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Global stability in switched recurrent neural networks with time-varying delay via nonlinear measure

Ping Li · Jinde Cao

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Abstract In this paper, based on switched systems and recurrent neural networks (RNNs) with time-varying delay, the model of switched RNNs is formulated. Global asymptotical stability (GAS) and global robust stability (GRS) for such switched neural networks are studied by employing nonlinear measure and linear matrix inequality (LMI) techniques. Some new sufficient conditions are obtained to ensure GAS or GRS of the unique equilibrium of the proposed switched system. Furthermore, the proposed LMI results are computationally efficient as it can be solved numerically with standard commercial software. Finally, three examples are provided to illustrate the usefulness of the results.

Keywords Switched system · Recurrent neural networks · Time-varying delay · Global asymptotic stability · Nonlinear measure

1 Introduction

In recent years, there has been increasing interest in the dynamic analysis of artificial neural networks. Among the most popular models in the previous literatures are the Hopfield neural networks (HNNs) proposed by Hopfield [8]. HNNs have proved to be essential in solving some classes of optimization problems and found fascinating applications in pattern recognition and associative memories. This model can be described by

$$\frac{\mathrm{d}u_i(t)}{\mathrm{d}t} = -c_i u_i(t) + \sum_{j=1}^{n} a_{ij} f_j(u_j(t)) + I_i, \quad i = 1, 2, \ldots, n,$$

or equivalently, in the following vector-matrix form:

$$\frac{\mathrm{d}u(t)}{\mathrm{d}t} = -Cu(t) + Af(u(t)) + I \tag{1}$$

where $u = [u_1, u_2, \ldots, u_n]^{\mathrm{T}}$ is the neuron state vector, $C = \mathrm{diag}(c_1, c_2, \ldots, c_n)$ is a positive diagonal matrix, $A = (a_{ij})_{n\times n}$ is the connection matrix representing the weight coefficients of the neurons, $f(u(t)) = [f_1(u_1(t)),\ f_2(u_2(t)), \ldots, f_n(u_n(t))]^{\mathrm{T}}$ is the activation functions vector and $I = [I_1, I_2, \ldots, I_n]^{\mathrm{T}}$ is the external input vector. Also, in the hardware implementation of a neural network, a time delay usually occurs due to finite switching speed of the amplifiers and communication time. In [16], the authors introduced the time delay τ into (1) and considered the model of neural system as follows:

$$\frac{\mathrm{d}u(t)}{\mathrm{d}t} = -Cu(t) + Af(u(t-\tau)) + I. \tag{2}$$

In [10], the author considered the recurrent neural networks (RNNs), which are the hybrid network model

P. Li · J. Cao (✉)
Department of Mathematics, Southeast University, Nanjing 210096, China
e-mail: jdcao@seu.edu.cn

of HNNs and cellular neural networks (CNNs). The model investigated in [10] is in the following form:

$$\frac{\mathrm{d}u(t)}{\mathrm{d}t} = -Cu(t) + Af(u(t)) + Bf(u(t-\tau(t))) + I. \quad (3)$$

The neural network models (1)–(3) have attracted numerous attention due to their promising application in the various engineering problems, for example, classification of patterns, solving optimization problems, designing associative memory. It has been observed that such applications greatly rely on the dynamical analysis of the neural network. Recently, a large number of results on this issue have been reported, see, e.g. [1, 3, 7, 13, 25, 28] and the reference therein. In [29], the authors investigated the global asymptotical stability of generalized recurrent neural networks with multiple discrete delays and distributed delays; a new Cohen-Grossberg type BAM neural networks with time-varying delays was proposed in [30], and several novel sufficient conditions ensuring the existence, uniqueness and global exponential stability of the equilibrium point are derived in the form of M-matrix. It should be also noted that many previous results require the activation functions to be monotonically nondecreasing [1, 3, 7, 13, 25]. However, Morita pointed out that for network's associative memory, its absolute capacity can be remarkably improved by replacing the usual sigmoid transfer functions with nonmonotonic transfer functions [17]. What's more, when a neural network is applied to optimization problems, the neural network model should be designed such that there is only one equilibrium point and it should be globally stable [22].

As is well known, when designing neural network, it is central to investigate stability problem of neural network where various types of stability have captured the attention of the researchers. However, parametric uncertainty which often breaks the stability of a neural network can be commonly encountered due to the modeling inaccuracies and changes in the environment of the model. For example, in the practical application of neural networks, some vital data such as the neuron firing rates and the weight coefficients are usually acquired and processed by means of the statistical estimates. Thus, the robust stability analysis of different uncertain neural networks has gained much research attention [4, 14, 24, 27].

