Nonlinear Dyn (2007) 50:363–371
DOI 10.1007/s11071-006-9164-3

ORIGINAL ARTICLE

Delay-independent exponential stability of stochastic Cohen–Grossberg neural networks with time-varying delays and reaction–diffusion terms

Xiaolin Li · Jinde Cao

Received: 13 July 2006 / Accepted: 2 November 2006 / Published online: 5 January 2007

Abstract Different from the approaches used in the earlier papers, in this paper, the Halanay inequality technique, in combination with the Lyapunov method, is exploited to establish a delay-independent sufficient condition for the exponential stability of stochastic Cohen–Grossberg neural networks with time-varying delays and reaction–diffusion terms. Moreover, for the deterministic delayed Cohen–Grossberg neural networks, with or without reaction–diffusion terms, sufficient criteria for their global exponential stability are also obtained. The proposed results improve and extend those in the earlier literature and are easier to verify. An example is also given to illustrate the correctness of our results.

Keywords Exponential stability · Stochastic Cohen–Grossberg neural networks · Time-varying delays · Reaction–diffusion terms · Halanay inequality · Lyapunov method

1 Introduction

The Cohen–Grossberg neural network model, first proposed and studied by Cohen and Grossberg in 1983 [1], can be described by the following ordinary differential equations

$$\frac{du_i(t)}{dt} = -a_i(u_i)\left[b_i(u_i(t)) - \sum_{j=1}^{n} a_{ij} g_j(u_j(t)) + I_i\right],\quad i = 1, \ldots, n, \tag{1}$$

where $n \geq 2$ is the number of neurons in the network, u_i denotes the state variable associated with the ith neuron, a_i represents an amplification function, and b_i is an appropriately behaved function. The matrix $A = (a_{ij})$ represents the connection strengths between neurons, and if the output from neuron j excites (respectively, inhibits) neuron i, then $a_{ij} \geq 0$ (respectively, $a_{ij} \leq 0$). The activation function g_j shows how neurons respond to each other.

As we know, the Cohen–Grossberg neural network model includes many models from evolutionary theory, population biology, and neurobiology. Also, it includes the well-known Hopfield neural network, cellular neural network, and bidirectional associative memory (BAM) neural network as its special cases. In the past few years, the stability of these neural networks has been studied extensively. In Refs. [2, 3], several stability conditions for BAM neural networks have been derived. In Refs. [4–6], delayed cellular neural networks have been extensively studied and a set of global stability criteria have been proposed.

In Ref. [29], the global asymptotical stability is investigated for a generalized recurrent neural network with hybrid delays based on LMI approach. In

X. Li · J. Cao (✉)
Department of Mathematics, Southeast University, Nanjing 210096, P.R. China
e-mail: jdcao@seu.edu.cn; jdcaoseu@gmail.com

Ref. [30], the authors proposed a new Cohen-Grossberg-type BAM neural networks with delays, and gave several novel sufficient conditions ensuring the existence, uniqueness and global exponential stability of the equilibrium point in the form of M-matrix.

In practice, time delays inevitably exist in a working network due to the finite speeds of the switching and transmission of signals in a network and thus should be taken into account within the model equations of the network. In Refs. [7–13], various delayed Cohen–Grossberg neural networks have been studied and several sufficient conditions have been obtained to check the stabilities for the delayed Cohen–Grossberg neural networks. However, strictly speaking, *diffusion effects* cannot be avoided in the neural networks when electrons are moving in asymmetric electromagnetic fields, so we must consider the space is varying with time. In Refs. [14, 15], the authors have considered the stability of neural networks with diffusion terms, which are expressed by partial differential equations.

In recent years, the dynamic behavior of *stochastic neural networks*, especially the stability of stochastic neural networks, has become a hot study topic. The main reason is that in practice, a real system is usually affected by external perturbations which, in many cases, are of great uncertainty and hence may be treated as random. As pointed out by Haykin [29], in real nervous systems, synaptic transmission is a noisy process brought on by random fluctuations from the release of neurotransmitters and other probabilistic causes. Therefore, it is of significant importance to consider stochastic effects for the stability of neural networks. In Refs. [17, 18], the authors studied the mean square exponential stability and instability of cellular neural networks. In Ref. [19], the exponential stability of linear stochastic delay interval systems was studied, and some sufficient criteria with respect to matrix norm were given by Razumikin-type theorem. In Ref. [20], the almost sure exponential stability for a class of stochastic cellular neural networks with discrete delays was discussed. In Ref. [21], a sufficient condition was established for the stochastic delay perturbed system. In Ref. [22], the mean square exponential stability for stochastic delayed Hopfield neural networks with discrete and continuously distributed delay is studied by means of a variation parameter.

Motivated by the above discussion, the main objective of this paper is to consider *stochastic Cohen–Grossberg neural networks with time-varying delays and reaction–diffusion terms*. To the best of our knowledge, few authors have considered the stochastic Cohen–Grossberg neural networks with time-varying delays and reaction–diffusion terms so far. Different from the previous approach, by means of Halanay inequality technique [23], a delay-independent sufficient condition is proposed to ensure the exponential stability of the equilibrium of the stochastic Cohen–Grossberg neural networks with time-varying delays and reaction–diffusion terms. Moreover, sufficient criteria to guarantee the global exponential stability of deterministically delayed Cohen–Grossberg neural networks with or without reaction–diffusion terms are also obtained. The results proposed in this paper extend and improve some previous ones.

