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ORIGINAL ARTICLE

On transversal vibrations of an axially moving string with a time-varying velocity

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Abstract In this paper an initial-boundary value problem for a linear equation describing an axially moving string will be considered for which the bending stiffness will be neglected. The velocity of the string is assumed to be time-varying and to be of the same order of magnitude as the wave speed. A two time-scales perturbation method and the Laplace transform method will be used to construct formal asymptotic approximations of the solutions. It will be shown that the linear axially moving string model already has complicated dynamical behavior and that the truncation method can not be applied to this problem in order to obtain approximations which are valid on long time-scales.

Keywords Axially moving string · Asymptotics · Internal resonances · Oscillations · Two-timescales perturbation method

1 Formulation of the problem

In this paper the dynamic behavior of an axially moving string without bending stiffness will be studied (see Fig. 1).

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The following linear equation of motion for a moving string will be considered

$$c^2 u_{xx} = u_{tt} + 2V u_{xt} + V^2 u_{xx} + V_t u_x,\\ 0 < x < l, \quad t > 0, \tag{1}$$

where, $u(x, t)$: the displacement of the string in vertical direction, $V(t)$: the time-varying string speed, c: the wave speed, x: the coordinate in horizontal direction, t: the time, and, l: the distance between the pulleys, and where $c = \sqrt{\frac{T_0}{\rho}}$, in which T_0 and ρ are assumed to be the constant tension and the constant mass density of the string, respectively. In this paper the case $V_0 < c$ is considered and it is assumed that $V(t) = V_0 + \varepsilon\alpha \sin(\omega t)$, where V_0, ω and α are some positive constants, and where ε is a small parameter with $0 < \varepsilon \ll 1$. The term $\varepsilon\alpha \sin(\omega t)$ can be seen as a small perturbation of the main belt speed V_0 due to different kinds of imperfections of the belt system. At the pulleys it is assumed that there is no displacement of the string in vertical direction. Equation (1) can also be found in [1], but now it is assumed that V_0 is not necessarily small compared to the wave speed c. Consequently (1) becomes:

$$\begin{aligned} &u_{tt} + 2V_0 u_{xt} + \left(V_0^2 - c^2\right) u_{xx} \\ &\quad = \varepsilon(-2\alpha \sin(\omega t) u_{xt} - 2V_0\alpha \sin(\omega t) u_{xx} \\ &\qquad - \alpha\omega \cos(\omega t) u_x) - \varepsilon^2(\alpha^2 \sin^2(\omega t) u_{xx}), \end{aligned} \tag{2}$$

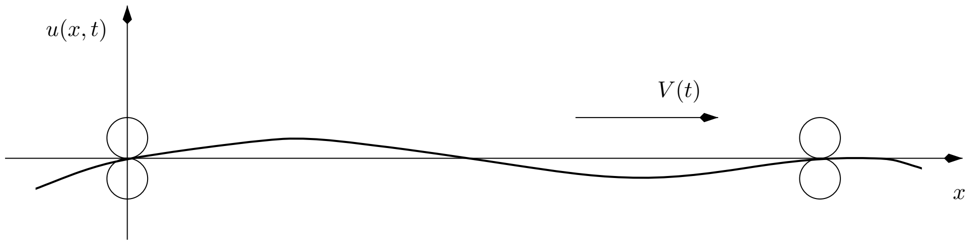

Fig. 1 An axially moving string system.

where the boundary and the initial conditions are given by

$$u(0,t;\varepsilon) = u(l,t;\varepsilon) = 0, \quad t \geqslant 0,$$
$$u(x,0;\varepsilon) = f(x), \quad \text{and} \quad u_t(x,0;\varepsilon) = r(x), \quad 0 < x < l, \tag{3}$$

where $f(x)$ and $r(x)$ represent the initial displacement and the initial velocity of the string, respectively. It is assumed that the functions $f(x)$ and $r(x)$ are sufficiently smooth such that a two times continuously differentiable solution for the initial-boundary value problem (2) and (3) exists. Moreover, it is assumed that the series representations which are used for the solution u (and its derivatives), and for the functions f and r are convergent. In the following section asymptotic approximations of the solution of the initial-boundary value problem (2) and (3) will be constructed using a two time-scales perturbation method. To study nonlinear, transversal vibrations of conveyor belt problems the solution of related linear problems always play an important role. In this paper not only approximations of these linear problems will be constructed, but also the (non-) applicability of the truncation method will be discussed. For a recent overview of the literature on axially moving linear and nonlinear strings the reader is referred to [2–5].

2 Application of the two time-scales perturbation method

Approximations of the solution of the initial-boundary value problem (2) and (3) which are constructed by means of a straight-forward expansion method become unbounded on long time-scales due to the occurrence of so-called secular terms. To avoid these secular terms two time-scales are introduced: $t_0 = t$ and $t_1 = \varepsilon t$, so that $u(x,t;\varepsilon) = v(x,t_0,t_1;\varepsilon)$. The introduction of these two time-scales defines the following transformations for the time derivatives:

$$\frac{\partial u}{\partial t} = \frac{\partial v}{\partial t_0} + \varepsilon \frac{\partial v}{\partial t_1},$$
$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 v}{\partial t_0^2} + 2\varepsilon \frac{\partial^2 v}{\partial t_0 \partial t_1} + \varepsilon^2 \frac{\partial^2 v}{\partial t_1^2}. \tag{4}$$