On the other hand, with the rapid development of the hybrid control, hybrid systems have been studied extensively for their significance both in theory and application. As an important class of the hybrid systems, switched systems, composed of two or more continuous (or discrete) subsystems and controlled by a switching law, are viewed as nonlinear systems. Recently, the investigation of switched systems is creating a novel and promising discipline bridging control engineering, mathematics and computer science. In general, a continuous switched system can be characterized by the following differential equation:

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = f_{\sigma}(x(t)). \quad (4)$$

Let $\mathcal{F} = \{f_p(x),\ p \in \mathcal{P}\}$, the parameter p takes the value in the index set $\mathcal{P}$, each map $f_p(x) : \mathbb{R}^n \to \mathbb{R}^n$ of $\mathcal{F}$ is assumed to be locally Lipschitz and the piecewise constant function of time $\sigma(\cdot) : [0, +\infty) \to \mathcal{P}$ is the switching signal. To switched systems, stability analysis is an important research field. In [15], Libezon and Morse summarize three basic problems regarding stability and design of the switched systems as follows:

(1) find conditions to ensure the GAS of the switched systems for arbitrary switching sequences;
(2) identify the GAS condition for certain useful classes of switching sequences;
(3) construct a switching signal that makes the system globally asymptotically stable.

Until now, a lot of efforts have been made for the stability analysis and design of the switched systems, and some results have been reported, see [11, 19, 21, 26] and the reference therein. In [5, 6, 23], a set of linear systems are used as individual subsystems of switched system. In [9], a set of delayed HNNs are introduced into switched system as its individual subsystems. In this paper, combined switched system with recurrent neural networks, the model of switched RNNs is proposed, which is more general than [9]. In addition, to the best of our knowledge, GAS and GRS of such switched RNNs are seldom considered. Hence, motivated by the above discussions, GAS and GRS are studied for switched RNNs in this paper by resorting to the Lyapunov stability theorem and nonlinear measure technique. Some new sufficient conditions given in the form of LMIs are obtained to ensure the GAS or GRS of the proposed switched model.

This paper is organized as follows. In Section 2, the model description and some preliminaries are introduced. In Section 3, the main results are presented. In Section 4, illustrative examples are given to show the effectiveness of the proposed results. Finally, concluding remarks are made in Section 5.

Notation: Throughout this paper, for real symmetric matrices P and Q, the notation $P \geq Q$ (respectively, $P > Q$) means that the matrix $P - Q$ is positive semidefinite (respectively, positive definite). The notation $\|u\|$ denotes a vector form defined by $\|u\|_2 = (\sum_{i=1}^{n} u_i^2)^{1/2}$ when u is a vector. For matrix X, X^{T} and X^{-1} denote its transpose and inverse, respectively. Matrix dimensions, if not explicitly stated, are assumed to be compatible for algebraic operations.

2 Model description and preliminaries

For convenience, we shift the equilibrium point $u^* = [u_1^*, u_2^*, \ldots, u_n^*]^{\mathrm{T}}$ of system (3) to the origin. The transformation $x(t) = u(t) - u^*$ makes system (3) into the following form:

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = -Cx(t) + Ag(x(t)) + Bg(x(t-\tau(t))) \quad (5)$$

where $x = [x_1, x_2, \ldots, x_n]^{\mathrm{T}}$ and $g_j(x_j) = f_j(x_j + u_j^*) - f_j(u_j^*)$ with $g_j(0) = 0,\ \forall\ j = 1, 2, \ldots, n$.

We considered the switched system composed of the RNNs in the form of (5):

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = \sum_{i=1}^{N} \xi_i(t)\big[-C_i x(t) + A_i g(x(t)) + B_i g(x(t-\tau(t)))\big] \quad (6)$$

where i is a switching signal which takes its value in the finite index set $\mathcal{P} = \{1, 2, \ldots, N\}$, $\xi(t) = [\xi_1(t), \xi_2(t), \ldots, \xi_N(t)]^{\mathrm{T}}$ is the indicator function, that is ,$\forall\ i \in \mathcal{P}$,

$$\xi_i(t) = \begin{cases} 1, & \text{when the substem } (A_i, B_i, C_i) \text{ is activated.} \\ 0, & \text{otherwise.} \end{cases}$$

As in [15] and [11], we assume the activation functions satisfying

(A1): Each g_i is globally Lipschitz continuous with $g_i(0) = 0$, that is, there exist a set of number l_i such that

$$|g_i(x_1) - g_i(x_2)| \leq l_i |x_1 - x_2|, \quad g_i(0) = 0,$$
$$i = 1, 2, \ldots, n$$

hold, for simplication, we denote $L = \mathrm{diag}(l_1, l_2, \ldots, l_n)$. In addition, the following assumption is further made:

(A2): The time-varying delay $\tau(t)$ is bounded and differential, that is, there exist $\hat{\tau} > 0$ and $\eta > 0$ such that $0 < \tau(t) < \hat{\tau}$ and $\dot{\tau}(t) \leq \eta < 1$ hold respectively. Obviously, this assumption is certainly satisfied if the transmission delay $\tau(t)$ is constant.

Remark 1. Some similar models of other switched neural networks with time-varying delay can be obtained in a similar way, such as switched HNNs, switched CNNs and switched BAM, that is, the individual systems are HNN, CNN and BAM model respectively. When $A_i = 0$, the switched RNNs degenerate into the switched HNNs model considered in [9]. Therefore, our proposed model is more general.

Next, we present some definitions and lemmas, which are needed in the next section.