2 Model description and preliminaries

In this paper, we consider the stochastic Cohen–Grossberg neural networks with time-varying delays and reaction–diffusion terms described by the following differential equations

$$\begin{aligned} dy_i(t,x) &= \sum_{k=1}^{m} \frac{\partial}{\partial x_k}\left(D_{ik}\frac{\partial y_i}{\partial x_k}\right)dt \\ &\quad - a_i(y_i(t,x))\Bigg[b_i(y_i(t,x)) \\ &\quad - \sum_{j=1}^{n} c_{ij} f_j(y_j(t,x)) \\ &\quad - \sum_{j=1}^{n} d_{ij} f_j(y_j(t-\tau_{ij}(t),x)) + J_i\Bigg]dt \\ &\quad + \sum_{j=1}^{n} \rho_{ij}(y_i(t,x), y_i(t-\tau_{ij}(t),x))\, d\omega_j(t), \\ &\qquad x \in \Omega, \end{aligned} \tag{2}$$

for $i = 1, 2, \ldots, n$ and $t \geq 0$, where $y_i(t,x)$ is the state of the ith neurons at time t and in space x; $f_j(y_j(t,x))$ denotes the activation function of the jth neurons at time t and in space x; J_i denotes the external bias on the ith neuron; c_{ij} and d_{ij} denote the connection weights; $\tau_{ij}(t)$ denotes the transmission delay along the axon of the jth neuron from the ith neuron and satisfies $0 \leq \tau_{ij}(t) \leq \tau$; $D_{ik} = D_{ik}(t,x,y) \geq 0$ denotes the diffusion operator; Ω is a bounded compact set with smooth boundary $\partial\Omega$ and $mes\Omega > 0$ in space R^m; ρ: $R^n \times R^n \to R^{n\times n}$, i.e., $\rho = (\rho_{ij})_{n\times n}$ is the diffusion coefficient matrix; $\omega(t) = (\omega_1(t), \ldots, \omega_n(t))^{\mathrm{T}}$ is an n-dimensional Brownian motion defined on a complete probability space $(\Omega, \mathcal{F}, P)$ with a natural filtration [24] $\{\mathcal{F}_t\}_{t\geq 0}$ (i.e., $\mathcal{F}_t = \sigma\{\omega(s) : 0 \leq s \leq t\}$).

The boundary condition and the initial condition are

$$\begin{cases} \dfrac{\partial y_i}{\partial n} = \left(\dfrac{\partial y_i}{\partial x_1}, \dfrac{\partial y_i}{\partial x_2}, \ldots, \dfrac{\partial y_i}{\partial x_m}\right)^{\mathrm{T}} = 0, \\ \quad t \geq t_0 \geq 0, \quad x \in \partial\Omega, \\ y_i(t_0 + s, x) = \varphi_i(s, x), \quad -\tau_{ij}(t_0) \leq s \leq 0, \end{cases} \tag{3}$$

where $\varphi(s, x) = \{(\varphi_1(s, x), \ldots, \varphi_n(s, x))^{\mathrm{T}} : -\tau \leq s \leq 0\}$ is $C([-\tau, 0] \times R^m; R^n)$-valued function and $\mathcal{F}_0$-measurable R^n-valued random variable.

Let $\rho_{ij} = 0, i, j = 1, 2, \ldots, n$, then system (2) becomes the following deterministic Cohen–Grossberg neural networks with time-varying delays and reaction–diffusion terms

$$\begin{aligned} \frac{\partial y_i(t, x)}{\partial t} &= \sum_{k=1}^{m} \frac{\partial}{\partial x_k}\left(D_{ik}\frac{\partial y_i}{\partial x_k}\right) - a_i(y_i(t, x)) \\ &\quad \times \Bigg[b_i(y_i(t, x)) - \sum_{j=1}^{n} c_{ij} f_j(y_j(t, x)) \\ &\quad - \sum_{j=1}^{n} d_{ij} f_j(y_j(t - \tau_{ij}(t), x)) + J_i\Bigg], \\ &\qquad x \in \Omega, \end{aligned} \tag{4}$$

Furthermore, let $D_{ik} = 0\,(i = 1, \ldots, n, k = 1, \ldots, m)$, then system (4) is reduced to the following Cohen–Grossberg neural networks with time-varying delays

$$\begin{aligned} \frac{dy_i(t)}{dt} &= -a_i(y_i(t))\Bigg[b_i(y_i(t)) - \sum_{j=1}^{n} c_{ij} f_j(y_j(t)) \\ &\quad - \sum_{j=1}^{n} d_{ij} f_j(y_j(t - \tau_{ij}(t))) + J_i\Bigg]. \end{aligned} \tag{5}$$

In this paper, we make the following assumptions:

(H_1) For each $i \in \{1, 2, \ldots, n\}$, $a_i(y_i)$ is bounded, positive, and locally Lipschitz continuous. Furthermore, $0 < \underline{\alpha}_i \leq a_i(y_i) \leq \bar{\alpha}_i$.