Considering the transformations (4), Equation (2) becomes:

$$\frac{\partial^2 v}{\partial t_0^2} + 2V_0 \frac{\partial^2 v}{\partial t_0 \partial x} + \left(V_0^2 - c^2\right) \frac{\partial^2 v}{\partial x^2} = \varepsilon \bigg(-2 \frac{\partial^2 v}{\partial t_0 \partial t_1} - 2V_0 \frac{\partial^2 v}{\partial t_1 \partial x} - 2\alpha \sin(\omega t) \frac{\partial^2 v}{\partial t_0 \partial x} - 2V_0 \alpha \sin(\omega t) \frac{\partial^2 v}{\partial x^2} - \alpha\omega \cos(\omega t) \frac{\partial v}{\partial x} \bigg) + \mathcal{O}(\varepsilon^2). \tag{5}$$

Assuming that $v(x,t_0,t_1;\varepsilon) = v_0(x,t_0,t_1) + \varepsilon v_1(x,t_0,t_1) + \cdots$, the following problems have to be solved in order to remove secular terms up to $\mathcal{O}(\varepsilon)$:

$$\mathcal{O}(1): \frac{\partial^2 v_0}{\partial t_0^2} + 2V_0 \frac{\partial^2 v_0}{\partial t_0 \partial x} + \left(V_0^2 - c^2\right) \frac{\partial^2 v_0}{\partial x^2} = 0,$$
$$\mathcal{O}(\varepsilon): \frac{\partial^2 v_1}{\partial t_0^2} + 2V_0 \frac{\partial^2 v_1}{\partial t_0 \partial x} + \left(V_0^2 - c^2\right) \frac{\partial^2 v_1}{\partial x^2} = -2 \frac{\partial^2 v_0}{\partial t_0 \partial t_1} - 2V_0 \frac{\partial^2 v_0}{\partial t_1 \partial x} - 2\alpha \sin(\omega t) \frac{\partial^2 v_0}{\partial t_0 \partial x} - 2V_0 \alpha \sin(\omega t) \frac{\partial^2 v_0}{\partial x^2} - \alpha\omega \cos(\omega t) \frac{\partial v_0}{\partial x}. \tag{6}$$

The solution of the $\mathcal{O}(1)$-problem can be found by means of the Laplace transform method. For a more detailed analysis of that problem the readers are referred to [6, 7]. The solution is given by:

$$v_0(x, t_0, t_1) = \sum_{n=1}^{\infty} \{ F_{[1]n}(x) (A_{n0}(t_1) \cos(\Omega_n t_0) - B_{n0}(t_1) \sin(\Omega_n t_0)) + F_{[2]n}(x) (A_{n0}(t_1) \sin(\Omega_n t_0) + B_{n0}(t_1) \cos(\Omega_n t_0)) \}, \quad (7)$$

where

$$F_{[1]n}(x) = \cos\left(\frac{\pi n(V_0 + c)x}{lc}\right) - \cos\left(\frac{\pi n(V_0 - c)x}{lc}\right),$$
$$F_{[2]n}(x) = \sin\left(\frac{\pi n(V_0 + c)x}{lc}\right) - \sin\left(\frac{\pi n(V_0 - c)x}{lc}\right), \quad (8)$$

and where $\Omega_n = \frac{n\pi(V_0^2 - c^2)}{lc}$ with $n \in \mathbb{Z}^+$ are the natural frequencies of the conveyor belt system. In Equation (7) $A_{n0}(t_1)$ and $B_{n0}(t_1)$ are still arbitrary functions that can be used in order to avoid secular terms in the solution of the $\mathcal{O}(\varepsilon)$-problem.

By substituting (7) into $\mathcal{O}(\varepsilon)$-problem (see (6)) it follows that:

$$\mathcal{O}(\varepsilon): \frac{\partial^2 v_1}{\partial t_0^2} + 2V_0 \frac{\partial^2 v_1}{\partial t_0 \partial x} + \left(V_0^2 - c^2\right) \frac{\partial^2 v_1}{\partial x^2}$$
$$= \sum_{n=1}^{\infty} \{\sin(\Omega_n t_0)\varphi_n(x, t_1) + \cos(\Omega_n t_0)\tilde{\varphi}_n(x, t_1)\}$$
$$+ \sum_{n=1}^{\infty} \{\sin(\Omega_n t_0) \sin(\omega t_0)\psi_n(x, t_1) + \sin(\Omega_n t_0) \cos(\omega t)\tilde{\psi}_n(x, t_1) + \cos(\Omega_n t_0) \sin(\omega t_0)\theta_n(x, t_1) + \cos(\Omega_n t_0) \cos(\omega t)\tilde{\theta}_n(x, t_1)\}, \quad (9)$$

where

$$\varphi_n(x, t_1) = 2 \left(\frac{dA_{n0}(t_1)}{dt_1} \left(F_{[1]n}(x)\Omega_n - V_0 \frac{dF_{[2]n}(x)}{dx} \right) + \frac{dB_{n0}(t_1)}{dt_1} \left(F_{[2]n}(x)\Omega_n + V_0 \frac{dF_{[1]n}(x)}{dx} \right) \right),$$

$$\tilde{\varphi}_n(x, t_1) = 2 \left(\frac{dA_{n0}(t_1)}{dt_1} \left(-F_{[2]n}(x)\Omega_n - V_0 \frac{dF_{[1]n}(x)}{dx} \right) + \frac{dB_{n0}(t_1)}{dt_1} \left(F_{[1]n}(x)\Omega_n - V_0 \frac{dF_{[2]n}(x)}{dx} \right) \right),$$

$$\psi_n(x, t_1) = 2\alpha \left(A_{n0}(t_1) \left(\frac{dF_{[1]n}(x)}{dx}\Omega_n - V_0 \frac{d^2F_{[2]n}(x)}{dx^2} \right) + B_{n0}(t_1) \left(\frac{dF_{[2]n}(x)}{dx}\Omega_n + V_0 \frac{d^2F_{[1]n}(x)}{dx^2} \right) \right),$$

$$\tilde{\psi}_n(x, t_1) = \alpha\omega \left(-A_{n0}(t_1) \frac{dF_{[2]n}(x)}{dx} + B_{n0}(t_1) \frac{dF_{[1]n}(x)}{dx} \right),$$

$$\theta_n(x, t_1) = 2\alpha \left(A_{n0}(t_1) \left(-\frac{dF_{[2]n}(x)}{dx}\Omega_n - V_0 \frac{d^2F_{[1]n}(x)}{dx^2} \right) + B_{n0}(t_1) \left(\frac{dF_{[1]n}(x)}{dx}\Omega_n - V_0 \frac{d^2F_{[2]n}(x)}{dx^2} \right) \right),$$

$$\tilde{\theta}_n(x, t_1) = \alpha\omega \left(-A_{n0}(t_1) \frac{dF_{[1]n}(x)}{dx} - B_{n0}(t_1) \frac{dF_{[2]n}(x)}{dx} \right). \quad (10)$$