Definition 1. A vector $x^* \in \mathbb{R}^n$ is said to be the equilibrium point of system (6) if it satisfies

$$\sum_{i=1}^{N} \xi_i(t)[-C_i x^* + A_i g(x^*) + B_i g(x^*)] = 0, \quad \forall\ t > t_0 - \hat{\tau}. \quad (7)$$

Definition 2. [12] Suppose that Ω is an open set of $\mathbb{R}^n$, and $G: \Omega \to \mathbb{R}^n$ is an operator. The constant

$$m_\Omega(G) \triangleq \sup_{\substack{x \neq y \\ x, y \in \Omega}} \frac{\langle G(x) - G(y), x - y\rangle}{\|x - y\|_2^2} = \sup_{\substack{x \neq y \\ x, y \in \Omega}} \frac{(x-y)^{\mathrm{T}}(G(x) - G(y))}{\|x - y\|_2^2} \quad (8)$$

is called the nonlinear measure of G on Ω with the norm $\|\cdot\|_2$.

Lemma 1. *[12] If $m_\Omega(G) < 0$, then G is an injective mapping on Ω. In addition, if $\Omega = \mathbb{R}^n$, then G is a homeomorphism of $\mathbb{R}^n$.*

Remark 2. Lemma 1 particularly shows that $G(x) = 0$ will have only one solution whenever $\Omega = \mathbb{R}^n$ and $m_\Omega(G) < 0$. By utilizing some matrix techniques, the nonlinear measure defined in the norm of $\|\cdot\|_2$ is convenient to use in real applications which will be shown later.

Lemma 2. *[2] The following LMI*

$$\begin{bmatrix} E(x) & H(x) \\ H^{\mathrm{T}}(x) & F(x) \end{bmatrix} > 0$$

where $E(x) = E^{\mathrm{T}}(x), F(x) = F^{\mathrm{T}}(x)$ and $H(x)$ depend affinely on x, is equivalent to each of the following conditions:
(1) $E(x) > 0, \quad F(x) - H^{\mathrm{T}}(x)E^{-1}(x)H(x) > 0$;
(2) $F(x) > 0, \quad E(x) - H(x)F^{-1}(x)H^{\mathrm{T}}(x) > 0$.

Lemma 3. *[20] Given any real matrices X, Y and $Q > 0$ with appropriate dimensions. Then the following matrix inequality holds:*

$$X^{\mathrm{T}}Y + Y^{\mathrm{T}}X \le X^{\mathrm{T}}QX + Y^{\mathrm{T}}Q^{-1}Y.$$

3 Stability analysis of switched RNN

This section discusses the GAS and GRS of the proposed model (6). Several new criteria are obtained in the form of LMIs. Firstly, GAS criteria are given in Section 3.1, then, based on the results in this section, a sufficient condition is given in Section 3.2.

3.1 GAS of the switched RNN

Theorem 1. *Under the assumptions (A1) and (A2), the origin of the switched RNN (6) is the unique equilibrium point and is globally asymptotically stable if there exist matrices $P = P^{\mathrm{T}} > 0$, $Q = Q^{\mathrm{T}} > 0$ and a diagonal matrix $K = diag(k_1, k_2, \ldots, k_n)$ such that the following LMIs hold for $i = 1, 2, \ldots, N$:*

$$\Sigma_i = \begin{bmatrix} -PC_i - C_iP + LKL & PA_i & PB_i \\ A_i^{\mathrm{T}}P & -K + Q & 0 \\ B_i^{\mathrm{T}}P & 0 & -(1-\eta)Q \end{bmatrix} < 0. \tag{9}$$

Proof: We shall prove this theorem in two steps. Firstly, we prove that the origin is the unique equilibrium by means of nonlinear measure. Secondly, we show the origin of (6) is globally asymptotically stable.
Step 1: Define an operator $G : \mathbb{R}^n \to \mathbb{R}^n$ by

$$G(x) = \sum_{i=1}^{N} \xi_i(t)\,[-C_i x + A_i g(x) + B_i g(x)] \tag{10}$$

where $x = [x_1, x_2, \ldots, x_n]^{\mathrm{T}} \in \mathbb{R}^n$.

Considering the following system

$$\frac{\mathrm{d}y(t)}{\mathrm{d}t} = PG(y(t)). \tag{11}$$

With the invertibility of the matrix P, we can conclude that the system (6) and the system (11) have the same equilibrium set. Next, we will show that $m_{\mathbb{R}^n}(PG) < 0$. Using Lemma 1, we obtain

$$\begin{aligned}
&(x-y)^{\mathrm{T}}P(G(x)-G(y)) \\
&\quad = (x-y)^{\mathrm{T}}P\left\{\sum_{i=1}^{N}\xi_i(t)[-C_i(x-y)\right. \\
&\qquad \left. + (A_i+B_i)(g(x)-g(y))]\right\} \\
&\quad \le \sum_{i=1}^{N}\xi_i(t)(x-y)^{\mathrm{T}}P[-C_i(x-y) \\
&\qquad + (A_i+B_i)(g(x)-g(y))] \\
&\quad = \sum_{i=1}^{N}\frac{\xi_i(t)}{2}[-(x-y)^{\mathrm{T}}(PC_i+C_iP)(x-y) \\
&\qquad + 2(x-y)^{\mathrm{T}}P(A_i+B_i)(g(x)-g(y))] \\
&\quad \le \sum_{i=1}^{N}\frac{\xi_i(t)}{2}[(x-y)^{\mathrm{T}}(-PC_i-C_iP)(x-y) \\
&\qquad + (g(x)-g(y))^{\mathrm{T}}K(g(x)-g(y))] \\
&\qquad + \sum_{i=1}^{N}\frac{\xi_i(t)}{2}(x-y)^{\mathrm{T}}P(A_i+B_i)K^{-1}(A_i+B_i)^{\mathrm{T}} \\
&\qquad \times P(x-y) = \sum_{i=1}^{N}\frac{\xi_i(t)}{2}(x-y)^{\mathrm{T}}(-PC_i-C_iP \\
&\qquad + P(A_i+B_i)K^{-1}(A_i+B_i)^{\mathrm{T}}P)(x-y) \\
&\qquad + \sum_{i=1}^{N}\frac{\xi_i(t)}{2}(g(x)-g(y))^{\mathrm{T}}K(g(x)-g(y)),
\end{aligned} \tag{12}$$