(H_2) For each $i \in \{1, 2, \ldots, n\}$, $b_i(y_i)$ is locally Lipschiz continuous and there exists $\gamma_i > 0$ such that $y_i b_i(y_i(t)) \geq \gamma_i y_i^2(t)$.

(H_3) The activation function f_j is bounded and there exist constants $L_j > 0$, such that

$$|f_j(u_1) - f_j(u_2)| \leq L_j|u_1 - u_2|, \quad j = 1, 2, \ldots, n,$$

for any $u_1, u_2 \in R$.

(H_4) There are nonnegative constants v_i, μ_i such that

$$\mathrm{trace}[\rho^{\mathrm{T}}(x, y)\rho(x, y)] \leq \sum_{i=1}^{n} \left(v_i x_i^2 + \mu_i y_i^2\right).$$

It should be noted that assumption (H_3) guarantees the existence of an equilibrium point for system (4) by the well-known Brouwer fixed-point theorem. For the detailed proof of the existence of the equilibrium, refer to Proposition 3.1 in Ref. [8]. Let $y^* = (y_1^*, \ldots, y_n^*)^{\mathrm{T}}$ be an equilibrium point of system (4). For the stability of equilibrium of system (2), we furthermore assume that

(H_5) $\rho_{ij}(y_i^*, y_i^*) = 0,\ i, j = 1, 2, \ldots, n.$

Then, system (2) has an equilibrium point $y^* = (y_1^*, \ldots, y_n^*)^{\mathrm{T}}$. By means of a coordinate translation $z_i = y_i - y_i^*$, system (2) is equivalent to

$$\begin{aligned} dz_i(t) &= \sum_{k=1}^{m} \frac{\partial}{\partial x_k}\left(D_{ik}\frac{\partial z_i(t)}{\partial x_k}\right) dt \\ &\quad - \alpha_i(z_i(t, x))\Bigg[\beta_i(z_i(t, x)) \\ &\quad - \sum_{j=1}^{n} c_{ij} g_j(z_j(t, x)) \\ &\quad - \sum_{j=1}^{n} d_{ij} g_j(z_j(t - \tau_{ij}(t), x))\Bigg] dt \\ &\quad + \sum_{j=1}^{n} \rho_{ij}(y_i(t, x), y_i(t - \tau_{ij}(t), x))\, d\omega_j(t), \\ &\qquad x \in \Omega, \end{aligned} \tag{6}$$

where $\alpha_i(z_i(t)) = a_i(z_i(t) + y_i^*)$, $\beta_i(z_i(t)) = b_i(z_i(t) + y_i^*) - b_i(y_i^*)$, $g_j(z_j(t)) = f_j(z_j(t) + y_j^*) - f_j(y_j^*)$.

It follows from Ref. [24] that under the assumptions (H_1)–(H_4), Equation (2) has a global solution for $t \geq 0$, which is denoted as $y(t, \varphi)$.

Notation. In this paper, E stands for the mathematical expectation operator with respect to the given probability measure P. For any $u(t, x) = (u_1(t, x), u_2(t, x), \ldots, u_n(t, x))^{\mathrm{T}} \in R^n$, define

$$\|u_i(t, x)\|_2 = \left[\int_\Omega |u_i(t, x)|^2 dx\right]^{\frac{1}{2}},$$

$$\|u(t, x)\| = \left[\sum_{i=1}^{n} \|u_i(t, x)\|_2^2\right]^{\frac{1}{2}}.$$

To prove our main theorem, we need the following preliminaries.

Definition 1. Let $r : \mathbb{R} \to \mathbb{R}$ be a continuous function, $D^+ r$, the upper right Dini derivative of $r(t)$ is defined as

$$D^+ r(t) = \limsup_{\triangle t \to 0^+} \frac{r(t + \triangle t) - r(t)}{\triangle t}.$$

Definition 2 ([30]). System (2) is said to be exponentially stable in the mean square if there exists a pair of positive constants λ and K such that

$$E\|y(t, t_0, \varphi) - y^*\|^2 \leq K E\|\varphi - y^*\|^2 e^{-\lambda(t - t_0)}, \quad t \geq t_0 \tag{7}$$

for any φ. In this case

$$\limsup_{t \to \infty} \frac{1}{t} \ln(E\|y(t, t_0, \varphi) - y^*\|^2) \leq -\lambda. \tag{8}$$

The left-hand side of Equation (8) is called the mean square Lyapunov exponent of the solution.

In the following, we introduce the Halanay inequality for a stochastic system which plays an important role in the proof of the main theorem. The detailed proof of the Halanay inequality can be found in Ref. [23].

Consider an n-dimensional stochastic functional differential equation

$$dx(t) = f(t, x_t)\, dt + g(t, x_t)\, d\omega(t), \quad t \geq t_0, \qquad x_{t_0} = \xi. \tag{9}$$

Here $\xi \in L^2_{\mathcal{F}_0}([t_0 - \tau, t_0], R^n)$ and $x_t = \{x(t + \theta)$: $-\tau \leq \theta \leq 0\}$ which is regarded as a R^n-valued stochastic process. Both f and g are functions from $R_+ \times C([-\tau, 0]; R^n)$ to R^n, satisfying the local Lipschitz condition and linear growth condition, which guarantee a unique global solution of Equation (9) denoted by $x(t, \xi)$.