Following the method which has been presented in [6] for the equations of the type (9), it follows that actually two cases have to be considered to eliminate the secular terms in the solution of Equation (9): (i) ω is not in a neighborhood of any Ω_n (the non-resonant case), and (ii) $\omega = \Omega_{m^*}$, where $m^* \in \mathbb{Z}^+$ and fixed (the resonant case).

2.1 Case (i): ω is not in a neighborhood of any Ω_n, the non-resonant case

In this case it is assumed that the frequency of the velocity-fluctuations of the axially moving string is not equal to any of its natural frequencies. In this case only terms in the first sum in the right hand side of (9) will lead to unbounded behavior in the solution of the $\mathcal{O}(\varepsilon)$-problem. After applying the Laplace transform method to (9), calculating the poles, and then the residues, and then after applying the convolution integral theorem to find the inverse Laplace transform, one obtains:

$$\begin{aligned} v_1(x,t_0,t_1) = \frac{1}{2}\sum_{n=1}^{\infty}\big\{ & t_0\sin(\Omega_n t_0)(f_{[1]n}(x,t_1) \\ & + \tilde{f}_{[2]n}(x,t_1)) \\ & + t_0\cos(\Omega_n t_0)(-f_{[2]n}(x,t_1) \\ & + \tilde{f}_{[1]n}(x,t_1))\big\} \\ & + \text{“terms with non-secular behavior”}, \end{aligned} \tag{11}$$

where

$$\begin{aligned} f_{[1]n}(x,t_1) &= w_n(t_1)F_{[1]n}(x) + p_n(t_1)F_{[2]n}(x), \\ f_{[2]n}(x,t_1) &= w_n(t_1)F_{[2]n}(x) - p_n(t_1)F_{[1]n}(x), \\ \tilde{f}_{[1]n}(x,t_1) &= \tilde{w}_n(t_1)F_{[1]n}(x) + \tilde{p}_n(t_1)F_{[2]n}(x), \\ \tilde{f}_{[2]n}(x,t_1) &= \tilde{w}_n(t_1)F_{[2]n}(x) - \tilde{p}_n(t_1)F_{[1]n}(x), \end{aligned} \tag{12}$$

and

$$\begin{aligned} w_n(t_1) &= \frac{1}{2c}\int_0^l \frac{\varphi_n(x,t_1)}{\pi n}(-F_{[2]n}(x))\,dx, \\ p_n(t_1) &= -\frac{1}{2c}\int_0^l \frac{\varphi_n(x,t_1)}{\pi n}(-F_{[1]n}(x))\,dx, \\ \tilde{w}_n(t_1) &= \frac{1}{2c}\int_0^l \frac{\tilde{\varphi}_n(x,t_1)}{\pi n}(-F_{[2]n}(x))\,dx, \\ \tilde{p}_n(t_1) &= -\frac{1}{2c}\int_0^l \frac{\tilde{\varphi}_n(x,t_1)}{\pi n}(-F_{[1]n}(x))\,dx. \end{aligned} \tag{13}$$

In (12) and (13) $F_{[1]n}(x)$ and $F_{[2]n}(x)$ are given by (9). To get rid of the secular terms in the solution (11) it is necessary to put $(f_{[1]n}(x,t_1) + \tilde{f}_{[2]n}(x,t_1))$ and $(-f_{[2]n}(x,t_1) + \tilde{f}_{[1]n}(x,t_1))$ equal to zero, or equivalently:

$$\begin{cases} \left(-\dfrac{dA_{n0}}{dt_1}\cos\left(\dfrac{\pi n x V_0}{lc}\right) - \dfrac{dB_{n0}}{dt_1}\sin\left(\dfrac{\pi n x V_0}{lc}\right)\right)\sin\left(\dfrac{\pi n x}{l}\right) = 0, \\ \left(\dfrac{dA_{n0}}{dt_1}\sin\left(\dfrac{\pi n x V_0}{lc}\right) - \dfrac{dB_{n0}}{dt_1}\cos\left(\dfrac{\pi n x V_0}{lc}\right)\right)\sin\left(\dfrac{\pi n x}{l}\right) = 0. \end{cases} \tag{14}$$

System (14) can be seen as a system for the two unknowns $\frac{dA_{n0}}{dt_1}$ and $\frac{dB_{n0}}{dt_1}$. The determinant of this system is non-zero for all $x \in (0,l)$. So the only solution is the trivial one that is, $\frac{dA_{n0}}{dt_1} = 0$ and $\frac{dB_{n0}}{dt_1} = 0$. It then follows that $A_{n0}(t_1)$ and $B_{n0}(t_1)$ are constants. A similar result has been obtained in [8] for a non-resonant case.