According to the assumption (A1), we have

$$(g(x) - g(y))^{\mathrm{T}} K (g(x) - g(y)) \\ \leq (x - y)^{\mathrm{T}} LKL(x - y), \tag{13}$$

which and (12) yield that

$$\begin{aligned}&(x - y)^{\mathrm{T}} P(G(x) - G(y)) \\ &\leq \sum_{i=1}^{N} \frac{\xi_i(t)}{2}(x - y)^{\mathrm{T}}(-PC_i - C_i P \\ &\quad + LKL + P(A_i + B_i)K^{-1} \\ &\quad \times (A_i + B_i)^{\mathrm{T}} P)(x - y).\end{aligned} \tag{14}$$

On the other hand, let

$$\Xi \triangleq \begin{bmatrix} I & 0 & 0 \\ 0 & I & I \end{bmatrix},$$

multiplying (9) by Ξ and Ξ^{T} on its left and right side, respectively, we obtain

$$\begin{bmatrix} PC_i + C_i P - LKL & -P(A_i + B_i) \\ -(A_i + B_i)^{\mathrm{T}} P & K - \eta Q \end{bmatrix} > 0, \tag{15}$$

which implies that

$$\begin{bmatrix} PC_i + C_i P - LKL & -P(A_i + B_i) \\ -(A_i + B_i)^{\mathrm{T}} P & K \end{bmatrix} > 0. \tag{16}$$

Based on Lemma 2, we can easily obtain

$$PC_i + C_i P - LKL \\ - P(A_i + B_i)K^{-1}(A_i + B_i)^{\mathrm{T}} P > 0. \tag{17}$$

From (14) and (17), we get

$$(x - y)^{\mathrm{T}} P(G(x) - G(y)) < 0. \tag{18}$$

According to Definition (2), we obtain $m_{\mathbb{R}^n}(PG) < 0$. Then by Lemma 1, we conclude that the origin is the unique equilibrium of system (11), which implies that the origin is the unique equilibrium of system (6).

Step 2: We choose the following Lyapunov functional

$$V(t, x_t) = x(t)^{\mathrm{T}} P x(t) \\ + \int_{t-\tau(t)}^{t} g^{\mathrm{T}}(x(s)) Q g(x(s))\, \mathrm{d}s. \tag{19}$$

Calculating the time derivative of V along the trajectory of (6), we obtain

$$\begin{aligned}\frac{\mathrm{d}V(t, x_t)}{\mathrm{d}t} &= 2x^{\mathrm{T}}(t) P \dot{x}(t) + g^{\mathrm{T}}(x(t)) Q g(x(t)) \\ &\quad - (1 - \dot{\tau}(t)) g^{\mathrm{T}}(x(t - \tau(t))) \\ &\quad \times Q g(x(t - \tau(t))) \\ &= 2x^{\mathrm{T}}(t) P \left[\sum_{i=1}^{N} \xi_i(t)(-C_i x + A_i g(x) \right. \\ &\quad \left. + B_i g(x(t - \tau(t)))) \right] + g^{\mathrm{T}}(x(t)) \\ &\quad \times Q g(x(t)) - (1 - \dot{\tau}(t)) g^{\mathrm{T}}(x(t - \tau(t))) \\ &\quad \times Q g(x(t - \tau(t))) \leq \sum_{i=1}^{N} \xi_i(t) \\ &\quad \times [-2x^{\mathrm{T}} PC_i x + 2x^{\mathrm{T}} PA_i g(x) \\ &\quad + 2x^{\mathrm{T}} PB_i g(x(t - \tau(t)))] \\ &\quad + \sum_{i=1}^{N} \xi_i(t)[g^{\mathrm{T}}(x(t)) Q g(x(t)) \\ &\quad - (1 - \eta) g^{\mathrm{T}}(x(t - \tau(t))) \\ &\quad \times Q g(x(t - \tau(t)))].\end{aligned} \tag{20}$$