Lemma 1 ([23]). *Let constants $\alpha > \beta \geq 0$. Assume that there exists a positive, continuous function $V(t, x)$ satisfying the following inequality*

$$D^+ E(V(t, x(t))) \leq -\alpha E(V(t, x(t))) + \beta \sup_{s \in [t - \tau, t]} E(V(s, x(s))), \quad t \geq t_0,$$

then

$$E(V(t, x(t))) \leq \sup_{s \in [t_0 - \tau, t_0]} E(V(s, x(s))) \exp(-\lambda(t - t_0)),$$

in which $\lambda \in (0, \alpha - \beta]$ is the unique positive solution of the following equation:

$$\lambda = \alpha - \beta e^{\lambda \tau}. \tag{10}$$

3 Main results

In this section, we will employ the Halanay inequality technique to present a sufficient criterion for the exponential stability of stochastic Cohen–Grossberg neural networks with time-varying delays and reaction–diffusion terms defined by Equation (2).

Theorem 1. *Under assumptions (H_1)–(H_5), system (2) is exponentially stable in the mean square if the following condition holds*

$$\min_{1 \leq i \leq n} \left(2\underline{\alpha}_i \gamma_i - \sum_{j=1}^{n} |c_{ij}| \bar{\alpha}_i L_j - \sum_{j=1}^{n} |c_{ji}| \bar{\alpha}_j L_i - \sum_{j=1}^{n} |d_{ij}| \bar{\alpha}_i L_j - v_i\right) > \max_{1 \leq i \leq n} \left(\sum_{j=1}^{n} |d_{ji}| \bar{\alpha}_j L_i + \mu_i\right). \tag{11}$$

Proof: Consider the following Lyapunov function

$$V(t, z(t)) = \int_\Omega \sum_{i=1}^{n} z_i^2(t)\,dx = \|z(t)\|^2. \tag{12}$$

Applying Ito's formula [26] to $V(t, z(t))$, we obtain

$$\begin{aligned} &V(t+\delta, z(t+\delta)) - V(t, z(t)) = \int_t^{t+\delta} dV(s, z(s)) \\ &= \int_t^{t+\delta} V_t(s, z(s))\,ds \\ &\quad + \int_t^{t+\delta} V_z(s, z(s)) \Bigg\{ \sum_{k=1}^{m} \frac{\partial}{\partial x_k}\left(D_{ik}\frac{\partial z_i(s)}{\partial x_k}\right) \\ &\quad - \alpha_i(z_i(s,x))\Bigg[\beta_i(z_i(s,x)) - \sum_{j=1}^{n} c_{ij} g_j(z_j(s,x)) \\ &\quad - \sum_{j=1}^{n} d_{ij} g_j(z_j(s-\tau_{ij}(s),x))\Bigg]\Bigg\}\,ds \\ &\quad + \int_t^{t+\delta} V_z(s, z(s))\rho_{ij}(y_i(s,x), \\ &\quad y_i(s-\tau_{ij}(s),x))\,d\omega_j(s) \\ &\quad + \frac{1}{2}\int_t^{t+\delta} \mathrm{trace}[\rho^T V_{zz}(s, z(s)))\rho]\,ds \\ &= \int_t^{t+\delta} \int_\Omega 2\sum_{i=1}^{n} z_i(s) \Bigg\{ \sum_{k=1}^{m} \frac{\partial}{\partial x_k}\left(D_{ik}\frac{\partial z_i(s)}{\partial x_k}\right) \\ &\quad - \alpha_i(z_i(s,x))\Bigg[\beta_i(z_i(s,x)) - \sum_{j=1}^{n} c_{ij} g_j(z_j(s,x)) \\ &\quad - \sum_{j=1}^{n} d_{ij} g_j(z_j(s-\tau_{ij}(s),x))\Bigg] dx\Bigg\}\,ds \\ &\quad + \int_t^{t+\delta} \int_\Omega 2\sum_{i=1}^{n} z_i(s)\rho_{ij}(y_i(s,x), \\ &\quad y_i(s-\tau_{ij}(s),x))\,dx\,d\omega_j(s) \\ &\quad + \int_t^{t+\delta} \int_\Omega \mathrm{trace}(\rho^T \rho)\,dx\,ds. \end{aligned} \tag{13}$$