2.2 Case (ii): $\omega = \frac{\pi(V_0^2-c^2)}{lc}$, the first resonant case

In this section it is assumed that ω is equal to the first natural frequency of the traveling string, that is, $\omega = \frac{\pi(V_0^2-c^2)}{lc}$. It this case terms in both sums in the right hand side of (9) will lead to unbounded behavior in the solution of the $\mathcal{O}(\varepsilon)$-problem. After introducing $\omega = \frac{\pi(V_0^2-c^2)}{lc}$ into (9), Equation (9) becomes:

$$\begin{aligned} \mathcal{O}(\varepsilon)\colon\ & \frac{\partial^2 v_1}{\partial t_0^2} + 2V_0\frac{\partial^2 v_1}{\partial t_0\partial x} + \left(V_0^2 - c^2\right)\frac{\partial^2 v_1}{\partial x^2} \\ &= \sum_{n=1}^{\infty}\left\{\sin(\Omega_n t_0)\varphi_n(x,t_1) + \cos(\Omega_n t_0)\tilde{\varphi}_n(x,t_1)\right\} \\ &\quad + \frac{1}{2}\sum_{n=1}^{\infty}\big\{\cos(\Omega_{n-1}t_0)(\psi_n(x,t_1) + \tilde{\theta}_n(x,t_1)) \\ &\quad + \cos(\Omega_{n+1}t_0)(-\psi_n(x,t_1) + \tilde{\theta}_n(x,t_1)) \\ &\quad + \sin(\Omega_{n-1}t_0)(\tilde{\psi}_n(x,t_1) - \theta_n(x,t_1)) \\ &\quad + \sin(\Omega_{n+1}t_0)(\tilde{\psi}_n(x,t_1) + \theta_n(x,t_1))\big\}, \end{aligned} \tag{15}$$

where the functions $\varphi_n(x,t_1)$, $\tilde{\varphi}_n(x,t_1)$, $\psi_n(x,t_1)$, $\tilde{\psi}_n(x,t_1)$, $\theta_n(x,t_1)$, $\tilde{\theta}_n(x,t_1)$ are given by (10). Following the same procedure as in the non-resonant case one finally obtains for v_1:

$$\begin{aligned} v_1(x,t_0,t_1) &= \Big\{\frac{1}{2}\big(f_{[1]1}(x,t_1)+\tilde{f}_{[2]1}(x,t_1)\big) \\ &\quad +\frac{1}{4}\big(f^{[1]}_{[2]1}(x,t_1) \\ &\quad +f^{[3]}_{[1]1}(x,t_1)\big)\Big\} t_0\sin(\Omega_1 t_0) \\ &\quad +\Big\{\frac{1}{2}\big(\tilde{f}_{[1]1}(x,t_1)-f_{[2]1}(x,t_1)\big) \\ &\quad +\frac{1}{4}\big(f^{[1]}_{[1]1}(x,t_1)-f^{[3]}_{[2]1}(x,t_1)\big)\Big\} t_0\cos(\Omega_1 t_0) \\ &\quad +\sum_{n=2}^{\infty}\Big[\Big\{\frac{1}{2}\big(f_{[1]n}(x,t_1)+\tilde{f}_{[2]n}(x,t_1)\big) \\ &\quad +\frac{1}{4}\big(f^{[1]}_{[2]n}(x,t_1)+f^{[3]}_{[1]n}(x,t_1)+f^{[2]}_{[2]n}(x,t_1) \\ &\quad +f^{[4]}_{[1]n}(x,t_1)\big)\Big\} t_0\sin(\Omega_n t_0) \\ &\quad +\Big\{\frac{1}{2}\big(\tilde{f}_{[1]n}(x,t_1)-f_{[2]n}(x,t_1)\big) \\ &\quad +\frac{1}{4}\big(f^{[1]}_{[1]n}(x,t_1)-f^{[3]}_{[2]n}(x,t_1)+f^{[2]}_{[1]n}(x,t_1) \\ &\quad -f^{[4]}_{[2]n}(x,t_1)\big)\Big\} t_0\cos(\Omega_n t_0)\Big] \\ &\quad +\text{“terms with non-secular behavior”}, \end{aligned} \tag{16}$$

where the functions $f_{[1]n}(x,t_1)$, $f_{[2]n}(x,t_1)$, $\tilde{f}_{[1]n}(x,t_1)$ and $\tilde{f}_{[2]n}(x,t_1)$ are given by (12), and the functions $f^{[k]}_{[1]n}(x,t_1)$ and $f^{[k]}_{[2]n}(x,t_1)$ are given by the following formulas:

$$\begin{aligned} f^{[k]}_{[1]n}(x,t_1) &= w^{[k]}_n(t_1)F_{[1]n}(x)+p^{[k]}_n(t_1)F_{[2]n}(x), \\ f^{[k]}_{[2]n}(x,t_1) &= w^{[k]}_n(t_1)F_{[2]n}(x)-p^{[k]}_n(t_1)F_{[1]n}(x), \end{aligned} \tag{17}$$

with the index $k=1,2,3,4$, respectively, where $F_{[1]n}(x)$ and $F_{[2]n}(x)$ are given by (9) and where $w^{[k]}_n(t_1)$, $p^{[k]}_n(t_1)$ are given by:

$$\begin{aligned} w^{[1]}_n(t_1) &= \frac{1}{2c}\int_0^l \frac{\psi_{n+1}(x,t_1)+\tilde{\theta}_{n+1}(x,t_1)}{\pi n}\big(-F_{[2]n}(x)\big)\,dx, \\ p^{[1]}_n(t_1) &= -\frac{1}{2c}\int_0^l \frac{\psi_{n+1}(x,t_1)+\tilde{\theta}_{n+1}(x,t_1)}{\pi n}\big(-F_{[1]n}(x)\big)\,dx, \\ w^{[2]}_n(t_1) &= \frac{1}{2c}\int_0^l \frac{-\psi_{n-1}(x,t_1)+\tilde{\theta}_{n-1}(x,t_1)}{\pi n}\big(-F_{[2]n}(x)\big)\,dx, \\ p^{[2]}_n(t_1) &= -\frac{1}{2c}\int_0^l \frac{-\psi_{n-1}(x,t_1)+\tilde{\theta}_{n-1}(x,t_1)}{\pi n}\big(-F_{[1]n}(x)\big)\,dx, \\ w^{[3]}_n(t_1) &= \frac{1}{2c}\int_0^l \frac{\tilde{\psi}_{n+1}(x,t_1)-\theta_{n+1}(x,t_1)}{\pi n}\big(-F_{[2]n}(x)\big)\,dx, \\ p^{[3]}_n(t_1) &= -\frac{1}{2c}\int_0^l \frac{\tilde{\psi}_{n+1}(x,t_1)-\theta_{n+1}(x,t_1)}{\pi n}\big(-F_{[1]n}(x)\big)\,dx, \\ w^{[4]}_n(t_1) &= \frac{1}{2c}\int_0^l \frac{\tilde{\psi}_{n-1}(x,t_1)+\theta_{n-1}(x,t_1)}{\pi n}\big(-F_{[2]n}(x)\big)\,dx, \\ p^{[4]}_n(t_1) &= -\frac{1}{2c}\int_0^l \frac{\tilde{\psi}_{n-1}(x,t_1)+\theta_{n-1}(x,t_1)}{\pi n}\big(-F_{[1]n}(x)\big)\,dx, \end{aligned} \tag{18}$$

where $\psi_n(x,t_1)$, $\tilde{\psi}_n(x,t_1)$, $\tilde{\theta}_n(x,t_1)$, $\theta_n(x,t_1)$ are given by (10). It follows from (16) that the solution of the $\mathcal{O}(\varepsilon)$-problem does not contain secular terms if and only if:

$$\begin{cases} f_{[1]1}(x,t_1)+\tilde{f}_{[2]1}(x,t_1) \\ \quad +\frac{1}{2}\big(f^{[1]}_{[2]1}(x,t_1)+f^{[3]}_{[1]1}(x,t_1)\big)=0, \\ \tilde{f}_{[1]1}(x,t_1)-f_{[2]1}(x,t_1) \\ \quad +\frac{1}{2}\big(f^{[1]}_{[1]1}(x,t_1)-f^{[3]}_{[2]1}(x,t_1)\big)=0, \end{cases}$$

and for $n \geqslant 2$:

$$\begin{cases} f_{[1]n}(x,t_1) + \tilde{f}_{[2]n}(x,t_1) + \frac{1}{2}\big(f^{[1]}_{[2]n}(x,t_1) \quad + f^{[3]}_{[1]n}(x,t_1) \\ + f^{[2]}_{[2]n}(x,t_1) + f^{[4]}_{[1]n}(x,t_1)\big) = 0, \\ \tilde{f}_{[1]n}(x,t_1) - f_{[2]n}(x,t_1) + \frac{1}{2}\big(f^{[1]}_{[1]n}(x,t_1) - f^{[3]}_{[2]n}(x,t_1) \\ \quad + f^{[2]}_{[1]n}(x,t_1) - f^{[4]}_{[2]n}(x,t_1)\big) = 0. \end{cases} \tag{19}$$

Defining $A_{00}(t_1) \equiv 0$ and $B_{00}(t_1) \equiv 0$ it follows that for all $n = 1, 2, \ldots$:

$$\begin{cases} F_{[1]n}(x)\sigma_{[1]n}(t_1) + F_{[2]n}(x)\sigma_{[2]n}(t_1) = 0, \\ F_{[1]n}(x)\sigma_{[2]n}(t_1) - F_{[2]n}(x)\sigma_{[1]n}(t_1) = 0, \end{cases} \tag{20}$$

where:

$$\sigma_{[1]n}(t_1) = w_n(t_1) - \tilde{p}_n(t_1) + \frac{1}{2}\left(w^{[3]}_n(t_1) - p^{[1]}_n(t_1) \right.$$
$$\left. + w^{[4]}_n(t_1) - p^{[2]}_n(t_1)\right),$$
$$\sigma_{[2]n}(t_1) = \tilde{w}_n(t_1) + p_n(t_1) + \frac{1}{2}\left(w^{[1]}_n(t_1) + p^{[3]}_n(t_1)\right.$$
$$\left. + w^{[2]}_n(t_1) + p^{[4]}_n(t_1)\right). \tag{21}$$

System (20) can be seen as a system for two unknown $\sigma_{[1]n}(t_1)$ and $\sigma_{[2]n}(t_1)$. For $x \in (0, l)$ the determinant of this system is equal to $(F_{[1]n})^2 + (F_{[2]n})^2 \neq 0$. From this it follows that $\sigma_{[1]n}(t_1) = 0$ and $\sigma_{[2]n}(t_1) = 0$, or equivalently

$$\begin{cases} w_n(t_1) - \tilde{p}_n(t_1) + \frac{1}{2}\left(w^{[3]}_n(t_1) - p^{[1]}_n(t_1) + w^{[4]}_n(t_1) \right. \\ \quad \left. - p^{[2]}_n(t_1)\right) = 0, \\ \tilde{w}_n(t_1) + p_n(t_1) + \frac{1}{2}\left(w^{[1]}_n(t_1) + p^{[3]}_n(t_1) + w^{[2]}_n(t_1) \right. \\ \quad \left. + p^{[4]}_n(t_1)\right) = 0. \end{cases} \tag{22}$$