Noticing the fact that

$$x^{\mathrm{T}} LKLx - g^{\mathrm{T}}(x) K g(x) \geq 0, \tag{21}$$

we obtain

$$\begin{aligned}\frac{\mathrm{d}V(t, x_t)}{\mathrm{d}t} &\leq \sum_{i=1}^{N} \xi_i(t)[x^{\mathrm{T}}(-PC_i - C_i P + LKL)x \\ &\quad + 2x^{\mathrm{T}} PA_i g(x) + 2x^{\mathrm{T}} PB_i g(x(t - \tau(t)))] \\ &\quad + \sum_{i=1}^{N} \xi_i(t)[g^{\mathrm{T}}(x(t))(-K + Q) g(x(t)) \\ &\quad - (1 - \eta) g^{\mathrm{T}}(x(t - \tau(t))) Q g(x(t - \tau(t)))]\end{aligned}$$

$$= \sum_{i=1}^{N} \xi_i(t)(x^{\mathrm{T}}(t), g^{\mathrm{T}}(x(t)), g^{\mathrm{T}}(x(t-\tau(t))))$$

$$\times \Sigma_i \begin{pmatrix} x(t) \\ g(x(t)) \\ g(x(t-\tau(t))) \end{pmatrix}. \tag{22}$$

Clearly, from (9), this implies $\frac{\mathrm{d}V(t,x_t)}{\mathrm{d}t} < 0$ for all $x(t) \neq 0$, and $\frac{\mathrm{d}V(t,x_t)}{\mathrm{d}t} = 0$ if and only if $x(t) = g(x(t)) = g(x(t-\tau(t))) = 0$. Hence, we can conclude that the origin of (6) is globally asymptotically stable. This completes the proof. □

Corollary 1. *Under the assumption (A1), the origin of the switched RNN with the constant transmission delay in (6) is the unique equilibrium point and is globally asymptotically stable if there exist matrices* $P = P^{\mathrm{T}} > 0$, $Q = Q^{\mathrm{T}} > 0$ *and a diagonal matrix* $K = diag(k_1, k_2, \ldots, k_n)$ *such that the following LMIs hold for* $i = 1, 2, \ldots, N$*:*

$$\begin{bmatrix} -PC_i - C_iP + LKL & PA_i & PB_i \\ A_i^{\mathrm{T}}P & -K+Q & 0 \\ B_i^{\mathrm{T}}P & 0 & -Q \end{bmatrix} < 0. \tag{23}$$

When f_i in (3) is globally Lipschitz continuous with Lipschitz constant l_i for all $i = 1, 2, \ldots, n$, we obtain the following corollary:

Corollary 2. *Under the assumption (A2), the RNN in (3) has and only has a unique equilibrium point which is globally asymptotically stable if there exist matrices* $P = P^{\mathrm{T}} > 0$, $Q = Q^{\mathrm{T}} > 0$ *and a diagonal matrix* $K = diag(k_1, k_2, \ldots, k_n)$ *such that the following LMI holds:*

$$\begin{bmatrix} -PC - CP + LKL & PA & PB \\ A^{\mathrm{T}}P & -K+Q & 0 \\ B^{\mathrm{T}}P & 0 & -(1-\eta)Q \end{bmatrix} < 0. \tag{24}$$

Proof: Here, $N = 1$. For system (3), defining the operator F from $\mathbb{R}^n$ to $\mathbb{R}^n$ as $F(x) = -Cx + Af(x) + Bf(x) + I$ where $x = [x_1, x_2, \ldots, x_n]^{\mathrm{T}} \in \mathbb{R}^n$, then similar to the proof of step 1 in Theorem 1, we can conclude that the RNN in (3) has and only has a unique equilibrium. Then we shift this equilibrium to the origin, and the GAS of the origin is directly obtained from the proof of step 2 in Theorem 1. This completes the proof. □

Remark 3. When $N = 1$, from Corollary 2, we obtain a sufficient condition to ensure the GAS of RNN (3). Compared with the previous literature, the activation functions here are only assumed to be Lipschitz continuous, not necessarily bounded, monotonic or differential. It is easy to verify the activation functions used in [1, 3, 7, 13, 25] are special cases of ours.

Remark 4. Clearly, a necessary condition for GAS under arbitrary switching rule is that all of the subsystems are globally asymptotically stable, but it is not sufficient. By employing nonlinear measure and Lyapunov approach, Theorem 1 provides a GAS criterion of the switched system (6). The results are given in the form of LMIs, which can be checked easily by the interior-point algorithm. This makes the design and application of the switched neural networks promising and practical.

3.2 GRS of the switched RNN

When the parameters uncertainties appear in the system (6), it becomes of the following model:

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = \sum_{i=1}^{N} \xi_i(t)[-(C_i + \triangle C_i(t))x(t) + (A_i + \triangle A_i(t))g(x(t)) + (B_i + \triangle B_i(t))g(x(t-\tau(t)))] \tag{25}$$

where $\triangle C_i(t)$, $\triangle A_i(t)$ and $\triangle B_i(t)$ represent the parameter uncertainties in the matrix C_i, A_i and B_i, respectively. In addition, we assume that the parameter uncertainties are unknown and time-varying, but norm-bounded, that is

$$(A3):\ [\triangle C_i(t), \triangle A_i(t), \triangle B_i(t)] = D_iF(t)\big[E_i^C, E_i^A, E_i^B\big] \tag{26}$$

where D_i, E_i^C, E_i^A and E_i^B are known real constant matrices with appropriate dimensions, the uncertain matrix $F(t)$ satisfies

$$F(t)F^{\mathrm{T}}(t) \leq I, \quad \text{for all } t \in \mathbb{R}. \tag{27}$$

Now, we present the GAS condition of the switched RNN in (25).