From the boundary condition, we get

$$\begin{aligned} &\sum_{k=1}^{m} \int_\Omega z_i \frac{\partial}{\partial x_k}\left(D_{ik}\frac{\partial z_i}{\partial x_k}\right) dx \\ &= \int_\Omega z_i \nabla \cdot \left(D_{ik}\frac{\partial z_i}{\partial x_k}\right)_{k=1}^{m} dx \\ &= \int_\Omega \nabla \cdot \left(z_i D_{ik}\frac{\partial z_i}{\partial x_k}\right)_{k=1}^{m} dx \\ &\quad - \int_\Omega \left(D_{ik}\frac{\partial z_i}{\partial x_k}\right)_{k=1}^{m} \cdot \nabla z_i\,dx \\ &= \int_{\partial\Omega} \left(z_i D_{ik}\frac{\partial z_i}{\partial x_k}\right)_{k=1}^{m} \cdot ds \\ &\quad - \sum_{k=1}^{m} \int_\Omega D_{ik}\left(\frac{\partial z_i}{\partial x_k}\right)^2 dx \\ &= -\sum_{k=1}^{m} \int_\Omega D_{ik}\left(\frac{\partial z_i}{\partial x_k}\right)^2 dx, \end{aligned} \tag{14}$$

in which $\nabla = (\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_m})^{\mathrm{T}}$ is the gradient operator, and

$$\left(D_{ik}\frac{\partial z_i}{\partial x_k}\right)_{k=1}^{m} = \left(D_{i1}\frac{\partial z_i}{\partial x_1}, \ldots, D_{im}\frac{\partial z_i}{\partial x_m}\right)^{\mathrm{T}}.$$

According to the assumptions (H_2), (H_3), and (H_4), we have

$$\begin{aligned} &|g_j(z_j(t))| \le L_j |z_j(t)|, \\ &z_i(t)|\beta_i(z_i(t))| \ge \gamma_i |z_i(t)|^2, \\ &\mathrm{trace}(\rho^T \rho) \le \sum_{i=1}^{n} (v_i|z_i(t)|^2 + \mu_i|z_i(t-\tau_{ij}(t))|^2). \end{aligned} \tag{15}$$

Substituting Equations (14) and (15) into Equation (13), we have

$$\begin{aligned} &V(t+\delta, z(t+\delta)) - V(t, z(t)) \\ &\le -2\int_t^{t+\delta} \int_\Omega \sum_{i=1}^{n} \underline{\alpha}_i \gamma_i z_i^2(s,x)\,dx\,ds \\ &\quad + 2\int_t^{t+\delta} \int_\Omega \sum_{i=1}^{n}\sum_{j=1}^{n} |c_{ij}|\bar{\alpha}_i L_j |z_i(s)||z_j(s)|\,dx\,ds \\ &\quad + 2\int_t^{t+\delta} \int_\Omega \sum_{i=1}^{n}\sum_{j=1}^{n} |d_{ij}|\bar{\alpha}_i L_j |z_i(s)|| \\ &\quad \times z_j(s-\tau_{ij}(s))|\,dx\,ds + \int_t^{t+\delta} \int_\Omega \Bigg(\sum_{i=1}^{n} v_i|z_i(s)|^2 \\ &\quad + \sum_{i=1}^{n} \mu_i |z_i(s-\tau_{ij}(s))|^2\Bigg)\,dx\,ds \end{aligned}$$

$$
\begin{aligned}
&+2\int_t^{t+\delta}\int_\Omega\sum_{i=1}^n z_i(s)\rho_{ij}(y_i(s,x),\\
&\quad\times y_i(s-\tau_{ij}(s),x))\,dx\,d\omega_j(s)\\
&\le\int_t^{t+\delta}\int_\Omega\sum_{i=1}^n\Big(-2\underline{\alpha}_i\gamma_i+\sum_{j=1}^n|c_{ij}|\bar{\alpha}_iL_j\\
&\quad+\sum_{j=1}^n|c_{ji}|\bar{\alpha}_jL_i+\sum_{j=1}^n|d_{ij}|\bar{\alpha}_iL_j+v_i\Big)\\
&\times|z_i(s)|^2dx\,ds\\
&\quad+t\int_t^{t+\delta}\int_\Omega\sum_{i=1}^n\Big(\sum_{j=1}^n|d_{ji}|\bar{\alpha}_jL_i+\mu_i\Big)\\
&\quad\times|z_j(s-\tau_{ij}(s))|^2dx\,ds\\
&\quad+2\int_t^{t+\delta}\int_\Omega\sum_{i=1}^n z_i(s)\rho_{ij}(y_i(s,x),\\
&y_i(s-\tau_{ij}(s),x))\,dx\,d\omega_j(s)\\
&\le-\int_t^{t+\delta}\min_{1\le i\le n}\Big(2\underline{\alpha}_i\gamma_i-\sum_{j=1}^n|c_{ij}|\bar{\alpha}_iL_j\\
&\quad-\sum_{j=1}^n|c_{ji}|\bar{\alpha}_jL_i-\sum_{j=1}^n|d_{ij}|\bar{\alpha}_iL_j-v_i\Big)\\
&\quad\times V(s,z(s))\,ds\\
&\quad+\int_t^{t+\delta}\max_{1\le i\le n}\Big(\sum_{j=1}^n|d_{ji}|\bar{\alpha}_jL_i+\mu_i\Big)\\
&\quad\times V(s-\tau_{ij}(s),z(s-\tau_{ij}(s)))\,ds\\
&\quad+2\int_t^{t+\delta}\int_\Omega\sum_{i=1}^n z_i(s)\rho_{ij}(y_i(s,x),\\
&y_i(s-\tau_{ij}(s),x))\,dx\,d\omega_j(s).
\end{aligned}\tag{16}
$$