System (22) involves the functions $\frac{dA_{n0}}{dt_1}$, $\frac{dB_{n0}}{dt_1}$, $A_{n0}(t_1)$ and $B_{n0}(t_1)$. Solving this system for $\frac{dA_{n0}}{dt_1}$, $\frac{dB_{n0}}{dt_1}$ finally yields for $n = 1, 2, 3, \ldots$:

$$\begin{cases} \frac{dA_{n0}}{dt_1} = \frac{\alpha \sin\left(\frac{\pi V_0}{c}\right)}{2l}(n+1)A_{(n+1)0} \\ \quad - \frac{\left(\cos\left(\frac{\pi V_0}{c}\right)+1\right)\alpha}{2l}(n+1)B_{(n+1)0} \\ \quad - \frac{\alpha \sin\left(\frac{\pi V_0}{c}\right)}{2l}(n-1)A_{(n-1)0} \\ \quad - \frac{\left(\cos\left(\frac{\pi V_0}{c}\right)+1\right)\alpha}{2l}(n-1)B_{(n-1)0}, \\ \frac{dB_{n0}}{dt_1} = \frac{\left(\cos\left(\frac{\pi V_0}{c}\right)+1\right)\alpha}{2l}(n+1)A_{(n+1)0} \\ \quad + \frac{\alpha \sin\left(\frac{\pi V_0}{c}\right)}{2l}(n+1)B_{(n+1)0} \\ \quad + \frac{\left(\cos\left(\frac{\pi V_0}{c}\right)+1\right)\alpha}{2l}(n-1)A_{(n-1)0} \\ \quad - \frac{\alpha \sin\left(\frac{\pi V_0}{c}\right)}{2l}(n-1)B_{(n-1)0}. \end{cases} \tag{23}$$

This system is an infinite dimensional system of ordinary differential equations. It can clearly be seen that for $\omega = \Omega_1$ all vibration modes are interacting, and it will be difficult to solve the system analytically. It can also be seen that in the limit case $V_0 = 0$ system (23) coincides with the system as studied in [1]. System (23) can be rewritten in the following way:

$$\begin{cases} \frac{dA_{n0}}{d\bar{t}_1} = \gamma(n+1)A_{(n+1)0} - \sigma(n+1)B_{(n+1)0} \\ \quad - \gamma(n-1)A_{(n-1)0} - \sigma(n-1)B_{(n-1)0}, \\ \frac{dB_{n0}}{d\bar{t}_1} = \sigma(n+1)A_{(n+1)0} + \gamma(n+1)B_{(n+1)0} \\ \quad + \sigma(n-1)A_{(n-1)0} - \gamma(n-1)B_{(n-1)0}, \end{cases} \tag{24}$$

where $\bar{t}_1 = \frac{\alpha}{2l}t_1$, $\gamma = \sin(\frac{\pi V_0}{c})$ and $\sigma = 1 + \cos(\frac{\pi V_0}{c})$. If the truncation method is applied to system (24), so only some first modes are taken into account and higher order modes are being neglected, the following system has to be solved:

$$\dot{X} = AX, \tag{25}$$

where vector $\dot{X}$ represents the derivatives of A_n and B_n with respect to $\bar{t}_1$ and the demension of the square matrix A is two times the number of modes which are considered. Table 1 represents the eigenvalues of the truncated system (24) up to 10 modes, which have been calculated by using the computer software package Maple. From this table it can be seen that the eigenvalues of the truncated system are always purely imaginary or zero. It is well known in mathematics that in this case no conclusions can be drawn for the infinite dimensional system.

2.2.1 *Analysis of the infinite dimensional system (24)*

By introducing $X_{n0}(\bar{t}_1) = nA_{n0}(\bar{t}_1)$ and $Y_{n0}(\bar{t}_1) = nB_{n0}(\bar{t}_1)$ system (24) becomes:

$$\begin{cases} \dfrac{dX_{n0}}{d\bar{t}_1} = n\big(\gamma X_{(n+1)0} - \sigma Y_{(n+1)0} \\ \qquad - \gamma X_{(n-1)0} - \sigma Y_{(n-1)0}\big), \\ \dfrac{dY_{n0}}{d\bar{t}_1} = n\big(\sigma X_{(n+1)0} + \gamma Y_{(n+1)0} \\ \qquad + \sigma X_{(n-1)0} - \gamma Y_{(n-1)0}\big), \end{cases} \tag{26}$$

for $n = 1, 2, \ldots,$ and $X_{00} = Y_{00} = 0$. Then it can be deduced that:

$$\begin{cases} X_{n0}\dot{X}_{n0} = n\big(\gamma X_{(n+1)0}X_{n0} - \sigma Y_{(n+1)0}X_{n0} \\ \qquad -\gamma X_{(n-1)0}X_{n0} - \sigma Y_{(n-1)0}X_{n0}\big), \\ Y_{n0}\dot{Y}_{n0} = n\big(\sigma X_{(n+1)0}Y_{n0} + \gamma Y_{(n+1)0}Y_{n0} \\ \qquad +\sigma X_{(n-1)0}Y_{n0} - \gamma Y_{(n-1)0}Y_{n0}\big). \end{cases} \tag{27}$$

By adding both equations in (27) and by taking the sum from $n = 1$ to ∞ it follows that:

$$\begin{aligned} &\frac{1}{2}\sum_{n=1}^{\infty}\frac{d}{d\bar{t}_1}\big(X_{n0}^2 + Y_{n0}^2\big) \\ &\quad = \sigma\sum_{n=1}^{\infty}\big\{Y_{(n+1)0}X_{n0} - X_{(n+1)0}Y_{n0}\big\} \\ &\qquad -\gamma\sum_{n=1}^{\infty}\big\{X_{(n+1)0}X_{n0} + Y_{(n+1)0}Y_{n0}\big\}. \end{aligned} \tag{28}$$