Theorem 2. *Under the assumptions (A1), (A2) and (A3), then the origin of the switched RNN in (25) is the unique equilibrium point and is globally asymptotically stable if there exist matrices* $P = P^{\mathrm{T}} > 0$, $Q = Q^{\mathrm{T}} > 0$, *a diagonal matrix* $K = diag(k_1, k_2, \ldots, k_n)$ *and three positive scalars* $\epsilon_1 > 0, \epsilon_2 > 0$ *and* $\epsilon_3 > 0$ *such that*

$$\Pi_i = \begin{bmatrix} -PC_i - C_iP + LKL + \epsilon_1(E_i^C)^{\mathrm{T}}E_i^C & PA_i & PB_i & PD_i \\ A_i^{\mathrm{T}}P & \Phi_{i2} & 0 & 0 \\ B_i^{\mathrm{T}}P & 0 & \Phi_{i3} & 0 \\ D_i^{\mathrm{T}}P & 0 & 0 & -\left(\epsilon_1{}^{-1} + \epsilon_2{}^{-1} + \epsilon_3{}^{-1}\right)^{-1}I \end{bmatrix} < 0, \tag{28}$$

or equivalently,

$$\tilde{\Pi}_i = \begin{bmatrix} -PC_i - C_iP + LKL + \epsilon_1\left(E_i^C\right)^{\mathrm{T}}E_i^C & PA_i & PB_i & PD_i & PD_i & PD_i \\ A_i^{\mathrm{T}}P & \Phi_{i2} & 0 & 0 & 0 & 0 \\ B_i^{\mathrm{T}}P & 0 & \Phi_{i3} & 0 & 0 & 0 \\ D_i^{\mathrm{T}}P & 0 & 0 & -\epsilon_1 I & 0 & 0 \\ D_i^{\mathrm{T}}P & 0 & 0 & 0 & -\epsilon_2 I & 0 \\ D_i^{\mathrm{T}}P & 0 & 0 & 0 & 0 & -\epsilon_3 I \end{bmatrix} < 0 \tag{29}$$

hold for $i = 1, 2, \ldots, N$, *where*

$$\Phi_{i2} = -K + Q + \epsilon_2\left(E_i^A\right)^{\mathrm{T}}E_i^A,$$

$$\Phi_{i3} = -(1-\eta)Q + \epsilon_3\left(E_i^B\right)^{\mathrm{T}}E_i^B.$$

Proof: Using Lemma 2, we can easily check that the condition (28) is equivalent to the condition (29). According to the assumption (A3), using Lemma 3, we have

$$\begin{bmatrix} -P\triangle C_i - \triangle C_iP & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} < \epsilon_1{}^{-1}\Omega_{i0}\Omega_{i0}^{\mathrm{T}} + \epsilon_1\Omega_{i1}\Omega_{i1}^{\mathrm{T}}, \tag{30}$$

$$\begin{bmatrix} 0 & P\triangle A_i & 0 \\ \triangle A_i^{\mathrm{T}}P & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} < \epsilon_2{}^{-1}\Omega_{i0}\Omega_{i0}^{\mathrm{T}} + \epsilon_2\Omega_{i2}\Omega_{i2}^{\mathrm{T}} \tag{31}$$

and

$$\begin{bmatrix} 0 & 0 & P\triangle B_i \\ 0 & 0 & 0 \\ \triangle B_i^{\mathrm{T}}P & 0 & 0 \end{bmatrix} < \epsilon_3{}^{-1}\Omega_{i0}\Omega_{i0}^{\mathrm{T}} + \epsilon_3\Omega_{i3}\Omega_{i3}^{\mathrm{T}} \tag{32}$$

where

$$\Omega_{i0} = \left[D_i^{\mathrm{T}}P, 0, 0\right]^{\mathrm{T}}, \quad \Omega_{i1} = \left[E_i^C, 0, 0\right]^{\mathrm{T}},$$

$$\Omega_{i2} = \left[0, E_i^A, 0\right], \quad \Omega_{i3} = \left[0, 0, E_i^B\right]^{\mathrm{T}}.$$

From (28), Using Lemma 1 again, we obtain

$$\Upsilon_i \triangleq \begin{bmatrix} \Phi_{i1} & PA_i & PB_i \\ A_i^{\mathrm{T}}P & \Phi_{i2} & 0 \\ B_i^{\mathrm{T}}P & 0 & \Phi_{i3} \end{bmatrix} < 0 \tag{33}$$

where $\Phi_{i1} = -PC_i - C_iP + LKL + \epsilon_1(E_i^C)^{\mathrm{T}}E_i^C + (\epsilon_1{}^{-1} + \epsilon_2{}^{-1} + \epsilon_3{}^{-1})PD_iD_i^{\mathrm{T}}P$.