By Theorem 4.2.8 [27], we obtain $E\int_t^{t+\delta}\int_\Omega\sum_{i=1}^n z_i(s)\rho_{ij}(y_i(s,x),y_i(s-\tau_{ij}(s),x))\,dx\;\;d\omega_j(s)=0$. Therefore, taking expectation on both sides of Equation (16), we get

$$
\begin{aligned}
&EV(t+\delta,z(t+\delta))-EV(t,z(t))\\
&\le-\int_t^{t+\delta}\min_{1\le i\le n}\Big(2\underline{\alpha}_i\gamma_i-\sum_{j=1}^n|c_{ij}|\bar{\alpha}_iL_j\\
&\quad-\sum_{j=1}^n|c_{ji}|\bar{\alpha}_jL_i-\sum_{j=1}^n|d_{ij}|\bar{\alpha}_iL_j-v_i\Big)\\
&\quad\times EV(s,z(s))\,ds\\
&\quad+\int_t^{t+\delta}\max_{1\le i\le n}\Big(\sum_{j=1}^n|d_{ji}|\bar{\alpha}_jL_i+\mu_i\Big)\\
&\quad\times EV(s-\tau_{ij}(s),z(s-\tau_{ij}(s)))\,ds.
\end{aligned}
$$

From the definition of the Dini derivative

$$
\begin{aligned}
&D^+EV(t,z(t))\\
&\quad=\limsup_{\delta\to0^+}\frac{EV(t+\delta,z(t+\delta))-EV(t,z(t))}{\delta},
\end{aligned}
$$

and also by the mean value theorem for integrals, we have

$$
\begin{aligned}
&D^+EV(t,z(t))\\
&\quad\le-\min_{1\le i\le n}\Big(2\underline{\alpha}_i\gamma_i-\sum_{j=1}^n|c_{ij}|\bar{\alpha}_iL_j-\sum_{j=1}^n|c_{ji}|\bar{\alpha}_jL_i\\
&\qquad-\sum_{j=1}^n|d_{ij}|\bar{\alpha}_iL_j-v_i\Big)EV(t,z(t))\\
&\qquad+\max_{1\le i\le n}\Big(\sum_{j=1}^n|d_{ji}|\bar{\alpha}_jL_i+\mu_i\Big)\\
&\qquad\times EV(t-\tau_{ij}(t),z(t-\tau_{ij}(t)))\\
&\quad\le-\min_{1\le i\le n}\Big(2\underline{\alpha}_i\gamma_i-\sum_{j=1}^n|c_{ij}|\bar{\alpha}_iL_j-\sum_{j=1}^n|c_{ji}|\bar{\alpha}_jL_i\\
&\qquad-\sum_{j=1}^n|d_{ij}|\bar{\alpha}_iL_j-v_i\Big)EV(t,z(t))\\
&\qquad+\max_{1\le i\le n}\Big(\sum_{j=1}^n|d_{ji}|\bar{\alpha}_jL_i+\mu_i\Big)\sup_{s\in[t-\tau,t]}EV(s,x(s)).
\end{aligned}
$$

By Lemma 1, we get

$$
EV(t,z(t))\le\sup_{s\in[t_0-\tau,t_0]}EV(s)\exp(-\lambda(t-t_0)),\quad t\ge t_0,
$$

i.e.,

$$
\begin{aligned}
&E\|y(t)-y^*\|^2\\
&\quad\le\sup_{s\in[-\tau,0]}E\|\varphi(s)-y^*\|^2\exp(-\lambda(t-t_0)),\ t\ge t_0.
\end{aligned}
$$

This shows that system (2) is globally exponentially stable in mean square. □

Remark 1. When $D_{ik} = 0, (i = 1, \ldots, n, k = i, \ldots, m)$, system (2) is reduced to the following stochastic Cohen–Grossberg neural networks:

$$\begin{aligned} dy_i(t,x) = &- a_i(y_i(t,x))\Big[b_i(y_i(t,x)) \\ &- \sum_{j=1}^{n} c_{ij} f_j(y_j(t,x)) \\ &- \sum_{j=1}^{n} d_{ij} f_j(y_j(t-\tau_{ij}(t),x)) + J_i\Big]dt \\ &+ \sum_{j=1}^{n} \rho_{ij}(y_i(t,x), \\ &\quad y_i(t-\tau_{ij}(t),x))\, d\omega_j(t), \\ &\qquad x \in \Omega, \quad i = 1, \ldots, n. \end{aligned} \tag{17}$$

This model has been studied in Ref. [28] and it is seen that Corollary 1 in Ref. [28] is a direct result of Theorem 1, which shows that the reaction–diffusion term has no influence on the stability for system (2).

In the following, we give two corollaries for the deterministic Cohen–Grossberg neural network with and without the reaction–diffusion term, respectively.