By differentiating (28) with respect to $\bar{t}_1$ one gets:

$$\frac{1}{2}\sum_{n=1}^{\infty}\frac{d^2}{d\bar{t}_1^2}\big(X_{n0}^2 + Y_{n0}^2\big) = 2(\sigma^2 + \gamma^2)\sum_{n=1}^{\infty}\big(X_{n0}^2 + Y_{n0}^2\big), \tag{29}$$

and then by putting $\sum_{n=1}^{\infty}(X_{n0}^2 + Y_{n0}^2) = W(\bar{t}_1)$ it follows that:

$$\frac{d^2W(\bar{t}_1)}{d\bar{t}_1^2} - 4(\sigma^2 + \gamma^2)W(\bar{t}_1) = 0. \tag{30}$$

The solution of (30) is $W(\bar{t}_1) = K_1 e^{2\sqrt{(\sigma^2+\gamma^2)}\bar{t}_1} + K_2 e^{-2\sqrt{(\sigma^2+\gamma^2)}\bar{t}_1}$, or equivalently:

$$\begin{aligned} &W(t_1) \\ &\quad = K_1 \exp\left(\frac{\alpha}{l}\sqrt{\left(\cos\left(\frac{\pi V_0}{c}\right)+1\right)^2 + \sin^2\left(\frac{\pi V_0}{c}\right)}\,t_1\right) \\ &\qquad + K_2 \exp\left(-\frac{\alpha}{l}\sqrt{\left(\cos\left(\frac{\pi V_0}{c}\right)+1\right)^2 + \sin^2\left(\frac{\pi V_0}{c}\right)}\,t_1\right), \end{aligned} \tag{31}$$

where K_1 and K_2 are both constants of integration. Now it should be observed that for $K_1 \neq 0$ $W(t_1)$ increases if t_1 increases. As it has been shown before (Section 2.2) the application of the truncation method to system (24) (that is, by considering only a finite number of vibration modes) that only purely imaginary eigenvalues or zero eigenvalues will be found. This implies that only oscillatory behavior will be found by applying the truncation method, whereas (31) clearly indicates that also exponential behavior should be included. Because of that the approximations obtained by the truncation method are not accurate on time-scales of order ε^{-1}. For example, the approximation of the solution that was obtained in [8] is not valid on long time-scales of order ε^{-1}.

2.3 Case (ii): $\omega = \dfrac{\pi m^*(V_0^2 - c^2)}{lc}$, a general resonant case

By considering the cases (i) and (ii), and by taking into account the results that have been obtained in [6] it follows, that if an external frequency is equal to a natural frequency of the moving string or equal to the sum or difference of those natural frequencies it will cause resonance in the conveyor belt system. Now it will be assumed that $\omega = \Omega_{m^*}$, that is, $\omega = \frac{\pi m^*(V_0^2-c^2)}{lc}$, where $m^* \in \mathbb{Z}^+$ and fixed. In this section the system of ordinary differential equations will be derived, which describes the interactions between the different oscillation modes.

Table 1 Approximations of the eigenvalues of the truncated system (24)

No. of modes	Eigenvalues of matrix A (all multiplicity 2)	Dimensi on eigen-space of A
1	0	2
2	$\pm\sqrt{2(\gamma^2+\sigma^2)}i$	4
3	$0, \pm 2\sqrt{2(\gamma^2+\sigma^2)}i$	6
4	$\pm 1.13\sqrt{(\gamma^2+\sigma^2)}i, \pm 4.33\sqrt{(\gamma^2+\sigma^2)}i$	8
5	$0,\pm 2.30\sqrt{(\gamma^2+\sigma^2)}i, \pm 5.89\sqrt{(\gamma^2+\sigma^2)}i$	10
6	$\pm 7.50\sqrt{(\gamma^2+\sigma^2)}i, \pm 1.00\sqrt{(\gamma^2+\sigma^2)}i, \pm 3.56\sqrt{(\gamma^2+\sigma^2)}i$	12
7	$0, \pm 9.15\sqrt{(\gamma^2+\sigma^2)}i,\pm 2.05\sqrt{(\gamma^2+\sigma^2)}i,\pm 4.90\sqrt{(\gamma^2+\sigma^2)}i$	14
8	$\pm 10.83\sqrt{(\gamma^2+\sigma^2)}i, \pm 0.93\sqrt{(\gamma^2+\sigma^2)}i, \pm 3.18\sqrt{(\gamma^2+\sigma^2)}i, \pm 6.30\sqrt{(\gamma^2+\sigma^2)}i$	16
9	$0, \pm 12.54\sqrt{(\gamma^2+\sigma^2)}i, \pm 1.89\sqrt{(\gamma^2+\sigma^2)}i, \pm 4.38\sqrt{(\gamma^2+\sigma^2)}i, \pm 7.74\sqrt{(\gamma^2+\sigma^2)}i$	18
10	$\pm 14.26\sqrt{(\gamma^2+\sigma^2)}i, \pm 0.87\sqrt{(\gamma^2+\sigma^2)}i, \pm 5.65\sqrt{(\gamma^2+\sigma^2)}i, \pm 9.23\sqrt{(\gamma^2+\sigma^2)}i$ $\pm 2.93\sqrt{(\gamma^2+\sigma^2)}i$	20

Substituting $\omega = \frac{\pi m^*(V_0^2-c^2)}{lc}$ into (9) yields:

$$\begin{aligned}
\mathcal{O}(\varepsilon): &\frac{\partial^2 v_1}{\partial t_0^2} + 2V_0\frac{\partial^2 v_1}{\partial t_0 \partial x} + \left(V_0^2 - c^2\right)\frac{\partial^2 v_1}{\partial x^2} \\
&= \sum_{n=1}^{\infty}\{\sin(\Omega_n t_0)\varphi_n(x,t_1) + \cos(\Omega_n t_0)\tilde{\varphi}_n(x,t_1)\} \\
&\quad + \frac{1}{2}\sum_{n=1}^{\infty}\Big\{\cos(\Omega_{n-m^*}t_0)(\psi_n(x,t_1) + \tilde{\theta}_n(x,t_1)) \\
&\quad + \cos(\Omega_{n+m^*}t_0)(-\psi_n(x,t_1) + \tilde{\theta}_n(x,t_1)) \\
&\quad + \sin(\Omega_{n-m^*}t_0)(\tilde{\psi}_n(x,t_1) - \theta_n(x,t_1)) \\
&\quad + \sin(\Omega_{n+m^*}t_0)(\tilde{\psi}_n(x,t_1) + \theta_n(x,t_1))\Big\},
\end{aligned} \quad (32)$$