Based on (30)–(33), we have the following equalities for $i = 1, 2, \dots, N$,

$$\begin{bmatrix} -P(C_i + \triangle C_i) - (C_i + \triangle C_i)^{\mathrm{T}} P + LKL & P(A_i + \triangle A_i) & P(B_i + \triangle B_i) \\ (A_i + \triangle A_i)^{\mathrm{T}} P & -K + Q & 0 \\ (B_i + \triangle B_i)^{\mathrm{T}} P & 0 & -(1-\eta)Q \end{bmatrix} < 0. \tag{34}$$

Hence, by (34) and Theorem 1, the desired result follows directly. □

Remark 5. In fact, Assumption (A3) can be generalized to the following: $\triangle C_i(t), \triangle A_i(t), \triangle B_i(t)$ to satisfy $\triangle C_i(t) = D_i^C F(t) E_i^C$, $\triangle A_i(t) = D_i^A F(t) E_i^A$, $\triangle B_i(t) = D_i^C F(t) E_i^B$ with $F(t)F^{\mathrm{T}}(t) \le I$. From the proof of Theorem 2, we can conclude this does not increase the difficulty in the robust analysis of the switched RNN model (25). Therefore, the assumption (A3) is reasonable.

4 Illustrative examples

In this section, we will give three examples to show the effectiveness of the above-obtained results.

Example 1. Consider the following recurrent neural networks with constant delay

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = -Cx(t) + Af(x(t)) + Bf(x(t-\tau)) + I \tag{35}$$

where the activation functions $f_i(x) = \frac{2}{3}\sin x + \frac{1}{3}x$ ($i = 1, 2$), and the parameters

$$A = \begin{bmatrix} 0.35 & -0.2 \\ -0.15 & 0.4 \end{bmatrix}, \quad B = \begin{bmatrix} 0.4 & 0.2 \\ 0.1 & 0.2 \end{bmatrix},$$
$$C = \begin{bmatrix} 0.9 & 0 \\ 0 & 0.8 \end{bmatrix}.$$

Obviously, $f_i(x)$ is Lipschitz continuous with Lipschitz constant $l_i = 1$ ($i = 1, 2$), that is $L = \mathrm{diag}(1, 1)$. On the other hand, we can check it easily that $f_i(x)$ is nonmonotone, so the conditions given in [1, 3, 7, 13, 25] are not satisfied. In addition, the matrix α in Theorem 3.1 of [28] is obtained as

$$\alpha = -[-CL^{-1} + |A| + |B|] = \begin{bmatrix} 0.15 & -0.4 \\ -0.25 & 0.2 \end{bmatrix},$$

which is not a M-matrix. This implies that criteria in [1, 3, 7, 13, 25, 28] all failed to conclude whether this system is globally asymptotically stable or not. However, by resorting to the Matlab LMI Control Toolbox to solve the (24) in Corollary 2, we can see that it is feasible and the solutions are obtained as follows:

$$P = \begin{bmatrix} 1.5316 & 0.0231 \\ 0.0231 & 1.8130 \end{bmatrix},$$
$$Q = \begin{bmatrix} 0.8042 & 0.1939 \\ 0.1939 & 0.7627 \end{bmatrix},$$
$$K = \begin{bmatrix} 1.5050 & 0 \\ 0 & 1.7072 \end{bmatrix}.$$

According to Corollary 2, we have that the system (35) has and only has a unique equilibrium point which is globally asymptotically stable.

For numerical simulation, let $I = [0.9, 1.2]^{\mathrm{T}}$ and the delay $\tau = 1$. Based on fourth-order Runge-Kutta method, Fig. 1 depicts the time responses of state variables $x_1(t)$ and $x_2(t)$ from the 30 random constant initial states in the set $[-2,4]\times[-2,4]$ with step $h = 0.01$, it can be seen from Fig. 1 that the trajectory of the system (35) asymptotically converges to a unique equilibrium $x^* = (2.0626, 2.3614)^{\mathrm{T}}$.

Example 2. Consider the following switched RNNs:

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = \sum_{i=1}^{2} \xi_i(t)[-C_i x(t) + A_i g(x(t)) + B_i g(x(t-\tau(t)))] \tag{36}$$

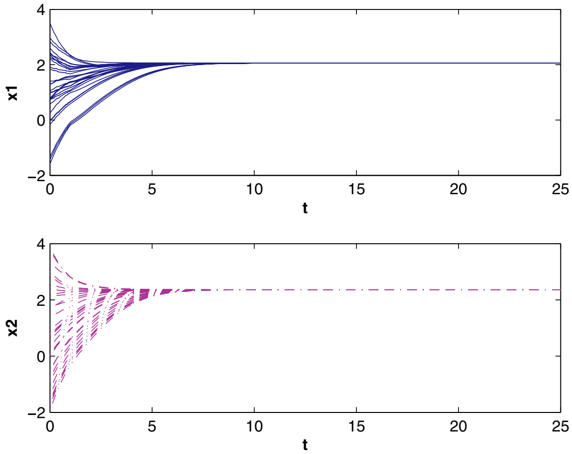

Fig. 1 Time responses of state variables $x_1(t)$ and $x_2(t)$ in Example 1

where $g_i(x) = -|x|$ $(i = 1, 2)$, $\tau(t) = 0.2\sin t + 0.8$ and the parameters are given as

$$C_1 = \begin{bmatrix} 0.900 & 0 \\ 0 & 0.900 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0.350 & -0.120 \\ -0.010 & 0.240 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 0.400 & 0.200 \\ 0.100 & 0.400 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 1.100 & 0 \\ 0 & 1.300 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} -0.430 & 0.211 \\ -0.122 & 0.651 \end{bmatrix},$$