Corollary 1. *Under assumptions (H_1)–(H_3), system (4) is globally exponentially stable if the following condition holds:*

$$\begin{aligned} \min_{1\le i\le n}\Big(&2\underline{\alpha}_i\gamma_i - \sum_{j=1}^{n}|c_{ij}|\bar{\alpha}_i L_j - \sum_{j=1}^{n}|c_{ji}|\bar{\alpha}_j L_i \\ &- \sum_{j=1}^{n}|d_{ij}|\bar{\alpha}_i L_j\Big) > \max_{1\le i\le n}\Big(\sum_{j=1}^{n}|d_{ji}|\bar{\alpha}_j L_i\Big). \end{aligned} \tag{18}$$

Corollary 2. *Under assumptions (H_1)–(H_3), system (5) is globally exponentially stable if the following condition holds:*

$$\begin{aligned} \min_{1\le i\le n}\Big(&2\underline{\alpha}_i\gamma_i - \sum_{j=1}^{n}|c_{ij}|\bar{\alpha}_i L_j - \sum_{j=1}^{n}|c_{ji}|\bar{\alpha}_j L_i \\ &- \sum_{j=1}^{n}|d_{ij}|\bar{\alpha}_i L_j\Big) > \max_{1\le i\le n}\Big(\sum_{j=1}^{n}|d_{ji}|\bar{\alpha}_j L_i\Big). \end{aligned} \tag{19}$$

Remark 2. In Ref. [13], if we let $r = 2$, then Corollary 4 is the same result as Corollary 2 in our paper.

Remark 3. It is noted that in Ref. [22], the stochastic perturbation term is independent of time delays. However, in practice, the stochastic perturbation is unavoidably influenced by time delays. Therefore, our results are more general.

4 An example

An example is presented here in order to illustrate the correctness of our main result.

Consider the following stochastic Cohen–Grossberg neural networks with time-varying delays and reaction–diffusion terms

$$\begin{aligned} dy_1(t,x) = &\sum_{k=1}^{2}\frac{\partial}{\partial x_k}\Big(D_{1k}\frac{\partial y_1}{\partial x_k}\Big)dt \\ &- a_1(y_1(t,x))\Big[b_1(y_1(t,x)) \\ &- \sum_{j=1}^{2} c_{1j} f_j(y_j(t,x)) \\ &- \sum_{j=1}^{2} d_{1j} f_j(y_j(t-\tau_{1j}(t),x)) + J_1\Big]dt \\ &+ \sum_{j=1}^{2}\rho_{1j}(y_1(t,x), \\ &\quad y_1(t-\tau_{1j}(t),x))\, d\omega_j(t), \\ dy_2(t,x) = &\sum_{k=1}^{2}\frac{\partial}{\partial x_k}\Big(D_{2k}\frac{\partial y_2}{\partial x_k}\Big)dt \\ &- a_2(y_2(t,x))\Big[b_2(y_2(t,x)) \\ &- \sum_{j=1}^{2} c_{2j} f_j(y_j(t,x)) \\ &- \sum_{j=1}^{2} d_{2j} f_j(y_j(t-\tau_{2j}(t),x)) + J_2\Big]dt \\ &+ \sum_{j=1}^{2}\rho_{2j}(y_2(t,x), \\ &\quad y_2(t-\tau_{2j}(t),x))\, d\omega_j(t), \end{aligned}$$

where $a_i(y_i(t)) = 2 + \sin(y_i(t))$, $f_j(y_j) = 0.5(|y_j + 1| - |y_j - 1|)$, $\tau_{ij}(t) = |\cos t|$, $i, j = 1, 2$. $b_1(y_1(t)) = 5y_1(t)$, $b_2(y_2(t)) = 6y_2(t)$, $c_{11} = 0.2$, $c_{12} = c_{21} = 0.5$, $c_{22} = -0.6$, $d_{11} = -0.02$, $d_{12} = -0.05$, $d_{21} = 0.06$, $d_{22} = 0.8$.

Obviously, we have $\underline{\alpha}_i = 1, \bar{\alpha}_i = 3, L_j = 1, \gamma_1 = 5, \gamma_2 = 6, i, j = 1, 2$.

Moreover, $\sigma: R^2 \times R^2 \to R^{2\times 2}$ satisfies

$$\mathrm{trace}[\rho^{\mathrm{T}}(x, y)\rho(x, y)] \leq x_1^2 + 0.1x_2^2 + 0.5y_1^2 + 0.15y_2^2.$$

Therefore, we obtain

$$\min_{1\leq i\leq 2}\left(2\underline{\alpha}_i\gamma_i - \sum_{j=1}^{n}|c_{ij}|\bar{\alpha}_i L_j - \sum_{j=1}^{n}|c_{ji}|\bar{\alpha}_j L_i - \sum_{j=1}^{n}|d_{ij}|\bar{\alpha}_i L_j - v_i\right) = 2.72,$$

$$\max_{1\leq i\leq 2}\left(\sum_{j=1}^{n}|d_{ji}|\bar{\alpha}_j L_i + \mu_i\right) = 2.7.$$

From Theorem 1, we know the neural networks are exponentially stable in mean square.

5 Conclusions

This paper is concerned with the exponential stability of stochastic Cohen–Grossberg neural networks with time-varying delays and reaction–diffusion terms. Different from the approaches employed in the previous literatures, the Halanay inequality technique in combination with Lyapunov method is exploited to establish a sufficient condition for the exponential stability of these networks. The results obtained in this paper are delay-independent, which implies that strong self-regulation is dominant in the networks. In addition, the methods used in this paper are also applicable to other neural networks, such as stochastic Hopfield neural networks with time-varying delays and reaction–diffusion terms, stochastic bidirectional associative memory (BAM) neural networks with or without time-varying delays and reaction–diffusion terms, etc.