where the functions $\varphi_n(x,t_1)$, $\tilde{\varphi}_n(x,t_1)$, $\psi_n(x,t_1)$, $\tilde{\psi}_n(x,t_1)$, $\theta_n(x,t_1)$, $\tilde{\theta}_n(x,t_1)$ are given again by (10). In (32) it should be observed that $\Omega_{-n} = -\Omega_n$ and $\Omega_0 = 0$.

Following the same procedure as in the previous cases to avoid secular terms in the solution of the $\mathcal{O}(\varepsilon)$-problem, one obtains after some lengthy, but elementary calculations the following system for $A_{n0}(t_1)$ and $B_{n0}(t_1)$:

$$\begin{cases}
\dfrac{dA_{n0}}{dt_1} = \dfrac{\alpha\mu_{m^*}}{m^*l}(n+m^*)A_{(n+m^*)0} + \dfrac{\alpha\eta_{m^*}}{m^*l}(n+m^*)B_{(n+m^*)0}, \\
\dfrac{dB_{n0}}{dt_1} = -\dfrac{\alpha\eta_{m^*}}{m^*l}(n+m^*)A_{(n+m^*)0} + \dfrac{\alpha\mu_{m^*}}{m^*l}(n+m^*)B_{(n+m^*)0},
\end{cases} \quad (33)$$

for $n = 1, 2, \ldots, m^* - 1$, and

$$\begin{cases}
\dfrac{dA_{n0}}{dt_1} = \dfrac{\alpha\mu_{m^*}}{2m^*l}(n+m^*)A_{(n+m^*)0} + \dfrac{\alpha\eta_{m^*}}{2m^*l}(n+m^*)B_{(n+m^*)0} - \dfrac{\alpha\mu_{m^*}}{2m^*l}(n-m^*)A_{(n-m^*)0} + \dfrac{\alpha\eta_{m^*}}{2m^*l}(n-m^*)B_{(n-m^*)0}, \\
\dfrac{dB_{n0}}{dt_1} = -\dfrac{\alpha\eta_{m^*}}{2m^*l}(n+m^*)A_{(n+m^*)0} + \dfrac{\alpha\mu_{m^*}}{2m^*l}(n+m^*)B_{(n+m^*)0} - \dfrac{\alpha\eta_{m^*}}{2m^*l}(n-m^*)A_{(n-m^*)0} - \dfrac{\alpha\mu_{m^*}}{2m^*l}(n-m^*)B_{(n-m^*)0},
\end{cases} \quad (34)$$

for $n = m^*, m^* + 1, \ldots$, where

$$\mu_{m^*} = (-1)^{1+m^*}\sin\left(\frac{\pi m^* V_0}{c}\right),$$
$$\eta_{m^*} = (-1)^{m^*}\left(\cos\left(\frac{\pi m^* V_0}{c}\right) - 1\right). \quad (35)$$

In system (33) and (34) A_{00} and B_{00} are defined to be identically equal to zero. The system (33) and (34) is an infinite dimensional system of coupled, ordinary differential equations. From the structure of the equations it can easily be seen that there are infinitely many interactions between the vibration modes. So, to apply the truncation method to system (33) and (34) (to find approximations of the solution which are valid on long time-scales of order ε^{-1}) can be a wrong procedure as has been shown for the case $m^* = 1$. How to obtain more information out of system (33) and (34) for arbitrary m^* is still an open subject for future research.

3 Conclusions and remarks

In this paper an initial-boundary value problem for a linear equation, describing an axially moving string has been studied. This equation can be used as a model for the lower frequency, transversal vibrations of a conveyor belt system. The axially moving string is assumed to move in one direction with a non-constant speed $V(t)$, that is, $V(t) = V_0 + \varepsilon\alpha \sin(\omega t)$, where $0 < \varepsilon \ll 1$ and where V_0, α and ω are positive constants. For V_0 it is assumed that $V_0 < c$, where c is the wave speed. Formal asymptotic approximations of the solution of the initial-boundary value problem have been constructed by using a combination of a two time-scales perturbation method and a Laplace transform method (see also [6] and [7]). It turns out that there are infinitely many values of ω that give rise to internal resonances in the axially moving string system. In fact, that happens when ω is equal to any natural frequency of the moving string, that is, $\omega = \Omega_n = \frac{\pi n(V_0^2 - c^2)}{lc}$, where $n = 1, 2, \ldots$. It turned out for $\omega = \Omega_1$ that only a one-mode approximation, as for instance has been used in [8] is not accurate on time-scales of order ε^{-1}, as the solution of the boundary-value problem (2) and (3) consists of infinitely many, interacting vibration modes. Moreover in [8] due to the application of the truncation method, the odd numbered resonance frequencies Ω_n were not found.

Three cases have been studied in this paper: ω is not in a neighborhood of any Ω_n, $\omega = \Omega_1$ and $\omega = \Omega_{m^*}$, where $m^* \in \mathbb{Z}^+$ and fixed. For the second case (that is when ω is equal to the lowest natural frequency of the moving string) a first integral has been found and it has been shown that the truncation method does not give accurate results on long time-scales. All approximations which are obtained by the method as introduced in this paper are valid on long time-scales, that is, on time-scales of order ε^{-1}. Moreover, the results as obtained in [1] are a special case of the results as obtained in this paper.

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