$$B_2 = \begin{bmatrix} 0.552 & -0.268 \\ -0.211 & 0.254 \end{bmatrix}.$$

Obviously, Assumptions (A1) and (A2) are satisfied with $L = \mathrm{diag}(1, 1)$ and $\eta = 0.2$, $\hat{\tau} = 1$. By resorting to the Matlab LMI Control Toolbox to solve the (9) in Theorem 1, we obtain

$$P = \begin{bmatrix} 23.8286 & 2.5860 \\ 2.5860 & 60.9644 \end{bmatrix},$$

$$Q = \begin{bmatrix} 15.4975 & 0.1145 \\ 0.1145 & 42.8986 \end{bmatrix},$$

$$K = \begin{bmatrix} 24.7755 & 0 \\ 0 & 68.3417 \end{bmatrix}.$$

From Theorem 1, it follows that the switched RNN (36) is globally asymptotically stable under arbitrary switching law.

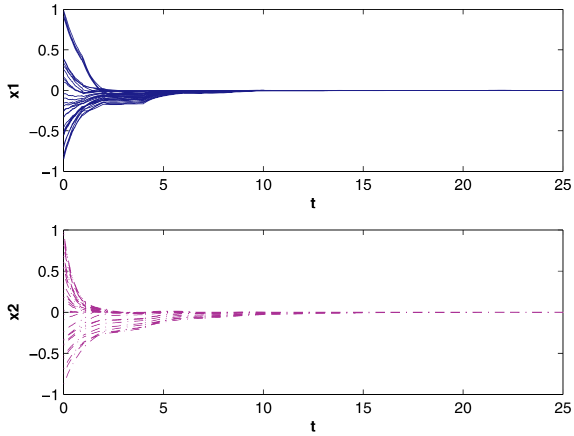

Fig. 2 Time responses of state variables $x_1(t)$ and $x_2(t)$ in Example 2

For numerical simulation. We assume that the two subsystems are switched every four seconds. Figure 2 depicts the time responses of state variables $x_1(t)$ and $x_2(t)$ from the 30 random constant initial states in the set $[-1, 1]\times[-1, 1]$ with step $h = 0.01$, which shows the switched system (36) asymptotically converges to the unique equilibrium $x^* = (0, 0)^{\mathrm{T}}$.

Example 3. Consider the following switched RNN with constant delay:

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = \sum_{i=1}^{2} \xi_i(t)[-(C_i + \triangle C_i(t))x(t) + (A_i + \triangle A_i(t))g(x(t)) + (B_i + \triangle B_i(t))g(x(t-\tau))] \tag{37}$$

where $g_i(x) = \frac{2}{3}\sin x + \frac{1}{3}x$ $(i = 1, 2)$ and the system parameters are given as

$$C_1 = \begin{bmatrix} 2.41 & 0 \\ 0 & 2.41 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0.95 & -0.24 \\ -0.31 & 1.02 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 0.81 & 0.20 \\ 0.100 & 0.64 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 2.26 & 0 \\ 0 & 2.26 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 0.87 & 0.21 \\ 0.32 & 0.75 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0.67 & -0.28 \\ -0.31 & 1.02 \end{bmatrix},$$

$$E_1^C = \begin{bmatrix} -0.5 & 0 \\ 0 & 0.4 \end{bmatrix}, \quad E_1^A = \begin{bmatrix} 0.3 & -0.2 \\ 0.3 & 0.4 \end{bmatrix},$$

$$E_1^B = \begin{bmatrix} 0.3 & 0.1 \\ 0.1 & -0.4 \end{bmatrix}, \quad E_2^C = \begin{bmatrix} -0.4 & 0 \\ 0 & 0.5 \end{bmatrix},$$

$$E_2^A = \begin{bmatrix} 0.5 & -0.1 \\ 0.2 & 0.5 \end{bmatrix}, \quad E_2^B = \begin{bmatrix} 0.4 & 0.2 \\ -0.2 & 0.4 \end{bmatrix}$$

and $D_1 = D_2 = 0.4I$, $F(t)$ is any matrix satisfying (27). By resorting to the Matlab LMI Control Toolbox to solve the (29) in Theorem 2, we obtain

$$P = \begin{bmatrix} 1.2082 & -0.0274 \\ -0.0274 & 1.1735 \end{bmatrix},$$

$$Q = \begin{bmatrix} 1.1413 & -0.0800 \\ -0.0800 & 1.1101 \end{bmatrix},$$

$$K = \begin{bmatrix} 2.7052 & 0 \\ 0 & 2.5994 \end{bmatrix}$$

and $\epsilon_1 = 1.1849, \epsilon_2 = 1.2203, \epsilon_3 = 1.2882$. From Theorem 2, it follows that the switched RNN (37) is globally asymptotically stable for all admissible uncertainties under the arbitrary switching law.

5 Conclusion

In this paper, a new switched RNN model has been presented by combining the theory of switched systems with recurrent neural networks. Moreover, several new sufficient conditions to ensure GAS and GRS of the switched RNNs have been derived by employing nonlinear measure and Lyapunov method. The results are given in the form of LMIs, which can be performed efficiently via numerical algorithms such as interior-point algorithms for solving LMIs. The new results obtained in this paper generalize and improve the earlier works, and the new conditions are easy to check and apply in practice. Three examples are also provided to show the effectiveness of the proposed results.

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