Acknowledgements This work was jointly supported by the National Natural Science Foundation of China under Grant 60574043 and the Natural Science Foundation of Jiangsu Province of China under Grant BK2006093.

References

1. Cohen, M.A., Grossberg, S.: Absolute stability and global pattern formation and parallel memory storage by competitive neural networks. IEEE Trans. Syst. Man Cybern. **13**, 815–821 (1983)
2. Gopalsamy, K., He, X.Z.: Delayed-independent stability in bidirectional associative memory networks. IEEE Trans. Neural Netw. **5**(6), 998–1002 (1994)
3. Cao, J., Wang, L.: Exponential stability and periodic oscillatory solution in BAM networks with delays. IEEE Trans. Neural Netw. **13**(2), 457–463 (2002)
4. Cao, J.: On exponential stability and periodic solutions of CNNs with delays. Phys. Lett. A **267**, 312–318 (2000)
5. Cao, J.: A set of stability criteria for delayed cellular neural networks. IEEE Trans. Circuits Syst. I **48**(4), 494–498 (2001)
6. Arik, S., Tavsanoglu, V.: On the global asymptotic stability of delayed cellular neural networks. IEEE Trans. Circuits Syst. I **47**, 571–574 (2000)
7. Ye, H., Michel, A.N., Wang, K.: Qualitative analysis of Cohen–Grossberg neural networks with multiple delays. Phys. Rev. E **51**, 2611–2618 (1995)
8. Wang, L., Zou, X.: Harmless delays in Cohen–Grossberg neural networks. Physica D **170**, 162–173 (2002)
9. Wang, L., Zou, X.: Exponential stability of Cohen–Grossberg neural networks. Neural Netw. **15**, 415–422 (2002)
10. Chen, T., Rong, L.: Delay-independent stability analysis of Cohen–Grossberg neural networks. Phys. Lett. A **317**(5–6), 436–449 (2003)
11. Yuan, K., Cao, J.: An analysis of global asymptotic stability of delayed Cohen–Grossberg neural networks via nonsmooth analysis. IEEE Trans. Circuits Syst. I **52**(9), 1854–1861 (2005)
12. Hwang, C., Cheng, C.J., Li, T.L.: Globally exponential stability of generalized Cohen–Grossberg neural networks with delays. Phys. Lett. A **319**, 157–166 (2003)
13. Cao, J., Liang, J.: Boundedness and stability for Cohen–Grossberg neural networks with time-varying delays. J. Math. Anal. Appl. **296**, 665–685
14. He, Q., Wang, L.: Existence and stability of global solution for generalized Hopfield neural network system. Neural Parallel Sci. Comput. **2**, 165–176 (1994)
15. Carpenter, G.A.: A geometric approach to singular perturbation problems with application to nerve impulse equations. J. Differ. Equ. **23**, 355–367 (1977)
16. Haykin, S.: Neural Networks. Prentice-Hall, Englewood Cliffs, NJ (1994)
17. Liao, X., Mao, X.: Exponential stability and instability of stochastic neural networks. Stoch. Anal. Appl. **14**, 165–168 (1996a)
18. Liao, X., Mao, X.: Stability of stochastic neural networks. Neural. Parallel Sci. Comput. **4**, 205–224 (1996b)
19. Liao, X., Mao, X.: Exponential stability of stochastic delay interval systems. Syst. Control Lett. **40**, 171–181 (2000)

20. Blythe, S., Mao, X., Liao, X.: Stability of stochastic delay neural networks. J. Franklin Inst. **338**, 481–495 (2001)
21. Mao, X., Colin, S.: Stability of stochastic interval systems with time delays. Syst. Control Lett. **42**, 279–290 (2001)
22. Wan, L., Sun, J.: Mean square exponential stability of stochastic delayed Hopfield neural networks. Phys. Lett. A **343**(4), 306–318 (2005)
23. Baker, C.T.H., Buckwar, E.: Exponential stability in p-th mean of solutions, and of convergent Euler-type solutions, of stochastic delay differential equations. J. Comput. Appl. Math. **184**(2), 404–427 (2005)
24. Arnold, L.: Stochastic Differential Equations: Theory and Applications. Wiley, New York (1974)
25. Mao, X.: Exponential Stability of Stochastic Differential Equations. Marcel Dekker, New York (1994)
26. Mao, X.: Stochastic Differential Equations and their Applications. Horwood, Chichester, UK (1997)
27. Friedman, A.: Stochastic Differential Equations and Applications, Vol. 1. Academic, New York (1975)
28. Li, X., Cao, J.: Exponential stability of stochastic Cohen–Grossberg neural networks with time-varying delays. In: Lecture Notes in Computer Science, Vol. 3496. Springer, Berlin, ISNN 2005, pp. 162–167 (2005)
29. Cao, J., Yuan, K., Li, H.: Global asymptotical stability of generalized recurrent neural networks with multiple discrete delays and distributed delays. IEEE Trans. Neural Networks **17**(6), 1646–1651 (2006)
30. Cao, J., Song, Q.: Stability in Cohen-Grossberg type BAM neural networks with time-varying delays. Nonlinearity **19**(7), 1601–1617 (2006)