom. Control **47**(7), 1108–1112 (2002)
14. Firoiu, V., Borden, M.: A study of active queue management for congestion control. In: Proceedings of the INFOCOM, Tel Aviv, Israel, Vol. 3, pp. 1435–1444 (2000)
15. Hassouneh, M.A.: Feedback Control of Border Collision Bifurcations in Piecewise Smooth Systems. PhD Thesis, University of Maryland, College Park, MD (2003)
16. Hassouneh, M.A., Abed, E.H., Nusse, H.E.: Robust dangerous border-collision bifurcations in piecewise smooth systems. Phys. Rev. Lett. **92**(7), 070201 (2004)
17. Hassouneh, M.A., Lee, H.-C., Abed, E.H.: Washout filters in feedback control: Benefits, limitations and extensions. In: Proceedings of the American Control Conference, Boston, MA, pp. 3950–3955, June/July (2004)
18. Hommes, C.H., Nusse, H.E.: Period 3 to period 2 bifurcation for piecewise linear-models. J. Econ. **54**(2), 157–169 (1991)
19. Johansson, M., Rantzer, A.: Computation of piecewise quadratic Lyapunov functions for hybrid systems. IEEE Trans. Autom. Control **43**, 555–559 (1998)
20. Kantner, M.: Robust stability of piecewise linear discrete time systems. In: Proceedings of the American Control Conference, Albuquerque, NM, pp. 1241–1245, June (1997)
21. Linsay, P.S.: Period doubling and chaotic behavior in a driven anharmonic-oscillator. Phys. Rev. Lett. **47**(19), 1349–1352 (1981)
22. Maistrenko, Y.L., Maistrenko, V.L., Vikul, S.I., Chua, L.O.: Bifurcations of attracting cycles from time-delayed Chua circuit. Int. J. Bifurcation Chaos **5**, 653–671 (1995)
23. Mignone, D., Ferrari-Trecate, G., Morari, M.: Stability and stabilization of piecewise affine and hybrid systems: An LMI approach. In: Proceedings of the Conference on Decision and Control, Sydney, Australia, Vol. 1, pp. 504–509, December (2000)
24. Nordmark, A.B.: Non-periodic motion caused by grazing incidence in impact oscillators. J. Sound Vib. **2**, 279–297 (1991)
25. Nusse, H.E., Ott, E., Yorke, J.A.: Border-collision bifurcation: An explanation for an observed phenomena. Phys. Rev. E **201**, 197–204 (1994)
26. Nusse, H.E., Yorke, J.A.: Border-collision bifurcations including "period two to period three" for piecewise smooth maps. Physica D **57**, 39–57 (1992)
27. Nusse, H.E., Yorke, J.A.: Border-collision bifurcations for piecewise smooth one-dimensional maps. Int. J. Bifurcation Chaos **5**, 189–207 (1995)
28. Ranjan, P., Abed, E.H., La, R.J.: Nonlinear instabilities in TCP-RED. In: Proceedings of the INFOCOM, New York, Vol. 1, pp. 249–258 (2002)

29. Sun, J., Amellal, F., Glass, L., Billette, J.: Alternans and period-doubling bifurcations in atrioventricular nodal conduction. J. Theor. Biol. **173**, 79–91 (1995)
30. Tanaka, H., Ushio, T.: Analysis of border-collision bifurcations in a flow model of a switching system. IEICE Trans. Fundam. Electron. Commun. Comput. Sci. **E85-A**(4), 734–739 (2002)
31. Yuan, G.H.: Shipboard Crane Control, Simulated Data Generation and Border Collision Bifurcations. PhD Thesis, University of Maryland, College Park, MD (1997)
32. Yuan, G.H., Banerjee, S., Ott, E., Yorke, J.A.: Border collision bifurcations in the buck converter. IEEE Trans. Circuits Syst. I **45**(7), 707–716 (1998)